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Research Article

Hölder Quasicontinuity in Variable Exponent Sobolev Spaces

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We show that a function in the variable exponent Sobolev spaces coincides with a Hölder continuous Sobolev function outside a small exceptional set. This gives us a method to approximate a Sobolev function with Hölder continuous functions in the Sobolev norm. Our argument is based on a Whitney-type extension and maximal function estimates. The size of the exceptional set is estimated in terms of Lebesgue measure and a capacity. In these estimates, we use the fractional maximal function as a test function for the capacity.

1. Introduction

Our main objective is to study the pointwise behaviour and Lusin-type approximation of functions which belong to a variable exponent Sobolev space. In particular, we are interested in the first-order Sobolev spaces. The standard Sobolev space $W^{1,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ consists of functions $u \in L^p(\mathbb{R}^n)$, whose distributional gradient $Du = (D_1 u,\dots,D_n u)$ also belongs to $L^p(\mathbb{R}^n)$. The rough philosophy behind the variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^n)$ is that the standard Lebesgue norm is replaced with the quantity

$$\int_{\mathbb{R}^n} |u(x)|^{p(x)} dx, \tag{1.1}$$

where p is a function of x. The exact definition is presented below, see also [1, 2]. Variable exponent Sobolev spaces have been used in the modeling of electrorheological fluids, see, for example, [3–7] and references therein. Very recently, Chen et al. have introduced a new variable exponent model for image restoration [8].

A somewhat unexpected feature of the variable exponent Sobolev spaces is that smooth functions need not be dense without additional assumptions on the exponent. This was observed by Zhikov in connection with the so-called Lavrentiev phenomenon. In [9], he introduced a logarithmic condition on modulus of continuity of the variable exponent. Variants of this condition have been expedient tools in the study of maximal functions, singular integral operators, and partial differential equations with nonstandard growth conditions on variable exponent spaces. This assumption is also important for us. Under this assumption, compactly supported smooth functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$.

Instead of approximating by smooth functions, we are interested in Lusin-type approximation of variable exponent Sobolev functions. By a Lusin-type approximation we mean that the Sobolev function coincides with a continuous Sobolev function outside a small exceptional set. The essential difference compared to the standard convolution approximation is that the mollification by convolution may differ from the original function at every point. In particular, our result implies that every variable exponent Sobolev function can be approximated in the Lusin sense by Hölder continuous Sobolev functions in the variable exponent Sobolev space norm. In the classical case this kind of question has been studied, for example, in [10–16]. For applications in calculus of variations and partial differential equations, we refer, for example, to [17, 18].

Our approach is based on maximal functions. For a different point of view, which is related to [15], in the variable exponent case, we refer to [19]. Bounds for maximal functions in variable exponent spaces have been obtained in [20–27]. The exceptional set is estimated in terms of Lebesgue measure and capacity. We apply the fact that the fractional maximal function is smoother than the original function and it can be used as a test function for the capacity.

2. Variable exponent spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $p : \Omega \to [1,\infty)$ be a measurable function (called the *variable exponent* on Ω). We write

$$p_\Omega^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x), \qquad p_\Omega^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \tag{2.1}$$

and abbreviate $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$. Throughout the work we assume that $1 < p^- \le p^+ < \infty$. Later we make further assumptions on the exponent p.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \to [-\infty,\infty]$ such that

$$\varrho_{p(\cdot),\Omega}(u) = \int_\Omega |u(x)|^{p(x)}\,dx < \infty. \tag{2.2}$$

The function $\varrho_{p(\cdot),\Omega}(\cdot) : L^{p(\cdot)}(\Omega) \to [0,\infty]$ is called the *modular* of the space $L^{p(\cdot)}(\Omega)$. We define the *Luxemburg norm* on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\}. \tag{2.3}$$

The variable exponent Lebesgue space is a special case of a more general Orlicz-Musielak space studied in [28]. For a constant function $p(\cdot)$, the variable exponent Lebesgue space coincides with the standard Lebesgue space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of all functions $u \in L^{p(\cdot)}(\Omega)$, whose distributional gradient $Du = (D_1 u,\dots,D_n u)$ belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)}. \tag{2.4}$$

For the basic theory of variable exponent spaces, we refer to [1], see also [2].

3. Capacities

We are interested in pointwise properties of variable exponent Sobolev functions and, for simplicity, we assume that our functions are defined in all of $\mathbb{R}^n$. Exceptional sets for Sobolev functions are measured in terms of the capacity. In the variable exponent case, the capacity has been studied in [29, Section 3]. Let us recall the definition here. The *Sobolev $p(\cdot)$-capacity* of $E \subset \mathbb{R}^n$ is defined by

$$C_{p(\cdot)}(E) = \inf \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |Du(x)|^{p(x)} \right) dx, \tag{3.1}$$

where the infimum is taken over all admissible functions $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u \geqslant 1$ in an open set containing E. If there are no admissible functions for E, we set $C_{p(\cdot)}(E) = \infty$. This capacity enjoys many standard properties of capacities, for example, it is an outer measure and a Choquet capacity, see [29, Corollaries 3.3 and 3.4].

We define yet another capacity of $E \subset \mathbb{R}^n$ by setting

$$\operatorname{Cap}_{p(\cdot)}(E) = \inf \int_{\mathbb{R}^n} \left(|u(x)|^{p^*(x)} + |Du(x)|^{p(x)} \right) dx, \tag{3.2}$$

where $p^*(x) = np(x)/(n - p(x))$ is the Sobolev conjugate of $p(x)$ and the infimum is taken over all functions u such that $u \in L^{p^*(\cdot)}(\mathbb{R}^n)$, $Du \in L^{p(\cdot)}(\mathbb{R}^n)$, and $u \geq 1$ in an open set containing E.

It is easy to see that

$$|E| \leq C_{p(\cdot)}(E), \qquad |E| \leq \operatorname{Cap}_{p(\cdot)}(E). \tag{3.3}$$

Thus both capacities are finer measures than Lebesgue measure. Next we study the relation of the capacities defined by (3.1) and (3.2).

