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Research Article

On the Precise Asymptotics of the Constant in Friedrich's Inequality for Functions Vanishing on the Part of the Boundary with Microinhomogeneous Structure

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We construct the asymptotics of the sharp constant in the Friedrich-type inequality for functions, which vanish on the small part of the boundary Γ_1^ε. It is assumed that Γ_1^ε consists of $(1/\delta)^{n-1}$ pieces with diameter of order $O(\varepsilon\delta)$. In addition, $\delta = \delta(\varepsilon)$ and $\delta \to 0$ as $\varepsilon \to 0$.

1. Introduction

The domain Ω is an open bounded set from the space $\mathbb{R}^n$. The Sobolev space $H^1(\Omega)$ is defined as the completion of the set of functions from the space $C^\infty(\overline{\Omega})$ by the norm $\sqrt{\int_\Omega (u^2 + |\nabla u|^2)\,dx}$. The space $\overset{\circ}{H}{}^1(\Omega)$ is the set of functions from the space $H^1(\Omega)$, with zero trace on $\partial\Omega$.

Let $\varepsilon = 1/N$, $N \in \mathbb{N}$, be a small positive parameter. Consider the set $\Gamma_\varepsilon \subset \partial\Omega$ which depends on the parameter ε. The space $H^1(\Omega, \Gamma_\varepsilon)$ is the set of functions from $H^1(\Omega)$, vanishing on Γ_ε.

The following estimate is known as Friedrich's inequality for functions $u \in \overset{\circ}{H}{}^1(\Omega)$:

$$\int_\Omega u^2\,dx \le K_0 \int_\Omega |\nabla u|^2\,dx, \tag{1.1}$$

where the constant K_0 depends on the domain Ω only and does not depend on the function u.

Inequality (1.1) is very important for several applications and it may be regarded as a special case of multidimensional Hardy-type inequalities. Such inequalities has attracted a lot of interest in particular during the last years; see, for example, the books [1–3] and the references given therein. We pronounce that not so much is known concerning the best constants in multidimensional Hardy-type inequalities and the aim of this paper is to study the asymptotic behavior of the constant in [4] for functions vanishing on a part of the boundary with microinhomogeneous structure. In particular, such result are useful in homogenization theory and in fact this was our original interest in the subject.

The paper is organized as follows. In Section 2, we present and discuss our main results. In Section 3, these results are proved via some auxiliary results, which are of independent interest. In Section 4, we consider partial cases, where it is possible to give the asymptotic expansion for the constant with respect to ε.

2. The main results

It is well known (see, e.g., [5]) that the Friedrich's inequality (1.1) is valid for functions $u \in H^1(\Omega,\Gamma_\varepsilon)$ and $K_0 = O(1/\operatorname{cap}\Gamma_\varepsilon)$, where we denote by $\operatorname{cap} F$ the capacity of $F \subset \mathbb{R}^n$ in $\mathbb{R}^n$:

$$\operatorname{cap} F = \inf\left\{\int_{\mathbb{R}^n} |\nabla\varphi|^2\,dx : \varphi \in C_0^\infty(\mathbb{R}^n),\ \varphi \geq 1 \text{ on } F\right\}. \tag{2.1}$$

Remark 2.1. Friedrich's inequality, when the functions vanishes on a part of the boundary is sometimes called "Poincaré's inequality," but we prefer to say "Friedrich's" or "Friedrich's type inequality" keeping the name "Poincaré's inequality" for the following (see, e.g., [6]):

$$\int_\Omega u^2\,dx \leq \left(\int_\Omega u\,dx\right)^2 + \int_\Omega |\nabla u|^2\,dx, \quad \forall u \in \overset{\circ}{H}{}^1(\Omega). \tag{2.2}$$

Further, it will be shown later on that K_0 is uniformly bounded under special assumptions on Γ_ε in the case when $\operatorname{mes}\Gamma_\varepsilon \to 0$ as $\varepsilon \to 0$.

Consider now the domain $\Omega \subset \mathbb{R}^2$ with smooth boundary of the length 1 such that

$$\begin{gathered} \partial\Omega = \overline{\Gamma_1^\varepsilon} \cup \Gamma_2^\varepsilon, \qquad \Gamma_1^\varepsilon = \bigcup_i (\Gamma_{1i}^\varepsilon), \qquad \Gamma_2^\varepsilon = \bigcup_i (\Gamma_{2i}^\varepsilon), \qquad \Gamma_{1i}^\varepsilon \cap \Gamma_{2j}^\varepsilon = \varnothing, \\ \operatorname{mes}\Gamma_{1i}^\varepsilon = \varepsilon\delta(\varepsilon), \quad \operatorname{mes}(\Gamma_{1i}^\varepsilon \cup \Gamma_{2i}^\varepsilon) = \delta(\varepsilon), \quad \delta(\varepsilon) = o\left(\frac{1}{|\ln\varepsilon|}\right) \quad \text{as } \varepsilon \longrightarrow 0, \end{gathered} \tag{2.3}$$

where Γ_{1i}^ε and Γ_{2i}^ε are alternating (see Figure 2.1).

