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Research Article

An Extragradient Method for Fixed Point Problems and Variational Inequality Problems

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We present an extragradient method for fixed point problems and variational inequality problems. Using this method, we can find the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for monotone mapping.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. Recall that a mapping A of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \tag{1.1}$$

for all $u, v \in C$. A is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{1.2}$$

for all $u, v \in C$. It is well known that the variational inequality problem $\mathrm{VI}(A, C)$ is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.3}$$

for all $v \in C$ (see [1–3]). The set of solutions of the variational inequality problem is denoted by Ω. The variational inequality has been extensively studied in the literature, see, for example, [4–6] and the references therein. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \tag{1.4}$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed points of S.

For finding an element of $F(S) \cap \Omega$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse strongly monotone, Takahashi and Toyoda [7] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where P_C is the metric projection of H onto C, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that if $F(S) \cap \Omega$ is nonempty, then the sequence $\{x_n\}$ generated by (1.5) converges weakly to some $z \in F(S) \cap \Omega$. Recently, Nadezhkina and Takahashi [8] introduced a so-called extragradient method motivated by the idea of Korpelevič [9] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They obtained the following weak convergence theorem.

Theorem 1.1 (see Nadezhkina and Takahashi [8]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone k-Lipschitz continuous mapping, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{aligned} x_0 &= x \in H, \\ y_n &= P_C (x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A y_n), \quad \forall n \geq 0, \end{aligned} \tag{1.6}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0,1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ converge weakly to the same point $P_{F(S)\cap\Omega}(x_0)$.

Very recently, Zeng and Yao [10] introduced a new extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They obtained the following strong convergence theorem.

Theorem 1.2 (see Zeng and Yao [10]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone k-Lipschitz continuous mapping, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$*

be generated by

$$x_0 = x \in H,$$
$$y_n = P_C(x_n - \lambda_n A x_n), \tag{1.7}$$
$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *satisfy the following conditions:*

(a) $\{\lambda_n k\} \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$*;*

(b) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $P_{F(S)\cap\Omega}(x_0)$ *provided that*

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{1.8}$$

Remark 1.3. The iterative scheme (1.6) in Theorem 1.1 has only weak convergence. The iterative scheme (1.7) in Theorem 1.2 has strong convergence but imposed the assumption (1.8) on the sequence $\{x_n\}$.

In this paper, motivated by the iterative schemes (1.6) and (1.7), we introduced a new extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for monotone mapping. We obtain a strong convergence theorem under some mild conditions.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. It is well known that for any $u \in H$, there exists unique $y_0 \in C$ such that

$$||u - y_0|| = \inf \{\|u - y\| : y \in C\}. \tag{2.1}$$

We denote y_0 by $P_C u$, where P_C is called the metric projection of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$,

(ii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$, for every $x, y \in H$,

(iii) $\langle x - P_C x, y - P_C x \rangle \leq 0$, for all $x \in H$, $y \in C$,

(iv) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$, for all $x \in H, y \in C$.

Such property of P_C will be crucial in the proof of our main results. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from (iv) that

$$u \in \Omega \Longleftrightarrow u = P_C(u - \lambda A u), \quad \forall \lambda > 0. \tag{2.2}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies that $f \in Tx$. Let A be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,\ \forall u \in C\}. \tag{2.3}$$

Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.4}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$ (see [11]).

Now, we introduce several lemmas for our main results in this paper.

Lemma 2.1 (see [12]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \tag{2.5}$$

Lemma 2.2 (see [13]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3 (see [14]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \tag{2.6}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone L-Lipschitz continuous mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \neq \emptyset$. For fixed $u \in H$ and given $x_0 \in H$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *satisfying the following conditions:*

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,

(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(C4) $\lim_{n\to\infty} \lambda_n = 0$.

Then $\{x_n\}$ *converges strongly to* $P_{F(S)\cap\Omega} u$.