By truncation it is easy to see that in (3.1) and (3.2) it is enough to test with admissible functions which satisfy $0 \leq u \leq 1$. For those functions, we have

$$|u(x)|^{p^*(x)} \leq |u(x)|^{p(x)}, \tag{3.4}$$

and hence

$$\mathrm{Cap}_{p(\cdot)}(E) \le C_{p(\cdot)}(E). \tag{3.5}$$

In particular, if $C_{p(\cdot)}(E) = 0$, then $\mathrm{Cap}_{p(\cdot)}(E) = 0$.

Assume then that $\mathrm{Cap}_{p(\cdot)}(E) = 0$. By the basic properties of Sobolev capacity, we have

$$C_{p(\cdot)}(E) = \lim_{i\to\infty} C_{p(\cdot)}(E \cap B(0,i)). \tag{3.6}$$

Hence, in order to show that $C_{p(\cdot)}(E) = 0$, it is enough to prove that $C_{p(\cdot)}(E \cap B(0,i)) = 0$ for every $i = 1,2,\dots$. Let $\varepsilon > 0$. Since $\mathrm{Cap}_{p(\cdot)}(E \cap B(0,i)) = 0$, there exists an admissible function $u \in L^{p^*(\cdot)}(\mathbb{R}^n)$, $Du \in L^{p(\cdot)}(\mathbb{R}^n)$, and $u \ge 1$ in an open set containing $E \cap B(0,i)$ for which

$$\int_{\mathbb{R}^n} \left(|u(x)|^{p^*(x)} + |Du(x)|^{p(x)} \right) dx < \varepsilon. \tag{3.7}$$

Let $\phi \in C_0^\infty(B(0,2i))$ be a cutoff function which is one in $E \cap B(0,i)$ and $|D\phi| \le c$. Now it is easy to show that ϕu is an admissible function for $C_{p(\cdot)}(E \cap B(0,i))$ and hence $C_{p(\cdot)}(E \cap B(0,i)) < c\varepsilon$. Letting $\varepsilon \to 0$, we see that $C_{p(\cdot)}(E \cap B(0,i)) = 0$. This implies that the capacities defined by (3.1) and (3.2) have the same sets of zero capacity.

Recall that a function $u : \mathbb{R}^n \to [-\infty, \infty]$ is said to be $p(\cdot)$-quasicontinuous with respect to capacity $C_{p(\cdot)}$ if for every $\varepsilon > 0$ there exists an open set U with $C_{p(\cdot)}(U) < \varepsilon$ such that the restriction of u to $\mathbb{R}^n \setminus U$ is continuous. We also say that a claim holds $p(\cdot)$-quasieverywhere with respect to capacity $C_{p(\cdot)}$ if it holds everywhere in $\mathbb{R}^n \setminus N$ with $C_{p(\cdot)}(N) = 0$. The corresponding notions can be defined with respect to capacity $\mathrm{Cap}_{p(\cdot)}$ in the obvious way.

By (3.5) we see that if a function is $p(\cdot)$-quasicontinuous with respect to capacity $C_{p(\cdot)}$, then it is $p(\cdot)$-quasicontinuous with respect to capacity $\mathrm{Cap}_{p(\cdot)}$. From now on, we will use the capacity defined by (3.2). It has certain advantages over the capacity defined by (3.1) which will become clear when we estimate the size of the exceptional set in our main result.

If continuous functions are dense in the variable exponent Sobolev space, then each function in $W^{1,p(\cdot)}(\mathbb{R}^n)$ has a $p(\cdot)$-quasicontinuous representative, see [29, Theorem 5.2]. It follows from our assumptions that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, which implies that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ [30, Corollary 2.5]. Usually a function $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is defined only up to a set of measure zero. We define u pointwise by setting

$$u^*(x) = \limsup_{r\to 0} \fint_{B(x,r)} u(y)\,dy. \tag{3.8}$$

Here the barred integral sign denotes the integral average. Observe that $u^* : \mathbb{R}^n \to [-\infty, \infty]$ is a Borel function which is defined everywhere in $\mathbb{R}^n$ and that it is independent of the choice of the representative of u. Instead of the limes superior the actual limes in (3.8) exists $p(\cdot)$-quasieverywhere in $\mathbb{R}^n$ and u^* is a quasicontinuous representative of u, see [31]. For every function $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, we take the representative given by (3.8).

4. Fractional maximal function

The fractional maximal operator of a locally integrable function f is defined by

$$\mathcal{M}_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f(y)|\,dy, \quad 0 \le \alpha < n. \tag{4.1}$$

Here $B(x,r)$ with $x \in \mathbb{R}^n$ and $r > 0$ denotes the open ball with center x and radius r. The restricted fractional maximal operator where the infimum is taken only over the radii $0 < r < R$ for some $R > 0$ is denoted by $\mathcal{M}_{\alpha,R} f(x)$. If $\alpha = 0$, then $\mathcal{M} f = \mathcal{M}_0 f$ is the Hardy-Littlewood maximal operator.

We say that the exponent $p : \mathbb{R}^n \to [1,\infty)$ is log-*Hölder continuous* if there exists a constant $c > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{c}{-\log|x-y|} \tag{4.2}$$

for every $x, y \in \mathbb{R}^n$ with $|x - y| \leqslant 1/2$. Assume that p is log-Hölder continuous and, in addition, that

$$|p(x) - p(y)| \le \frac{c}{\log(e + |x|)} \tag{4.3}$$

for every $x, y \in \mathbb{R}^n$ with $|y| \ge |x|$. Let us briefly discuss conditions (4.2) and (4.3) here. Under these assumptions on p, Cruz-Uribe, Fiorenza, and Neugebauer have proved that the Hardy-Littlewood maximal operator $\mathcal{M} : L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n)$ is bounded, see [21, 22]. This is an improvement of earlier work by Diening [24] and Nekvinda [27]. In [32], Pick and Růžička have given an example which shows that if log-Hölder continuity is replaced by a slightly weaker continuity condition, then the Hardy-Littlewood maximal operator need not be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Lerner has shown that the Hardy-Littlewood maximal operator may be bounded even if the exponent is discontinuous [26].