Here $\Gamma_\varepsilon = \Gamma_1^\varepsilon$. Our first main result reads as follows.

Theorem 2.2. *Suppose $n = 2$. For $u \in H^1(\Omega,\Gamma_1^\varepsilon)$, the following Friedrich's inequality holds true:*

$$\int_\Omega u^2\,dx \leq K_\varepsilon \int_\Omega |\nabla u|^2\,dx, \quad K_\varepsilon = K_0 + \varphi(\varepsilon), \tag{2.4}$$

where K_0 is a constant in Friedrich's inequality (1.1) for functions $u \in \overset{\circ}{H}{}^1(\Omega)$, and $\varphi(\varepsilon) \sim (|\ln\varepsilon|)^{-1/2} + (\delta(\varepsilon)|\ln\varepsilon|)^{1/2}$ as $\varepsilon \to 0$.

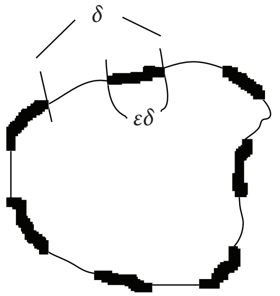

FIGURE 2.1. Plane domain.

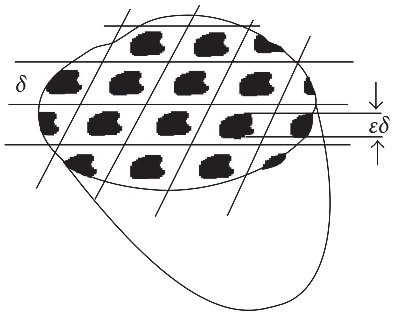

FIGURE 2.2. Spatial domain.

For the case $n \geq 3$, the geometrical constructions are similar. We assume that $\partial\Omega = \overline{S} \cup \Gamma$, $S \cap \Gamma = \varnothing$, Γ belongs to the hyperplane $x_n = 0$, and $\overline{\Gamma} = \overline{\Gamma_1^\varepsilon} \cup \overline{\Gamma_2^\varepsilon}$, $\Gamma_1^\varepsilon \cap \Gamma_2^\varepsilon = \varnothing$.

Denote by ω a bounded domain in the hyperplane $x_n = 0$, which contains the origin. Without loss of generality $\omega \in \square$, where $\square = \{\widehat{x} : -1/2 < x_i < 1/2,\ i = 1,\dots,n-1\}$, $x = (\widehat{x}, x_n)$. Let ω_ε be the domain $\{\widehat{x} : \widehat{x}/\varepsilon \in \omega\}$. Denote by $\widetilde{\Gamma}$ the integer translations of ω_ε on the hyperplane in x_i direction, $i = 1,\dots,n-1$. Finally, $\Gamma_1^\varepsilon = \{x : x/\delta \in \widetilde{\Gamma}\} \cap \Gamma$, $\Gamma_2^\varepsilon = \Gamma \setminus \overline{\Gamma_1^\varepsilon}$ (see Figure 2.2). In other words, Γ_1^ε is a translations of vectors $m\delta(\varepsilon)\mathbf{e}_i$ ($m \in \mathbb{Z}$, $i = 1,\dots,n-1$) of a set diameter $\varepsilon\delta(\varepsilon)$ contained in a ball of radius $\delta(\varepsilon)$. Here we assume that $\delta(\varepsilon) = o(\varepsilon^{n-2})$ as $\varepsilon \to 0$. Also we suppose that $\Gamma_\varepsilon = \Gamma_1^\varepsilon \cup S$.

In this case our main result reads as follows.

THEOREM 2.3. *Suppose $n \geq 3$. For $u \in H^1(\Omega, \Gamma_1^\varepsilon \cup S)$, the following Friedrich's inequality is valid:*

$$\int_\Omega u^2\,dx \leq K_\varepsilon \int_\Omega |\nabla u|^2\,dx, \quad K_\varepsilon = K_0 + \varphi(\varepsilon), \tag{2.5}$$

where K_0 is a constant in Friedrich's inequality (1.1) for functions $u \in \overset{\circ}{H^1}(\Omega)$, and $\varphi(\varepsilon) \sim \varepsilon^{n/2-1} + (\delta(\varepsilon)\varepsilon^{2-n})^{1/2}$ as $\varepsilon \to 0$.

Thus the precise dependence of the constant in Friedrich's inequality of the small parameter ε will be established. Hence, it is possible to construct the lower and the upper bounds for K_ε.

3. Proofs of the main results and some auxiliary results

In Sections 3.1 and 3.2 we discuss, present, and prove some auxiliary results, which are of independent interest but also crucial for the proof of the main results in Section 3.3.