Proof. Let $x^* \in F(S) \cap \Omega$, then $x^* = P_C(x^* - \lambda_n A x^*)$. Put $t_n = P_C(x_n - \lambda_n A y_n)$. Substituting x by $x_n - \lambda_n A y_n$ and y by x^* in (iv), we have

$$\begin{aligned}
||t_n - x^*||^2 &\le ||x_n - \lambda_n A y_n - x^*||^2 - ||x_n - \lambda_n A y_n - t_n||^2 \\
&= ||x_n - x^*||^2 - 2\lambda_n \langle A y_n, x_n - x^*\rangle + \lambda_n^2 ||A y_n||^2 \\
&\quad - ||x_n - t_n||^2 + 2\lambda_n \langle A y_n, x_n - t_n\rangle - \lambda_n^2 ||A y_n||^2 \\
&= ||x_n - x^*||^2 + 2\lambda_n \langle A y_n, x^* - t_n\rangle - ||x_n - t_n||^2 \\
&= ||x_n - x^*||^2 - ||x_n - t_n||^2 + 2\lambda_n \langle A y_n - A x^*, x^* - y_n\rangle \\
&\quad + 2\lambda_n \langle A x^*, x^* - y_n\rangle + 2\lambda_n \langle A y_n, y_n - t_n\rangle.
\end{aligned} \tag{3.2}$$

Using the fact that A is monotonic and x^* is a solution of the variational inequality problem $\mathrm{VI}(A,C)$, we have

$$\langle A y_n - A x^*, x^* - y_n\rangle \le 0, \qquad \langle A x^*, x^* - y_n\rangle \le 0. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}
||t_n - x^*||^2 &\le ||x_n - x^*||^2 - ||x_n - t_n||^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\
&= ||x_n - x^*||^2 - ||(x_n - y_n) + (y_n - t_n)||^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\
&= ||x_n - x^*||^2 - ||x_n - y_n||^2 - 2\langle x_n - y_n, y_n - t_n\rangle \\
&\quad - ||y_n - t_n||^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\
&= ||x_n - x^*||^2 - ||x_n - y_n||^2 - ||y_n - t_n||^2 + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle.
\end{aligned} \tag{3.4}$$

Substituting x by $x_n - \lambda_n A x_n$ and y by t_n in (iii), we have

$$\langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle \le 0. \tag{3.5}$$

It follows that

$$\begin{aligned}
\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \le \lambda_n L ||x_n - y_n|| ||t_n - y_n||.
\end{aligned} \tag{3.6}$$

By (3.4) and (3.6), we obtain

$$\begin{aligned}\|t_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n L\|x_n - y_n\|\|t_n - y_n\| \\ &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n L^2(\|x_n - y_n\|^2 + \|y_n - t_n\|^2) \\ &\le \|x_n - x^*\|^2 + (\lambda_n^2 L^2 - 1)\|x_n - y_n\|^2 + (\lambda_n^2 L^2 - 1)\|y_n - t_n\|^2.\end{aligned} \tag{3.7}$$

Since $\lambda_n \to 0$ as $n \to \infty$, there exists a positive integer N_0 such that $\lambda_n^2 L^2 - 1 \le -1/2$ when $n \ge N_0$. It follows from (3.7) that

$$\|t_n - x^*\| \le \|x_n - x^*\|. \tag{3.8}$$

By (3.1), we have

$$\begin{aligned}\|x_{n+1} - x^*\| = \|\alpha_n u + \beta_n x_n + \gamma_n S t_n - x^*\| &\le \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|t_n - x^*\| \\ &\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \le \max\{\|u - x^*\|, \|x_0 - x^*\|\}.\end{aligned} \tag{3.9}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{t_n\}$, $\{St_n\}$, $\{Ax_n\}$, and $\{Ay_n\}$ are also bounded.

For all $x, y \in C$, we get

$$\begin{aligned}\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 = \|(x - y) - \lambda_n(Ax - Ay)\|^2 &= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle \\ &\quad + \lambda_n^2\|Ax - Ay\|^2 \le \|x - y\|^2 + \lambda_n^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 + \lambda_n^2 L^2\|x - y\|^2 = (1 + L^2\lambda_n^2)\|x - y\|^2,\end{aligned} \tag{3.10}$$

which implies that

$$\|(I - \lambda_n A)x - (I - \lambda_n A)y\| \le (1 + L\lambda_n)\|x - y\|. \tag{3.11}$$

By (3.1) and (3.11), we have

$$\begin{aligned}\|t_{n+1} - t_n\| &= \|P_C(x_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(x_n - \lambda_n Ay_n)\| \\ &\le \|(x_{n+1} - \lambda_{n+1}Ay_{n+1}) - (x_n - \lambda_n Ay_n)\| \\ &= \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) \\ &\qquad + \lambda_{n+1}(Ax_{n+1} - Ay_{n+1} - Ax_n) + \lambda_n Ay_n\| \\ &\le \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n)\| \\ &\quad + \lambda_{n+1}(\|Ax_{n+1}\| + \|Ay_{n+1}\| + \|Ax_n\|) + \lambda_n\|Ay_n\| \\ &\le (1 + \lambda_{n+1}L)\|x_{n+1} - x_n\| \\ &\quad + \lambda_{n+1}(\|Ax_{n+1}\| + \|Ay_{n+1}\| + \|Ax_n\|) + \lambda_n\|Ay_n\|.\end{aligned} \tag{3.12}$$