There is also a Sobolev embedding theorem for the fractional maximal function in variable exponent spaces. If $1 < p^- \le p^+ < n$, (4.2), (4.3) hold, and $0 \le \alpha < n/p^+$, then Capone, Cruz-Uribe, and Fiorenza have proved in [20, Theorem 1.4] that

$$\mathcal{M}_\alpha : L^{p(\cdot)}(\mathbb{R}^n) \longrightarrow L^{np(\cdot)/(n-\alpha p(\cdot))}(\mathbb{R}^n) \tag{4.4}$$

is bounded. Observe that when $\alpha = 0$, then this reduces to the fact that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

A simple modification of a result of Kinnunen and Saksman [33, Theorem 3.1] shows that if (4.4) holds, $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $1 < p^- \le p^+ < n$, $1 \le \alpha < n/p^+$, then

$$\mathcal{M}_\alpha f \in L^{q^*(\cdot)}(\mathbb{R}^n), \qquad D_i \mathcal{M}_\alpha f \in L^{q(\cdot)}(\mathbb{R}^n), \quad i = 1,2,\dots,n. \tag{4.5}$$

Moreover, we have

$$\|\mathcal{M}_\alpha f\|_{L^{q^*(\cdot)}(\mathbb{R}^n)} \le c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{4.6}$$

$$\|D\mathcal{M}_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{4.7}$$

where

$$q(x)=\frac{np(x)}{n-(\alpha-1)p(x)},\qquad q^*(x)=\frac{np(x)}{n-\alpha p(x)}. \tag{4.8}$$

Estimate (4.7) follows from the pointwise inequality

$$|D_i\mathcal{M}_\alpha f(x)|\le c\mathcal{M}_{\alpha-1}f(x),\quad i=1,2,\dots,n, \tag{4.9}$$

for almost every $x\in\mathbb{R}^n$ and the Sobolev embedding (4.4), see [33, Theorem 3.1]. Roughly speaking, this means that the fractional maximal operator is a smoothing operator and it usually belongs to certain Sobolev space. This enables us to use the fractional maximal function as a test function for certain capacities.

5. Hölder-type quasicontinuity

In this section, we assume that $1<p^-\le p^+<\infty$ and that the Hardy-Littlewood maximal operator $\mathcal{M}:L^{p(\cdot)}(\mathbb{R}^n)\to L^{p(\cdot)}(\mathbb{R}^n)$ is bounded. We begin by recalling the well-known estimates for the oscillation of the function in terms of the fractional maximal function of the gradient. The proof of our main result is based on these estimates.

Let $x_0\in\mathbb{R}^n$ and $R>0$. If $u\in C^1(\mathbb{R}^n)$, then

$$⨍_{B(x_0,R)}|u(z)-u(y)|\,dy\le c(n)\int_{B(x_0,R)}\frac{|Du(y)|}{|z-y|^{n-1}}dy \tag{5.1}$$

for every $z\in B(x_0,R)$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, we find that the inequality (5.1) holds for almost every $x\in B(x_0,R)$ for each $u\in W^{1,p(\cdot)}(\mathbb{R}^n)$.

Let $B(x,r)\subset B(x_0,R)$. We integrate (5.1) over the ball $B(x,r)$ and obtain

$$\begin{aligned}⨍_{B(x_0,R)}|u_{B(x,r)}-u(y)|\,dy&\le ⨍_{B(x,r)}⨍_{B(x_0,R)}|u(z)-u(y)|\,dy\,dz\\&\le c(n)⨍_{B(x,r)}\int_{B(x_0,R)}\frac{|Du(y)|}{|z-y|^{n-1}}dy\,dz\\&\le c(n)\int_{B(x_0,R)}⨍_{B(x,r)}|z-y|^{1-n}dz\,|Du(y)|\,dy\\&\le c(n)\int_{B(x_0,R)}\frac{|Du(y)|}{|x-y|^{n-1}}dy.\end{aligned} \tag{5.2}$$

Here we also used the simple fact that

$$⨍_{B(x,r)}|z-y|^{1-n}dz\le c(n)|x-y|^{1-n}. \tag{5.3}$$

From this, we conclude that

$$⨍_{B(x_0,R)}\left|\limsup_{r\to 0}⨍_{B(x,r)}u(z)dz-u(y)\right|dy\le c(n)\int_{B(x_0,R)}\frac{|Du(y)|}{|x-y|^{n-1}}dy. \tag{5.4}$$

This shows that the inequality (5.1) is true at every $x \in B(x_0,R)$ for $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, which is defined pointwise by (3.8). A Hedberg-type zooming argument gives

$$\begin{aligned}
\int_{B(x_0,R)} \frac{|Du(y)|}{|x-y|^{n-1}}dy &\le \int_{B(x,2R)} \frac{|Du(y)|}{|x-y|^{n-1}}dy \\
&\le \sum_{i=0}^{\infty} \int_{B(x,2^{1-i}R)\setminus B(x,2^{-i}R)} \frac{|Du(y)|}{|x-y|^{n-1}}dy \\
&\le \sum_{i=0}^{\infty} 2^{i(n-1)}R^{1-n} \int_{B(x,2^{1-i}R)} |Du(y)|\,dy \\
&\le c(n)R\sum_{i=0}^{\infty} 2^{-i} \fint_{B(x,2^{1-i}R)} |Du(y)|\,dy \\
&= c(n)R^{1-\alpha/q}\sum_{i=0}^{\infty} 2^{-i}R^{\alpha/q} \fint_{B(x,2^{1-i}R)} |Du(y)|\,dy \\
&\le c(n)R^{1-\alpha/q}\mathcal{M}_{\alpha/q,2R}|Du|(x),
\end{aligned} \tag{5.5}$$

where $0 \le \alpha < q$.