3.1. The relation between the constant in Friedrich's inequality and the first eigenvalue of a boundary value problem. Let Ω be some bounded domain with smooth boundary $\partial\Omega$, $\Gamma_\varepsilon \subset \partial\Omega$. Suppose that the function u belongs to $H^1(\Omega)$. Consider the following problem:

$$\begin{aligned} \Delta u &= -\lambda_\varepsilon u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \setminus \Gamma_\varepsilon. \end{aligned} \tag{3.1}$$

Definition 3.1. The function $u \in H^1(\Omega, \Gamma_\varepsilon)$ is a solution of problem (3.1), if the following integral identity is valid:

$$\int_\Omega \nabla u \nabla v \, dx = \lambda_\varepsilon \int_\Omega uv \, dx \tag{3.2}$$

for all functions $v \in H^1(\Omega, \Gamma_\varepsilon)$.

The operator of problem (3.1) is positive and selfadjoint (it follows directly from the integral identity). According to the general theory (see, e.g., [7]), all eigenvalues of the problem are real, positive, and satisfy

$$0 \le \lambda_\varepsilon^1 \le \lambda_\varepsilon^2 \le \cdots, \qquad \lambda_\varepsilon^k \longrightarrow \infty \text{ as } k \longrightarrow \infty. \tag{3.3}$$

Here we assume that the eigenvalues λ_ε^k are repeated according to their multiplicities.

Denote by μ_ε the following value:

$$\mu_\varepsilon = \inf_{v \in H^1(\Omega, \Gamma_\varepsilon) \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 \, dx}. \tag{3.4}$$

We need the following lemma (see the analogous lemma in [4]).

Lemma 3.2. *The number μ_ε is the first eigenvalue λ_ε^1 of the problem (3.1).*

For the convenience of the reader we present the details of the proof.

Proof. It is sufficient to show that there exists such eigenfunction u^1 of problem (3.1), corresponding to the first eigenvalue λ_ε^1, that it satisfies

$$\mu_\varepsilon = \frac{\int_\Omega \left| \nabla u^1 \right|^2 dx}{\int_\Omega \left(u^1 \right)^2 dx}. \tag{3.5}$$

Let $\{v^{(k)}\}$ be a minimization sequence for (3.4), that is,

$$v^{(k)} \in H^1(\Omega,\Gamma_\varepsilon), \qquad \|v^{(k)}\|^2_{L_2(\Omega)} = 1,$$
$$\int_\Omega |\nabla v^{(k)}|^2 dx \longrightarrow \mu_\varepsilon, \quad \text{as } k \longrightarrow \infty. \tag{3.6}$$

It is obvious that the sequence $\{v^{(k)}\}$ is bounded in $H^1(\Omega,\Gamma_\varepsilon)$. Hence, according to the Rellich theorem, there exists a subsequence of $\{v^{(k)}\}$, converging weekly in $H^1(\Omega,\Gamma_\varepsilon)$ and strongly in $L_2(\Omega)$. For this subsequence, we keep the same notation $\{v^{(k)}\}$. We have that

$$\|v^{(k)} - v^{(l)}\|^2_{L_2(\Omega)} < \eta \quad \text{as } k,l > k_0(\eta). \tag{3.7}$$

Using the following formula:

$$\left\|\frac{v^{(k)}+v^{(l)}}{2}\right\|^2_{L_2(\Omega)} = \frac{1}{2}\|v^{(k)}\|^2_{L_2(\Omega)} + \frac{1}{2}\|v^{(l)}\|^2_{L_2(\Omega)} - \left\|\frac{v^{(k)}-v^{(l)}}{2}\right\|^2_{L_2(\Omega)}, \tag{3.8}$$

we obtain that

$$\left\|\frac{v^{(k)}+v^{(l)}}{2}\right\|^2_{L_2(\Omega)} > 1 - \frac{\eta}{4}. \tag{3.9}$$

From the definition of μ_ε we conclude that

$$\int_\Omega |\nabla v|^2 dx \geq \mu_\varepsilon \|v\|^2_{L_2(\Omega)} \tag{3.10}$$

for all function $v \in H^1(\Omega,\Gamma_\varepsilon)$. Inequalities (3.9) and (3.10) give the following estimate:

$$\int_\Omega \left|\nabla\left(\frac{v^{(k)}+v^{(l)}}{2}\right)\right|^2 dx > \mu_\varepsilon\left(1-\frac{\eta}{4}\right). \tag{3.11}$$

If $k,l > k_0(\eta)$, it follows that

$$\int_\Omega |\nabla v^{(k)}|^2 dx < \mu_\varepsilon + \eta, \qquad \int_\Omega |\nabla v^{(l)}|^2 dx < \mu_\varepsilon + \eta. \tag{3.12}$$

Hence,

$$\begin{aligned}
&\int_\Omega \left|\nabla\left(\frac{v^{(k)}-v^{(l)}}{2}\right)\right|^2 dx \\
&\quad = \frac{1}{2}\int_\Omega |\nabla v^{(k)}|^2 dx + \frac{1}{2}\int_\Omega |\nabla v^{(l)}|^2 dx - \int_\Omega \left|\nabla\left(\frac{v^{(k)}+v^{(l)}}{2}\right)\right|^2 dx \\
&\quad \leq \frac{\mu_\varepsilon+\eta}{2} + \frac{\mu_\varepsilon+\eta}{2} - \mu_\varepsilon\left(1-\frac{\eta}{4}\right) = \eta\left(1+\frac{\mu_\varepsilon}{4}\right) \longrightarrow 0, \quad \eta \longrightarrow 0.
\end{aligned} \tag{3.13}$$