Set $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$. Then, we obtain

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}St_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n St_n}{1-\beta_n} \\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(St_{n+1} - St_n) \\
&\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)St_n.
\end{aligned} \tag{3.13}
$$

Combining (3.12) and (3.13), we have

$$
\begin{aligned}
&||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| \\
&\quad \le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(1+\lambda_{n+1}L)||x_{n+1} - x_n|| \\
&\qquad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\{\lambda_{n+1}(||Ax_{n+1}|| + ||Ay_{n+1}|| + ||Ax_n||) + \lambda_n||Ay_n||\} \\
&\qquad + \left|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right| ||St_n|| - ||x_{n+1} - x_n|| \\
&\quad \le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| (\|u\| + ||St_n||) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\lambda_{n+1}L||x_{n+1} - x_n|| \\
&\qquad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\{\lambda_{n+1}(||Ax_{n+1}|| + ||Ay_{n+1}|| + ||Ax_n||) + \lambda_n||Ay_n||\},
\end{aligned} \tag{3.14}
$$

this together with (C2) and (C4) imply that

$$
\limsup_{n\to\infty} (||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \le 0. \tag{3.15}
$$

Hence by Lemma 2.2, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$
\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} (1-\beta_n)||z_n - x_n|| = 0. \tag{3.16}
$$

From (C4) and (3.12), we also have $\|t_{n+1} - t_n\| \to 0$ as $n \to \infty$.

For $x^* \in F(S) \cap \Omega$, from Lemma 2.1, (3.1), and (3.7), we obtain when $n \ge N_0$ that

$$
\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\alpha_n u + \beta_n x_n + \gamma_n St_n - x^*||^2 \\
&\le \alpha_n||u - x^*||^2 + \beta_n||x_n - x^*||^2 + \gamma_n||St_n - x^*||^2 \\
&\le \alpha_n||u - x^*||^2 + \beta_n||x_n - x^*||^2 + \gamma_n||t_n - x^*||^2 \\
&\le \alpha_n||u - x^*||^2 + \beta_n||x_n - x^*||^2 \\
&\quad + \gamma_n\{(||x_n - x^*||^2 + (\lambda_n^2L^2 - 1)||x_n - y_n||^2) + (\lambda_n^2L^2 - 1)||y_n - t_n||^2\} \\
&\le \alpha_n||u - x^*||^2 + ||x_n - x^*||^2 - \frac{1}{2}||x_n - y_n||^2,
\end{aligned} \tag{3.17}
$$

which implies that

$$\begin{aligned}\frac{1}{2}\|x_n-y_n\|^2 &\le \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &=\alpha_n\|u-x^*\|^2+(\|x_n-x^*\|-\|x_{n+1}-x^*\|)\\ &\quad\times(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\ &\le \alpha_n\|u-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|.\end{aligned} \tag{3.18}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$, from (3.18), we have $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Noting that

$$\begin{aligned}\|y_n-t_n\| &= \|P_C(x_n-\lambda_nAx_n)-P_C(x_n-\lambda_nAy_n)\|\\ &\le \lambda_n\|Ax_n-Ay_n\|\le\lambda_nL\|x_n-y_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty,\\ \|t_n-x_n\| &\le \|t_n-y_n\|+\|y_n-x_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty,\\ \|Sy_n-x_{n+1}\| &\le \|Sy_n-St_n\|+\|St_n-x_{n+1}\|\le\|y_n-t_n\|+\alpha_n\|St_n-u\|+\beta_n\|St_n-x_n\|\\ &\le \|y_n-t_n\|+\alpha_n\|St_n-u\|+\beta_n\|St_n-Sx_n\|+\beta_n\|Sx_n-x_n\|\\ &\le \|y_n-t_n\|+\alpha_n\|St_n-u\|+\beta_n\|t_n-x_n\|+\beta_n\|Sx_n-x_n\|.\end{aligned} \tag{3.19}$$