Let $R = |x-y|$ and choose $x_0 \in \mathbb{R}^n$ so that $x, y \in B(x_0,R)$. A simple computation gives

$$\begin{aligned}
|u(x)-u(y)| &\le |u(x)-u_{B(x_0,R)}| + |u(y)-u_{B(x_0,R)}| \\
&\le \fint_{B(x_0,R)} |u(x)-u(z)|\,dz + \fint_{B(x_0,R)} |u(y)-u(z)|\,dz \\
&\le c(n)|x-y|^{1-\alpha/q}\big(\mathcal{M}_{\alpha/q}|Du|(x) + \mathcal{M}_{\alpha/q}|Du|(y)\big)
\end{aligned} \tag{5.6}$$

for every $x, y \in \mathbb{R}^n$, if u is defined pointwise by (3.8).

Remark 5.1. It follows from the previous considerations that

$$\fint_{B(x,R)} |u(x)-u(z)|\,dz \le c(n)R^{1-\alpha/q}\mathcal{M}_{\alpha/q}|Du|(x) \tag{5.7}$$

for every $x \in \mathbb{R}^n$, if u is defined pointwise by (3.8). Thus all points which belong to the set

$$\{x \in \mathbb{R}^n : \mathcal{M}_{\alpha/q}|Du|(x) < \infty\} \tag{5.8}$$

are Lebesgue points of u. Next we provide a more quantitative version of this statement.

The following theorem is our main result. Later we give a sharper estimate on the size of the exceptional set in the theorem.

Theorem 5.2. *Assume that $1 < p^- \le p^+ < \infty$, $0 \le \alpha < q$, and that the Hardy-Littlewood maximal operator $\mathcal{M} : L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n)$ is bounded. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be defined* *pointwisely by (3.8). Then there exists $\lambda_0 \geq 1$ such that for every $\lambda \geq \lambda_0$, there are an open set U_λ and a function u_λ with the following properties:*

(i) $u(x) = u_\lambda(x)$ *for every* $x \in \mathbb{R}^n \setminus U_\lambda$,

(ii) $\|u - u_\lambda\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} \to 0$ *as* $\lambda \to 0$,

(iii) u_λ *is locally* $(1-\alpha/q)$*-Hölder continuous,*

(iv) $|U_\lambda| \to 0$ *as* $\lambda \to \infty$.

Remark 5.3. If $\alpha = 0$, then the theorem says that every function in the variable exponent Sobolev space coincides with a Lipschitz function outside a set of arbitrarily small Lebesgue measure. The obtained Lipchitz function approximates the original Sobolev function also in the Sobolev norm.

Proof. First we assume that the support of u is contained in a ball $B(x_0,2)$ for some $x_0 \in \mathbb{R}^n$. Later we show that the general case follows from this by a partition of unity.

We denote

$$U_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}_{\alpha/q}|Du|(x) > \lambda\}, \tag{5.9}$$

where $\lambda > 0$. We claim that there is $\lambda_0 \geq 1$ such that for every $x \in \mathbb{R}^n$ and $r > 1$ we have

$$r^{\alpha/q} \fint_{B(x,r)} |Du(y)|\,dy \leq \lambda_0. \tag{5.10}$$

Indeed, if $B(x,r) \cap B(x_0,2) \neq \emptyset$ and $r > 1$, then

$$\begin{aligned} r^{\alpha/q} \fint_{B(x,r)} |Du(y)|\,dy &= c(n) r^{\alpha/q-n} \int_{B(x,r)} |Du(y)|\,dy \\ &\leq c(n) \int_{B(x_0,2)} |Du(y)|\,dy, \end{aligned} \tag{5.11}$$

and hence we may choose

$$\lambda_0 = c(n) \int_{\mathbb{R}^n} |Du(y)|\,dy. \tag{5.12}$$

Taking a larger number if necessary, we may assume that $\lambda_0 \geq 1$. In particular, this implies that

$$U_\lambda \subset \{x \in B(x_0,3) : \mathcal{M}_{\alpha/q,1}|Du|(x) > \lambda\} \tag{5.13}$$

when $\lambda \geq \lambda_0$, where

$$\mathcal{M}_{\alpha/q,1}|Du|(x) = \sup_{0<r<1} r^{\alpha/q} \fint_{B(x,r)} |Du(y)|\,dy \leq \mathcal{M}|Du|(x). \tag{5.14}$$

From this, we conclude that

$$|U_\lambda| \leq \int_{U_\lambda} (\lambda^{-1}\mathcal{M}|Du|(x))^{p(x)} dx \leq \lambda^{-p^-} \int_{\mathbb{R}^n} (\mathcal{M}|Du|(x))^{p(x)} dx \tag{5.15}$$

for $\lambda \geq \lambda_0$. This proves claim (iv), since the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

The set U_λ is open, since $\mathcal{M}_\alpha$ is lower semicontinuous. By (5.6) we find that

$$|u(x) - u(y)| \leq c(n)\lambda|x - y|^{1-\alpha/q} \tag{5.16}$$

for every $x, y \in \mathbb{R}^n \setminus U_\lambda$. Hence, $u|_{\mathbb{R}^n \setminus U_\lambda}$ is $(1 - \alpha/q)$-Hölder continuous with the constant $c(n)\lambda$.

Let Q_i, $i = 1,2,\ldots$, be a Whitney decomposition of U_λ with the following properties:

(i) each Q_i is open,
(ii) cubes Q_i, $i = 1,2,\ldots$, are disjoint,
(iii) $U_\lambda = \bigcup_{i=1}^\infty \overline{Q}_i$,
(iv) $\sum_{i=1}^\infty \chi_{2Q_i} \leq N < \infty$,
(v) $4Q_i \subset U_\lambda$, $i = 1,2,\ldots$,
(vi) $c_1 \operatorname{dist}(Q_i, \mathbb{R}^n \setminus U_\lambda) \leq \operatorname{diam}(Q_i) \leq c_2 \operatorname{dist}(Q_i, \mathbb{R}^n \setminus U_\lambda)$.