Finally, according to the Cauchy condition the sequence $\{v^{(k)}\}$ converges to some function $v^* \in H^1(\Omega,\Gamma_\varepsilon)$ in the space $H^1(\Omega,\Gamma_\varepsilon)$, and

$$\int_\Omega |\nabla v^*|^2\,dx = \mu_\varepsilon, \qquad \|v^*\|^2_{L_2(\Omega)} = 1. \tag{3.14}$$

Assume that $v \in H^1(\Omega,\Gamma_\varepsilon)$ is an arbitrary function. Denote

$$g(t) = \frac{\int_\Omega |\nabla(v^* + tv)|^2\,dx}{\|v^* + tv\|^2_{L_2(\Omega)}}. \tag{3.15}$$

The function $g(t)$ is continuously differentiable in some neighborhood of $t = 0$. This ratio has the minimum, which is equal to μ_ε. Using the Fermat theorem, we obtain that

$$\begin{aligned} 0 = g'|_{t=0} &= \frac{2\|v^*\|^2_{L_2(\Omega)}\int_\Omega(\nabla v^*,\nabla v)\,dx - 2\int_\Omega v^* v\,dx\int_\Omega|\nabla v^*|^2\,dx}{\|v^*\|^4_{L_2(\Omega)}} \\ &= 2\int_\Omega(\nabla v^*,\nabla v)\,dx - 2\mu_\varepsilon\int_\Omega v^* v\,dx. \end{aligned} \tag{3.16}$$

Thus, we have proved that

$$\int_\Omega(\nabla v^*,\nabla v)\,dx = \mu_\varepsilon\int_\Omega v^* v\,dx \tag{3.17}$$

for $v \in H^1(\Omega,\Gamma^\varepsilon)$, that is, v^* satisfies the integral identity (3.1). This means that we have found a function v^* such that

$$\frac{\int_\Omega|\nabla v^*|^2\,dx}{\int_\Omega(v^*)^2\,dx} = \inf_{v\in H^1(\Omega,\Gamma_\varepsilon)\setminus\{0\}}\frac{\int_\Omega|\nabla v|^2\,dx}{\int_\Omega v^2\,dx} = \mu_\varepsilon. \tag{3.18}$$

Keeping in mind that $u^1 = v^*$ we conclude that $\mu_\varepsilon = \lambda^1_\varepsilon$. The proof is complete. □

Lemma 3.3. *The following Friedrich inequality holds true:*

$$\int_\Omega U^2\,dx \le K_\varepsilon\int_\Omega|\nabla U|^2\,dx, \quad U \in H^1(\Omega,\Gamma_\varepsilon), \tag{3.19}$$

where $K_\varepsilon = 1/\lambda^1_\varepsilon$.

Proof. From Lemma 3.2 we get that

$$\lambda^1_\varepsilon \le \frac{\int_\Omega|\nabla U|^2\,dx}{\int_\Omega U^2\,dx} \quad \text{for any } U \in H^1(\Omega,\Gamma_\varepsilon). \tag{3.20}$$

Using this estimate, we deduce that

$$\int_\Omega U^2\,dx \le \frac{1}{\lambda^1_\varepsilon}\int_\Omega|\nabla U|^2\,dx. \tag{3.21}$$

Denoting by K_ε the value $1/\lambda^1_\varepsilon$, we conclude the statement of our lemma. □

In the following section we will estimate λ^1_ε.

3.2. Auxiliary boundary value problems. Assume that $f \in L_2(\Omega)$ and consider the following boundary value problems:

$$\begin{aligned} -\Delta u_\varepsilon &= f \quad \text{in } \Omega, \\ u_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} &= 0 \quad \text{on } \Gamma_2^\varepsilon, \end{aligned} \tag{3.22}$$

$$\begin{aligned} -\Delta u_0 &= f \quad \text{in } \Omega, \\ u_0 &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.23}$$

Note that for $n = 2$ we assume that $\Gamma_\varepsilon = \Gamma_1^\varepsilon$ and for $n \geq 3$ we assume that $\Gamma_\varepsilon = \Gamma_1^\varepsilon \cup S$. Problem (3.23) is the homogenized (limit as $\varepsilon \to 0$) problem for problem (3.22) (a proof of this fact can be found in [4, 8], see also [6]).

Consider now the respective spectral problems:

$$\begin{aligned} -\Delta u_\varepsilon^k &= \lambda_\varepsilon^k u_\varepsilon^k \quad \text{in } \Omega, \\ u_\varepsilon^k &= 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_\varepsilon^k}{\partial \nu} &= 0 \quad \text{on } \Gamma_2^\varepsilon, \\ -\Delta u_0^k &= \lambda_0^k u_0^k \quad \text{in } \Omega, \\ u_0^k &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.24}$$

Next, let us estimate the difference $|1/\lambda_\varepsilon^k - 1/\lambda_0^k|$. We will use the method introduced by Oleĭnik et al. (see [9, 10]).