Then we construct a partition of unity associated with the covering $2Q_i$, $i = 1,2,\ldots$. This can be done in two steps.

First, let $\varphi_i \in C_0^\infty(2Q_i)$ be such that $0 \leq \varphi_i \leq 1$, $\varphi_i = 1$ in Q_i, and

$$|D\varphi_i| \leq \frac{c}{\operatorname{diam}(Q_i)} \tag{5.17}$$

for $i = 1,2,\ldots$. Then we define

$$\phi_i(x) = \frac{\varphi_i(x)}{\sum_{j=1}^\infty \varphi_j(x)} \tag{5.18}$$

for every $i = 1,2,\ldots$. Observe that the sum is over finitely many terms only since $\varphi_i \in C_0^\infty(2Q_i)$ and the cubes $2Q_i$, $i = 1,2,\ldots$, are of bounded overlap. The functions ϕ_i have the property

$$\sum_{i=1}^\infty \phi_i(x) = \chi_{U_\lambda}(x) \tag{5.19}$$

for every $x \in \mathbb{R}^n$.

Then we define the function u_λ by

$$u_\lambda(x) = \begin{cases} u(x), & x \in \mathbb{R}^n \setminus U_\lambda, \\ \displaystyle\sum_{i=1}^\infty \phi_i(x) u_{2Q_i}, & x \in U_\lambda, \end{cases} \tag{5.20}$$

and claim (i) holds. The function u_λ is a Whitney-type extension of $u|_{\mathbb{R}^n \setminus U_\lambda}$ to the set U_λ. We claim that u_λ has the desired properties. If $U_\lambda = \emptyset$, we are done. Hence, we may assume that $U_\lambda \neq \emptyset$.

Claim (iii). We show that the function u_λ is Hölder continuous with the exponent $1-\alpha/q$. Recall that we assumed that the support of u is contained in a ball $B(x_0,2)$ for some $x_0\in\mathbb{R}^n$. For every $x\in U_\lambda$, there is $\overline{x}\in\mathbb{R}^n\setminus U_\lambda$ such that $|x-\overline{x}|=\operatorname{dist}(x,\mathbb{R}^n\setminus U_\lambda)$. Then using the partition of unity we have

$$\left|u_\lambda(\overline{x})-u_\lambda(x)\right| = \left|\sum_{i=1}^{\infty}\phi_i(x)\left(u(\overline{x})-u_{2Q_i}\right)\right| \le \sum_{i\in I_x}\left|u(\overline{x})-u_{2Q_i}\right|, \tag{5.21}$$

where $i\in I_x$ if and only if x belongs to the support of ϕ_i. Observe that for every $i\in I_x$ we have $2Q_i\subset B(\overline{x},r_i)$, where $r_i=c\operatorname{diam}(Q_i)$ by the properties of the Whitney decomposition. Hence, we obtain

$$\left|u(\overline{x})-u_{2Q_i}\right| \le \left|u(\overline{x})-u_{B(\overline{x},r_i)}\right| + \left|u_{B(\overline{x},r_i)}-u_{2Q_i}\right|, \tag{5.22}$$

where, again by the properties of the Whitney decomposition, we have

$$\left|u(\overline{x})-u_{B(\overline{x},r_i)}\right| \le cr_i^{1-\alpha/q}\mathcal{M}_{\alpha/q}|Du|(\overline{x}) \le c\lambda|x-\overline{x}|^{1-\alpha/q}. \tag{5.23}$$

Here we also used (5.1), (5.5) and the fact that $\overline{x}\in\mathbb{R}^n\setminus U_\lambda$.

On the other hand, by the properties of the Whitney decomposition and the Poincaré inequality, we have

$$\begin{aligned}\left|u_{B(\overline{x},r_i)}-u_{2Q_i}\right| &\le \fint_{2Q_i}\left|u(z)-u_{B(\overline{x},r_i)}\right|dz \le c\fint_{B(\overline{x},r_i)}\left|u(z)-u_{B(\overline{x},r_i)}\right|dz\\ &\le cr_i\fint_{B(\overline{x},r_i)}\left|Du(z)\right|dz \le cr_i^{1-\alpha/q}\mathcal{M}_{\alpha/q}|Du|(\overline{x})\\ &\le c|x-\overline{x}|^{1-\alpha/q}\lambda.\end{aligned} \tag{5.24}$$

It follows that

$$\left|u_\lambda(\overline{x})-u_\lambda(x)\right| \le c\lambda|\overline{x}-x|^{1-\alpha/q} \tag{5.25}$$

whenever $x\in U_\lambda$ and $\overline{x}\in\mathbb{R}^n\setminus U_\lambda$ such that $|x-\overline{x}|=\operatorname{dist}(x,\mathbb{R}^n\setminus U_\lambda)$.

From this, we conclude easily that

$$\left|u_\lambda(x)-u_\lambda(y)\right| \le c\lambda|x-y|^{1-\alpha/q} \tag{5.26}$$

for every $x \in U_\lambda$ and $y \in \mathbb{R}^n \setminus U_\lambda$. Indeed, by (5.6), we have

$$\begin{aligned} |u_\lambda(x) - u_\lambda(y)| &\leq |u_\lambda(x) - u_\lambda(\overline{x})| + |u_\lambda(\overline{x}) - u_\lambda(y)| \\ &\leq c\lambda|x-\overline{x}|^{1-\alpha/q} + c\lambda|\overline{x}-y|^{1-\alpha/q}, \end{aligned} \tag{5.27}$$

where $|\overline{x}-y| \leq |\overline{x}-x| + |x-y| \leq 2|x-y|$.

Then we consider the case $x, y \in U_\lambda$. First we assume that

$$\max\{|x-\overline{x}|, |y-\overline{y}|\} < |x-y|. \tag{5.28}$$

By the previously considered cases, we have

$$\begin{aligned} |u_\lambda(x) - u_\lambda(y)| &\leq |u_\lambda(x) - u_\lambda(\overline{x})| + |u_\lambda(\overline{x}) - u_\lambda(\overline{y})| + |u_\lambda(\overline{y}) - u_\lambda(y)| \\ &\leq c\lambda\big(|x-\overline{x}|^{1-\alpha/q} + |\overline{x}-\overline{y}|^{1-\alpha/q} + |\overline{y}-y|^{1-\alpha/q}\big) \\ &\leq c\lambda|x-y|^{1-\alpha/q}. \end{aligned} \tag{5.29}$$

In the last inequality we used (5.28) and the fact that

$$|\overline{x}-\overline{y}| \leq |\overline{x}-x| + |x-y| + |y-\overline{y}| \leq 3|x-y|. \tag{5.30}$$

Then we consider the case $x, y \in U_\lambda$ with

$$|x-y| \leq \max\{|x-\overline{x}|, |y-\overline{y}|\}. \tag{5.31}$$

First we assume, in addition, that

$$\max\{|x-\overline{x}|, |y-\overline{y}|\} \leq 2\min\{|x-\overline{x}|, |y-\overline{y}|\}. \tag{5.32}$$

Since

$$\sum_{i=1}^{\infty}\big(\phi_i(x) - \phi_i(y)\big) = 0, \tag{5.33}$$

we obtain

$$\begin{aligned} |u_\lambda(x) - u_\lambda(y)| &= \left|\sum_{i=1}^{\infty}\phi_i(x)u_{2Q_i} - \sum_{i=1}^{\infty}\phi_i(y)u_{2Q_i}\right| \\ &= \left|\sum_{i=1}^{\infty}\big(\phi_i(x)-\phi_i(y)\big)\big(u(\overline{x}) - u_{2Q_i}\big)\right| \\ &\leq c|x-y|\sum_{i\in I_x\cup I_y}\operatorname{diam}(Q_i)^{-1}\,|u(\overline{x}) - u_{2Q_i}|. \end{aligned} \tag{5.34}$$

We have already proved in (5.23) that

$$|u(\overline{x}) - u_{2Q_i}| \leq c\operatorname{diam}(Q_i)^{1-\alpha/q}\mathcal{M}_{\alpha/q}|Du|(\overline{x}) \tag{5.35}$$

if $i \in I_x$. On the other hand, if $i \in I_y$, then (5.31) and (5.32) imply that $2Q_i \subset B(\overline{x}, r_i)$, where $r_i = c\,\mathrm{diam}(Q_i)$ by the properties of the Whitney decomposition. Therefore, we obtain (5.35) for all indices $i \in I_x \cup I_y$. From this, we conclude that

$$\begin{aligned} |u_\lambda(x) - u_\lambda(y)| &\le c|x-y|^{1-\alpha/q} \sum_{i \in I_x \cup I_y} \frac{|x-y|^{\alpha/q}}{\mathrm{diam}(Q_i)^{\alpha/q}} \mathcal{M}_{\alpha/q}|Du|(\overline{x}) \\ &\le c\lambda |x-y|^{1-\alpha/q}. \end{aligned} \tag{5.36}$$

Here we used (5.31) and (5.32), the properties of the Whitney decomposition, and the fact that $\overline{x} \in \mathbb{R}^n \setminus U_\lambda$.

Assume then that $x, y \in U_\lambda$ such that (5.31) holds and

$$\max\{|x-\overline{x}|, |y-\overline{y}|\} > 2\min\{|x-\overline{x}|, |y-\overline{y}|\}. \tag{5.37}$$

If $|x-\overline{x}| \le |y-\overline{y}|$, then

$$|x-y| \ge |y-\overline{y}| - |x-\overline{x}| > \frac{1}{2}|y-\overline{y}|, \tag{5.38}$$

where we used (5.37) and the fact that the distance function $\mathrm{dist}(x, \mathbb{R}^n \setminus U_\lambda)$ is Lipschitz continuous with constant one. This implies that $2|x-y| \ge |y-\overline{y}|$. Now we have

$$\begin{aligned} |u_\lambda(x) - u_\lambda(y)| &\le |u_\lambda(x) - u_\lambda(\overline{x})| + |u_\lambda(\overline{x}) - u_\lambda(\overline{y})| + |u_\lambda(\overline{y}) - u_\lambda(y)| \\ &\le c\lambda\left(|x-\overline{x}|^{1-\alpha/q} + |\overline{x}-\overline{y}|^{1-\alpha/q} + |\overline{y}-y|^{1-\alpha/q}\right) \\ &\le c\lambda|x-y|^{1-\alpha/q}. \end{aligned} \tag{5.39}$$

By switching the roles of x and y, we see that the same estimate holds also if $|x-\overline{x}| > |y-\overline{y}|$. This completes the proof of claim (iii).

We prove the claim (ii) in two steps.

Step 5.4. First we claim that

$$\|u_\lambda\|_{W^{1,p(\cdot)}(U_\lambda)} \le c\|u\|_{W^{1,p(\cdot)}(U_\lambda)}. \tag{5.40}$$

Since

$$\mathcal{M}(u\chi_{U_\lambda})(x) \ge c|u|_{2Q_i} \tag{5.41}$$

for every $x \in 2Q_i$ and the cubes $2Q_i$, $i = 1, 2, \dots$, are of bounded overlap, we have

$$|u_\lambda(x)| \le \sum_{i=1}^{\infty} \phi_i(x)|u|_{2Q_i} \le \sum_{i=1}^{\infty} \phi_i(x)\mathcal{M}(u\chi_{U_\lambda})(x) \le c\mathcal{M}(u\chi_{U_\lambda})(x). \tag{5.42}$$

Since the maximal function is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, we obtain

$$\begin{aligned} \|u_\lambda\|_{L^{p(\cdot)}(U_\lambda)} &\le c\|\mathcal{M}(u\chi_{U_\lambda})\|_{L^{p(\cdot)}(U_\lambda)} \le c\|\mathcal{M}(u\chi_{U_\lambda})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le c\|u\chi_{U_\lambda}\|_{L^{p(\cdot)}(\mathbb{R}^n)} = c\|u\|_{L^{p(\cdot)}(U_\lambda)}. \end{aligned} \tag{5.43}$$