Let H_ε, H_0 be separable Hilbert spaces with the inner products $(u^\varepsilon, v^\varepsilon)_{H_\varepsilon}$ and $(u,v)_{H_0}$, and the norms $\|u^\varepsilon\|_{H_\varepsilon}$ and $\|u\|_{H_0}$, respectively; assume that ε is a small parameter, $A_\varepsilon \in \mathscr{L}(H_\varepsilon)$, $A_0 \in \mathscr{L}(H_0)$ are linear continuous operators and $\operatorname{Im} A_0 \subset V \subset H_0$, where V is a linear subspace of H_0.

(C1) There exist linear continuous operators $R_\varepsilon : H_\varepsilon \to H_0$ such that for all $f \in V$ we have $(R_\varepsilon f, R_\varepsilon f)_{H_\varepsilon} \to c(f,f)_{H_0}$ as $\varepsilon \to 0$, where $c = \text{const} > 0$ does not depend on f.

(C2) The operators A_ε, A_0 are positive, compact and selfadjoint in H_ε, H_0, respectively, and $\sup_\varepsilon \|A_\varepsilon\|_{\mathscr{L}(H_\varepsilon)} < +\infty$.

(C3) For all $f \in V$ we have $\|A_\varepsilon R_\varepsilon f - R_\varepsilon A_0 f\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$.

(C4) The sequence of operators A_ε is uniformly compact in the following sense: if a sequence $f^\varepsilon \in H_\varepsilon$ is such that $\sup_\varepsilon \|f^\varepsilon\|_{\mathscr{L}(H_\varepsilon)} < +\infty$, then there exist a subsequence $f^{\varepsilon'}$ and a vector $w^0 \in V$ such that $\|A_{\varepsilon'} f^{\varepsilon'} - R_{\varepsilon'} w^0\|_{H_{\varepsilon'}} \to 0$ as $\varepsilon' \to 0$.

Assume that the spectral problems for the operators A_ε, A_0 are

$$\begin{aligned} &A_\varepsilon u_\varepsilon^k = \mu_\varepsilon^k u_\varepsilon^k, \quad k = 1,2,\ldots, \quad \mu_\varepsilon^1 \geq \mu_\varepsilon^2 \geq \cdots > 0, \quad (u_\varepsilon^l, u_\varepsilon^m) = \delta_{lm}, \\ &A_0 u_0^k = \mu_0^k u_0^k, \quad k = 1,2,\ldots, \quad \mu_0^1 \geq \mu_0^2 \geq \cdots > 0, \quad (u_0^l, u_0^m) = \delta_{lm}, \end{aligned} \tag{3.25}$$

where δ_{lm} is the Kronecker symbol and the eigenvalues μ_ε^k, μ_0^k are repeated according to their multiplicities.

The following theorem holds true (see [9]).

THEOREM 3.4 (Oleĭnik et al. [9, 10]). *Suppose that the conditions (C1)–(C4) are valid. Then μ_ε^k converges to μ_0^k as $\varepsilon \to 0$, and the following estimate takes place:*

$$|\mu_\varepsilon^k - \mu_0^k| \leq c^{-1/2} \sup_{f \in N(\mu_0^k, A_0), \|f\|_{H_0}=1} ||A_\varepsilon R_\varepsilon f - R_\varepsilon A_0 f||_{H_\varepsilon}, \tag{3.26}$$

where $N(\mu_0^k, A_0) = \{u \in H_0, A_0 u = \mu_0^k u\}$. Assume also that $k \geq 1$, $s \geq 1$, are the integer numbers, $\mu_0^k = \cdots = \mu_0^{k+s-1}$ and the multiplicity of μ_0^k is equal to s. Then there exist linear combinations U_ε of the eigenfunctions $u_\varepsilon^k, \ldots, u_\varepsilon^{k+s-1}$ to problem (3.22) such that for all $w \in N(\mu_0^k, A_0)$ we get $\|U_\varepsilon - R_\varepsilon w\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$.

To use the method of Oleĭnik et al. [9, 10], we define the spaces H_ε and H_0 and the operators A_ε, A_0, and R_ε in an appropriate way.

Assume that $H_\varepsilon = H_0 = V = L_2(\Omega)$, and R_ε is the identity operator. The operators A_ε, A_0 are defined in the following way: $A_\varepsilon f = u_\varepsilon$, $A_0 f = u_0$, where u_ε, u_0 are the solutions to problems (3.22) and (3.23), respectively. Let us verify the conditions (C1)–(C4).