Then we consider an estimate for the gradient. We recall that

$$\Phi(x) = \sum_{i=1}^{\infty} \phi_i(x) = 1 \tag{5.44}$$

for every $x \in U_\lambda$. Since the cubes $2Q_i$, $i = 1,2,\ldots$, are of bounded overlap, we see that $\Phi \in C^\infty(U_\lambda)$ and

$$D_j\Phi(x) = \sum_{i=1}^{\infty} D_j\phi_i(x) = 0, \quad j = 1,2,\ldots,n, \tag{5.45}$$

for every $x \in U_\lambda$. Hence we obtain

$$\begin{aligned} |D_j u_\lambda(x)| &= \left|\sum_{i=1}^{\infty} D_j\phi_i(x) u_{2Q_i}\right| = \left|\sum_{i=1}^{\infty} D_j\phi_i(x)\big(u(x) - u_{2Q_i}\big)\right| \\ &\leq c\sum_{i=1}^{\infty} \operatorname{diam}(Q_i)^{-1} |u(x) - u_{2Q_i}| \chi_{2Q_i}(x). \end{aligned} \tag{5.46}$$

Let $B(x_i,R_i)$ be the smallest ball containing $2Q_i$ with $R_i = \operatorname{diam}(2Q_i)/2$. By the pointwise inequalities (5.1) and (5.5) with $\alpha = 0$, we obtain

$$\begin{aligned} |u(x) - u_{2Q_i}| &\leq \fint_{2Q_i} |u(x) - u(y)|\,dy \leq c\fint_{B(x_i,R_i)} |u(x) - u(y)|\,dy \\ &\leq cR_i \sup_{0<r<2R_i} \fint_{B(x,r)} |Du(y)|\,dy \\ &\leq c\operatorname{diam}(2Q_i) \sup_{0<r<\operatorname{diam}(2Q_i)} \fint_{B(x,r)} |Du(y)|\,dy. \end{aligned} \tag{5.47}$$

This implies that for every $j = 1,2,\ldots,n$,

$$\begin{aligned} |D_j u_\lambda(x)| &\leq c\sum_{i=1}^{\infty} \chi_{2Q_i}(x) \sup_{0<r<\operatorname{diam}(2Q_i)} \fint_{B(x,r)} |Du(y)|\,dy \leq c\mathcal{M}(|Du|\chi_{U_\lambda})(x)\sum_{i=1}^{\infty} \chi_{2Q_i}(x) \\ &\leq c\mathcal{M}(|Du|\chi_{U_\lambda})(x). \end{aligned} \tag{5.48}$$

We again used the facts that $B(x_i,R_i) \subset U_\lambda$ and the cubes $2Q_i$, $i = 1,2,\ldots$, are of bounded overlap. This implies

$$\begin{aligned} \|D_j u_\lambda\|_{L^{p(\cdot)}(U_\lambda)} &\leq c\|\mathcal{M}(|Du|\chi_{U_\lambda})\|_{L^{p(\cdot)}(U_\lambda)} \leq c\|\mathcal{M}(|Du|\chi_{U_\lambda})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq c\||Du|\chi_{U_\lambda}\|_{L^{p(\cdot)}(\mathbb{R}^n)} = c\|Du\|_{L^{p(\cdot)}(U_\lambda)}. \end{aligned} \tag{5.49}$$

This completes the proof of Step 5.4.

Step 5.5. We show that $u_\lambda \in W^{1,p(\cdot)}(\mathbb{R}^n)$. We know that $u_\lambda \in W^{1,p(\cdot)}(U_\lambda)$ and that it is Hölder continuous in $\mathbb{R}^n$. Moreover, $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and $u = u_\lambda$ in $\mathbb{R}^n \setminus U_\lambda$ by (i). This implies that

$$w = u - u_\lambda \in W^{1,p(\cdot)}(U_\lambda) \tag{5.50}$$

and that $w = 0$ in $\mathbb{R}^n \setminus U_\lambda$. By the ACL-property, u is absolutely continuous on almost every line segment parallel to the coordinate axes. Take any such line. Now w is absolutely continuous on the part of the line segment which intersects U_λ. On the other hand, $w = 0$ in the complement of U_λ. Hence, the continuity of w in the line segment implies that w is absolutely continuous on the whole line segment. This completes the proof of Step 5.5.

By the claim (i) and Steps 5.4 and 5.5, we obtain

$$\begin{aligned} ||u - u_\lambda||_{W^{1,p(\cdot)}(\mathbb{R}^n)} &= ||u - u_\lambda||_{W^{1,p(\cdot)}(U_\lambda)} \le \|u\|_{W^{1,p(\cdot)}(U_\lambda)} + ||u_\lambda||_{W^{1,p(\cdot)}(U_\lambda)} \\ &\le c||u||_{W^{1,p(\cdot)}(U_\lambda)}. \end{aligned} \tag{5.51}$$

This completes the proof of the claim (ii).

Finally, we remove the assumption that the support of the function is contained in a ball $B(x_0, 2)$ for some $x_0 \in \mathbb{R}^n$. Let $B(x_i, 2)$, $i = 1, 2, \ldots$, be a family of balls which are of bounded overlap and which cover $\mathbb{R}^n$. Then we construct a partition of unity as before and we obtain functions $\psi_i \in C_0^\infty(B(x_i, 2))$, $i = 1, 2, \ldots$, such that

$$\sum_{i=1}^{\infty} \psi_i(x) = 1 \tag{5.52}$$

for every $x \in \mathbb{R}^n$.