The condition (C1) is fulfilled automatically because R_ε is the identity operator, $c = 1$. Let us verify the selfadjointness of the operator A_ε. Define $A_\varepsilon f = u_\varepsilon$, $A_\varepsilon g = v_\varepsilon$, $f, g \in L_2(\Omega)$. Because of the integral identity of problem (3.22) the following identities are valid:

$$\int_\Omega f v_\varepsilon \, dx = \int_\Omega \nabla v_\varepsilon \nabla u_\varepsilon \, dx = \int_\Omega g u_\varepsilon \, dx. \tag{3.27}$$

Hence,

$$\begin{aligned}(A_\varepsilon f, g)_{L_2(\Omega)} &= (u_\varepsilon, g)_{L_2(\Omega)} = \int_\Omega u_\varepsilon g \, dx = \int_\Omega \nabla v_\varepsilon \nabla u_\varepsilon \, dx \\ &= \int_\Omega f v_\varepsilon \, dx = (f, v_\varepsilon)_{L_2(\Omega)} = (f, A_\varepsilon g)_{L_2(\Omega)}.\end{aligned} \tag{3.28}$$

The selfadjointness of the operator A_0 can be proved in an analogous way.

It is easy to prove the positiveness of the operator A_ε:

$$(A_\varepsilon f, f)_{L_2(\Omega)} = (u_\varepsilon, f)_{L_2(\Omega)} = \int_\Omega u_\varepsilon f \, dx = \int_\Omega |\nabla u_\varepsilon|^2 \, dx \geq 0, \tag{3.29}$$

and $\int_\Omega |\nabla u_\varepsilon|^2 \, dx > 0$ if $f \neq 0$. The positiveness of A_0 may be proved in the same way.

Next, we prove that A_ε, A_0 are compact operators: let the sequence $\{f_\theta\}$ be bounded in $L_2(\Omega)$. It is evident that the sequence $\{A_\varepsilon f_\theta\} = \{u_{\varepsilon,\theta}\}$ is bounded in $H^1(\Omega, \Gamma_\varepsilon)$ and the sequence $\{A_0 f_\theta\} = \{u_{0,\theta}\}$ is bounded in $\overset{\circ}{H}{}^1(\Omega)$. Note that $\{u_{\varepsilon,\theta}\}$ is bounded uniformly on ε (for a proof see [4]). Because of compact embedding of the space $H^1(\Omega)$ to $L_2(\Omega)$, we conclude that A_ε and A_0 are compact operators. Moreover,

$$||A_\varepsilon||_{\mathscr{L}(H_\varepsilon)} \leq ||u_\varepsilon||_{H^1(\Omega, \Gamma_\varepsilon)} < \mathcal{M} < +\infty \tag{3.30}$$

and, consequently,

$$\sup_{\varepsilon} ||A_\varepsilon||_{\mathcal{L}(H_\varepsilon)} < \mathcal{M} < +\infty. \tag{3.31}$$

Let us verify the condition (C3). The operator R_ε is the identity operator and, thus, it is sufficient to prove that for all $f \in L_2(\Omega)$ we have that $\|A_\varepsilon f - A_0 f\|_{L_2(\Omega)} \to 0$ as $\varepsilon \to 0$, that is, $\|u_\varepsilon - u_0\|_{L_2(\Omega)} \to 0$ as $\varepsilon \to 0$. It is enough to prove that $u_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$. (The week convergence in $H^1(\Omega)$ gives the strong convergence in $L_2(\Omega)$.) The sequence u_ε is uniformly bounded in $H^1(\Omega)$. Consequently, there exists a subsequence $u_{\varepsilon'}$, such that $u_{\varepsilon'} \rightharpoonup u_*$ in $H^1(\Omega)$. Further we will set that u_ε is the same subsequence. Let us show that $u_* \equiv u_0$, that is, for all $v \in \overset{\circ}{H}{}^1(\Omega)$,

$$\int_\Omega f v \, dx = \int_\Omega \nabla u_* \nabla v \, dx. \tag{3.32}$$

The integral identity for problem (3.22) gives that

$$\int_\Omega f v \, dx = \int_\Omega \nabla u_\varepsilon \nabla v \, dx. \tag{3.33}$$

Because u_* is a week limit of u_ε in $H^1(\Omega)$, the following is valid:

$$\int_\Omega \nabla u_\varepsilon \nabla v \, dx \longrightarrow \int_\Omega \nabla u_* \nabla v \, dx \quad \text{when } \varepsilon \longrightarrow 0 \ \forall v \in H^1(\Omega), \tag{3.34}$$

and this gives us the desired result, because the integrals $\int_\Omega \nabla u_* \nabla v \, dx$ and $\int_\Omega f v \, dx$ do not depend on ε.

Let us verify the condition (C4). Consider the sequence $\{f_\varepsilon\}$, which is bounded in $L_2(\Omega)$. Then $\|A_\varepsilon f_\varepsilon\|_{H^1(\Omega,\Gamma_\varepsilon)} = \|u_\varepsilon\|_{H^1(\Omega,\Gamma_\varepsilon)} \le \text{const}$, that is, the sequence $\{A_\varepsilon f_\varepsilon\}$ is compact in $L_2(\Omega)$ and, consequently, there exists a subsequence ε' such that

$$A_{\varepsilon'} f_{\varepsilon'} \longrightarrow w^0 \quad \text{as } \varepsilon' \longrightarrow 0, \text{ where } w^0 \in L_2(\Omega). \tag{3.35}$$

Hence, we have that $\|A_{\varepsilon'} f_{\varepsilon'} - w^0\|_{L_2(\Omega)} \to 0$ as $\varepsilon' \to 0$.