If $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, then

$$u(x) = \sum_{i=1}^{\infty} u(x)\psi_i(x) \tag{5.53}$$

for every $x \in \mathbb{R}^n$. Let $\varepsilon > 0$. Now the support of $u\psi_i$ is contained in $B(x_i, 2)$ for every $i = 1, 2, \ldots$. For every $i = 1, 2, \ldots$, let v_i be a Hölder continuous function with the exponent $1 - \alpha/q$ such that

$$||v_i - u\psi_i||_{W^{1,p(\cdot)}(\mathbb{R}^n)} \le 2^{-i}\varepsilon \tag{5.54}$$

and that the support of v_i is contained in $B(x_i, 3)$. Since every bounded set can be covered by finitely many balls $B(x_i, 2)$, it is easy to see that the function

$$v(x) = \sum_{i=1}^{\infty} v_i(x) \tag{5.55}$$

has the desired properties. This completes the proof. □

6. Size of the exceptional set

In this section, we give a sharper estimate for the size of the set U_λ in Theorem 5.2 in the case $1 < p^- \le p^+ < n$ and p is globally log-Hölder continuous.

THEOREM 6.1. *Let $1 < p^- \le p^+ < n$, $1 \le \alpha < q < p^-$. Assume that (4.2) and (4.3) hold. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and*

$$t(x) = \frac{np(x)}{nq - (\alpha - 1)p(x)}. \tag{6.1}$$

Then

$$\operatorname{Cap}_{t(\cdot)}\left(\{x \in \mathbb{R}^n : \mathcal{M}_{\alpha/q}|Du|(x) > \lambda\}\right) \longrightarrow 0 \tag{6.2}$$

as $\lambda \to \infty$.

Proof. By Hölder's inequality,

$$\mathcal{M}_{\alpha/q}|Du|(x) \le c(n,q)\left(\mathcal{M}_\alpha |Du|^q(x)\right)^{1/q} \tag{6.3}$$

for every $x \in \mathbb{R}^n$. It follows that

$$\{x \in \mathbb{R}^n : \mathcal{M}_{\alpha/q}|Du|(x) > \lambda\} \subset \{x \in \mathbb{R}^n : \mathcal{M}_\alpha |Du|^q(x) > \lambda^q\}. \tag{6.4}$$

Since $|Du| \in L^{p(\cdot)}(\mathbb{R}^n)$, we have $|Du|^q \in L^{p(\cdot)/q}(\mathbb{R}^n)$. Let

$$t(x) = \frac{np(x)}{nq - (\alpha - 1)p(x)}, \qquad t^*(x) = \frac{np(x)}{nq - \alpha p(x)}. \tag{6.5}$$

Since $p(\cdot)$ satisfies (4.2) and (4.3), the exponent $p(\cdot)/q$ does it as well. Hence, the fractional maximal function $\mathcal{M}_\alpha : L^{p(\cdot)/q}(\mathbb{R}^n) \to L^{t(\cdot)}(\mathbb{R}^n)$ is bounded. From this, we conclude as in (4.6) and (4.7) that

$$\mathcal{M}_\alpha |Du|^q \in L^{t^*(\cdot)}(\mathbb{R}^n), \qquad D\mathcal{M}_\alpha |Du|^q \in L^{t(\cdot)}(\mathbb{R}^n). \tag{6.6}$$

Moreover, we have

$$\begin{aligned}
\left\|\mathcal{M}_\alpha |Du|^q\right\|_{L^{t^*(\cdot)}(\mathbb{R}^n)} &\le c\left\||Du|^q\right\|_{L^{p(\cdot)/q}(\mathbb{R}^n)} = c\|Du\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
\left\|D\mathcal{M}_\alpha |Du|^q\right\|_{L^{t(\cdot)}(\mathbb{R}^n)} &\le c\left\|\mathcal{M}_{\alpha-1}|Du|^q\right\|_{L^{t(\cdot)}(\mathbb{R}^n)} \\
&\le c\left\||Du|^q\right\|_{L^{p(\cdot)/q}(\mathbb{R}^n)} = c\|Du\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{6.7}$$

Let

$$F_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}_\alpha |Du|^q(x) > \lambda^q\}. \tag{6.8}$$

Since the fractional maximal function is lower semicontinuous, the set F_λ is open. By (4.6) and (4.7), the function $\lambda^{-1}\mathcal{M}_\alpha|Du|^q$ is an admissible function for $\operatorname{Cap}_{t(\cdot)}(F_\lambda)$. Hence, we have

$$\begin{aligned}\operatorname{Cap}_{q(\cdot)}(U_\lambda) &\le \int_{\mathbb{R}^n}\left(\lambda^{-t^*(x)}\left(\mathcal{M}_\alpha|Du|^q(x)\right)^{t^*(x)}+\lambda^{-t(x)}\left|D\mathcal{M}_\alpha|Du|^q(x)\right|^{t(x)}\right)dx\\ &\le \lambda^{-np^-/(nq-\alpha p^-)}\int_{\mathbb{R}^n}\left(\mathcal{M}_\alpha|Du|^q(x)\right)^{t^*(x)}dx\\ &\quad+\lambda^{-np^-/(nq-(\alpha-1)p^-)}\int_{\mathbb{R}^n}\left|D\mathcal{M}_\alpha|Du|^q(x)\right|^{t(x)}dx.\end{aligned} \tag{6.9}$$

Since by (6.7) the integrals in the right-hand side are finite and since the exponents are negative, we find that the right-hand side tends to zero as λ tends to infinity. □

By Theorem 5.2 and Theorem 6.1, we obtain the following theorem.

Theorem 6.2. *Let $1 < p^- \le p^+ < n$, $1 < \alpha < q < p^-$. Assume that p satisfies conditions (4.2) and (4.3). Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be defined pointwisely by (3.8). Then for each $\varepsilon > 0$, there exist an open set U and a function v so that*

(i) *$u(x) = v(x)$ for every $x \in \mathbb{R}^n \setminus U$,*

(ii) *$\|u - v\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} \le \varepsilon$,*

(iii) *v is $(1-\alpha/q)$-Hölder continuous,*

(iv)

$$\operatorname{Cap}_{np(\cdot)/(nq-(\alpha-1)p(\cdot))}(U) \le \varepsilon. \tag{6.10}$$

Proof. By Theorem 5.2 and Theorem 6.1, we can choose $U = U_\lambda$ for sufficiently large λ. □

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