Thus, the conditions (C1)–(C4) are valid.

It is evident that $\lambda_\varepsilon^k = 1/\mu_\varepsilon^k$, $\lambda_0^k = 1/\mu_0^k$. Using the estimate (3.26) we have

$$\begin{aligned} \left| \frac{1}{\lambda_\varepsilon^k} - \frac{1}{\lambda_0^k} \right| &\le \sup_{f \in N(\lambda_0^k, A_0), \|f\|_{L_2(\Omega)}=1} ||A_\varepsilon f - A_0 f||_{H^1(\Omega,\Gamma_\varepsilon)} \\ &= \sup_{f \in N(\lambda_0^k, A_0), \|f\|_{L_2(\Omega)}=1} ||u_\varepsilon - u_0||_{H^1(\Omega,\Gamma_\varepsilon)}, \end{aligned} \tag{3.36}$$

where u_ε, u_0 are the solutions of problems (3.22) and (3.23), respectively.

The following inequality was established in [4]:

$$||u_\varepsilon - u_0||_{H^1(\Omega,\Gamma_\varepsilon)} \le K \|f\|_{L_2(\Omega)} \left((\mu_\varepsilon)^{1/2} + \left(\frac{\delta}{\mu_\varepsilon} \right)^{1/2} \right), \tag{3.37}$$

where $\mu_\varepsilon = \inf_{u\in H^1(\Omega,\Gamma_1^\varepsilon)\setminus\{0\}}(\int_\Omega |\nabla u|^2\,dx/\int_\Omega u^2\,dx)$ and the constant K depends only on the domain Ω. Moreover, the following asymptotics was proved in [4] (the case $n=2$) and in [11] (the case $n\geq 3$) (see also [8, 12]):

$$\mu_\varepsilon = \begin{cases} \dfrac{\pi}{|\ln\varepsilon|} + O\left(\dfrac{1}{|\ln\varepsilon|^2}\right), & \text{if } n=2, \\ \varepsilon^{n-2}\dfrac{\sigma_n}{2}c_\omega + O(\varepsilon^{n-1}), & \text{if } n>2, \end{cases} \tag{3.38}$$

as $\varepsilon\to 0$.

Here σ_n is the area of the unit sphere in $\mathbb{R}^n$, and $c_\omega > 0$ is the capacity of the $(n-1)$-dimensional "disk" ω (see [13, 14]).

Define the following value:

$$\varphi(\varepsilon) \equiv K\left((\mu_\varepsilon)^{1/2} + \left(\frac{\delta}{\mu_\varepsilon}\right)^{1/2}\right). \tag{3.39}$$

Note that if $n=2$, it implies that $\mu_\varepsilon \sim 1/|\ln\varepsilon|$ and $\delta = O(1/|\ln\varepsilon|)$. Consequently, we have

$$\frac{\delta}{\mu_\varepsilon} = \frac{o(1/|\ln\varepsilon|)|\ln\varepsilon|}{\pi} = o(1) \tag{3.40}$$

as $\varepsilon\to 0$.

In the same way, if $n>2$ it yields that

$$\frac{\delta}{\mu_\varepsilon} \sim o(1) \tag{3.41}$$

as $\varepsilon\to 0$.

Using this asymptotics, we deduce

$$\varphi(\varepsilon) = K\begin{cases} (\ln\varepsilon)^{-1/2} + (\delta(\varepsilon)|\ln\varepsilon|)^{1/2} + o\left((\ln\varepsilon)^{-1/2} + (\delta(\varepsilon)|\ln\varepsilon|)^{1/2}\right), & \text{if } n=2, \\ \varepsilon^{n/2-1} + (\delta(\varepsilon)\varepsilon^{n-2})^{1/2} + o\left(\varepsilon^{n/2-1} + (\delta(\varepsilon)\varepsilon^{n-2})^{1/2}\right), & \text{if } n>2, \end{cases} \tag{3.42}$$

as $\varepsilon\to 0$.

Finally, due to (3.36), (3.37), and (3.38) we get that

$$\left|\frac{1}{\lambda_\varepsilon^1} - \frac{1}{\lambda_0^1}\right| \leq \varphi(\varepsilon), \tag{3.43}$$

where $\varphi(\varepsilon)$ has the asymptotics (3.42).

3.3. Proofs of the main results.

Proof of Theorem 2.2. Actually, because of estimate (3.19), Friedrich's inequality

$$\int_\Omega u^2\,dx \leq \frac{1}{\lambda_\varepsilon^1}\int_\Omega |\nabla u|^2\,dx, \quad u\in H^1(\Omega,\Gamma_\varepsilon) \tag{3.44}$$

is valid. The estimate (3.43) implies that

$$\frac{1}{\lambda_\varepsilon^1} \leq \frac{1}{\lambda_0^1} + \varphi(\varepsilon). \tag{3.45}$$

By rewriting inequality (3.44), using the established relations between λ_ε^1 and λ_0^1 we find that

$$\int_\Omega u^2\,dx \leq \frac{1}{\lambda_\varepsilon^1}\int_\Omega |\nabla u|^2\,dx \leq \left(\frac{1}{\lambda_0^1} + \varphi(\varepsilon)\right)\int_\Omega |\nabla u|^2\,dx. \tag{3.46}$$

According to our notations, $1/\lambda_0^1 = K_0$. Thus, for $u \in H^1(\Omega,\Gamma_\varepsilon)$, the following Friedrich inequality holds true:

$$\int_\Omega u^2\,dx \leq K_\varepsilon \int_\Omega |\nabla u|^2\,dx, \quad K_\varepsilon = K_0 + \varphi(\varepsilon), \tag{3.47}$$

where $\varphi(\varepsilon) \sim (|\ln\varepsilon|)^{-1/2} + (\delta(\varepsilon)|\ln\varepsilon|)^{1/2}$ as $\varepsilon \to 0$, if $n = 2$. Hence, the proof is complete. □

Proof of Theorem 2.3. The proof is completely analogous to that of Theorem 2.2: using inequalities (3.19), (3.42), and (3.43), we obtain the asymptotics of the constant K_ε, hence we leave out the details. Note only that in this case $\varphi(\varepsilon) \sim \varepsilon^{n/2-1} + (\delta(\varepsilon)\varepsilon^{n-2})^{1/2}$ as $\varepsilon \to 0$. □

4. Special cases

In this section, we consider domains with special geometry.

Let ∂G be the boundary of the unit disk G centered at the origin. Assume that $\omega^\varepsilon = \{(r,\theta) : r = 1, -\delta(\varepsilon)(\pi/2) < \theta < \delta(\varepsilon)(\pi/2)\}$ is the arc, where (r,θ) are the polar coordinates. Suppose also that $\eta_\varepsilon = (-\varepsilon\delta, \varepsilon\delta)$. Denote $\gamma^\varepsilon = \partial G \setminus \overline{\Gamma^\varepsilon}$, where Γ^ε is the union of the sets obtained from η_ε by rotation about the origin through the angle $\varepsilon\pi$ and its multiples. For simplicity we assume here that $\varepsilon = 2/N$, $N \in \mathbb{N}$. Let $\mathcal{P}$ be an arbitrary conformal mapping of a disk with radius exceeding 1, let Ω be the image of the unit disk G and let $\Gamma_1^\varepsilon = \mathcal{P}(\Gamma^\varepsilon)$, $\Gamma_2^\varepsilon = \mathcal{P}(\gamma^\varepsilon)$.

For this domain we have the following theorem (see [15–17]).

Theorem 4.1. *Suppose that $n = 2$, the domain Ω is the image of the unit disk G as defined at the beginning of the section and $\delta\ln\varepsilon \to 0$ as $\varepsilon \to 0$. For $u \in H^1(\Omega,\Gamma_1^\varepsilon)$, the following Friedrich inequality holds true:*

$$\int_\Omega u^2\,dx \leq K_\varepsilon \int_\Omega |\nabla u|^2\,dx, \quad K_\varepsilon = K_0 + \varphi(\varepsilon), \tag{4.1}$$

where K_0 is a constant in Friedrich's inequality (1.1) for functions $u \in \overset{\circ}{H}{}^1(\Omega)$, and $\varphi(\varepsilon) = \delta\ln\sin\varepsilon \int_{\partial\Omega}(\partial\psi_0/\partial\nu)^2|\mathcal{F}'|\,ds + o(\delta)$ as $\varepsilon \to 0$, where ψ_0 is the first normalized in $L_2(\Omega)$ eigenfunction of the problem

$$-\Delta\psi_0 = K_0\psi_0, \quad x \in \Omega; \qquad \psi_0 = 0, \quad x \in \partial\Omega, \tag{4.2}$$

and $\mathcal{F} = \mathcal{P}^{-1}$.

If the mapping $\mathcal{P}$ is identical, then the following statement takes place (see [15–18]).

Theorem 4.2. *Suppose that $n = 2$, the domain Ω is the unit disk and $\delta \ln \varepsilon \to 0$ as $\varepsilon \to 0$. For $u \in H^1(\Omega, \Gamma_1^\varepsilon)$, the following Friedrich inequality holds true:*

$$\int_\Omega u^2\,dx \le K_\varepsilon \int_\Omega |\nabla u|^2\,dx, \qquad K_\varepsilon = K_0\left(1 + 2\delta \ln \sin\varepsilon + 2\delta^2 (\ln \sin\varepsilon)^2 + o(\delta^2)\right) \tag{4.3}$$

as $\varepsilon \to 0$, where K_0 is a constant in Friedrich's inequality (1.1) for functions $u \in \overset{\circ}{H}{}^1(\Omega)$.

Note that nonperiodic geometry are considered in [19]. In the paper the author constructed and verified the asymptotic expansions of eigenvalues. Keeping in mind these results it is possible to obtain sharp bounds for the constant in Friedrich's inequality.

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