Consequently, from (3.19), we can infer that

$$\begin{aligned}\|Sx_n-x_n\| &\le \|Sx_n-St_n\|+\|St_n-Sy_n\|+\|Sy_n-x_{n+1}\|+\|x_{n+1}-x_n\|\\ &\le (1+\beta_n)\|x_n-t_n\|+2\|t_n-y_n\|+\alpha_n\|St_n-u\|+\beta_n\|Sx_n-x_n\|+\|x_{n+1}-x_n\|,\end{aligned} \tag{3.20}$$

which implies that

$$\|Sx_n-x_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.21}$$

Also we have

$$\begin{aligned}\|St_n-t_n\| &\le \|St_n-Sx_n\|+\|Sx_n-x_n\|+\|x_n-t_n\|\\ &\le 2\|t_n-x_n\|+\|Sx_n-x_n\|\longrightarrow\infty \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{3.22}$$

Next we show that

$$\limsup_{n\to\infty}\langle u-z_0, x_n-z_0\rangle\le 0, \tag{3.23}$$

where $z_0 = P_{F(S)\cap\Omega}u$.

To show it, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z_0, St_n-z_0\rangle=\lim_{i\to\infty}\langle u-z_0, St_{n_i}-z_0\rangle. \tag{3.24}$$

As $\{t_{n_i}\}$ is bounded, we have that a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $t_{n_i} \rightharpoonup z$. Since $\|St_n - t_n\| \to 0$, we obtain $St_{n_i} \rightharpoonup z$ as $i \to \infty$. Then we can obtain $z \in F(S) \cap \Omega$. In fact, let us first show that $z \in \Omega$.

Let

$$Uv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.25}$$

Then U is maximal monotone. Let $(v, w) \in G(U)$. Since $w - Av \in N_C v$ and $t_n \in C$, we have $\langle v - t_n, w - Av \rangle \geq 0$. On the other hand, from $t_n = P_C(x_n - \lambda_n A y_n)$, we have

$$\langle v - t_n, t_n - (x_n - \lambda_n A y_n) \rangle \geq 0, \tag{3.26}$$

that is,

$$\left\langle v - t_n, \frac{t_n - y_n}{\lambda_n} + A y_n \right\rangle \geq 0. \tag{3.27}$$

Therefore, we have

$$\begin{aligned} \langle v - t_{n_i}, w \rangle &\geq \langle v - t_{n_i}, Av \rangle \geq \langle v - t_{n_i}, Av \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + A y_{n_i} \right\rangle \\ &= \left\langle v - t_{n_i}, Av - A y_{n_i} - \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - A y_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + A y_{n_i} \right\rangle. \end{aligned} \tag{3.28}$$

Noting that $\|t_{n_i} - y_{n_i}\| \to 0$ as $i \to \infty$ and A is Lipschitz continuous, hence from (3.28), we obtain $\langle v - z, w \rangle \geq 0$ as $i \to \infty$. Since U is maximal monotone, we have $z \in U^{-1}0$, and hence $z \in \Omega$.

Let us show that $z \in F(S)$. Assume that $z \notin F(S)$. From Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty} \|t_{n_i} - z\| &< \liminf_{i\to\infty} \|t_{n_i} - Sz\| = \liminf_{i\to\infty} \|t_{n_i} - St_{n_i} + St_{n_i} - Sz\| \\ &\leq \liminf_{i\to\infty} (\|t_{n_i} - St_{n_i}\| + \|St_{n_i} - Sz\|) = \liminf_{i\to\infty} \|St_{n_i} - Sz\| \\ &\leq \liminf_{i\to\infty} \|t_{n_i} - z\|. \end{aligned} \tag{3.29}$$

This is a contradiction. Thus, we obtain $z \in F(S)$.

Hence, from (iii), we have

$$\begin{aligned} \limsup_{n\to\infty} \langle u - z_0, x_n - z_0 \rangle &= \limsup_{n\to\infty} \langle u - z_0, St_n - z_0 \rangle \\ &= \lim_{i\to\infty} \langle u - z_0, St_{n_i} - z_0 \rangle = \langle u - z_0, z - z_0 \rangle \leq 0. \end{aligned} \tag{3.30}$$

Therefore,

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &= \langle \alpha_n u+\beta_n x_n+\gamma_n St_n - z_0, x_{n+1}-z_0\rangle\\ &= \alpha_n\langle u-z_0, x_{n+1}-z_0\rangle+\beta_n\langle x_n-z_0, x_{n+1}-z_0\rangle+\gamma_n\langle St_n-z_0, x_{n+1}-z_0\rangle\\ &\le \frac{1}{2}\beta_n\left(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2\right)+\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle\\ &\quad+\frac{1}{2}\gamma_n\left(\|t_n-z_0\|^2+\|x_{n+1}-z_0\|^2\right)\\ &\le \frac{1}{2}(1-\alpha_n)\left(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2\right)+\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle,\end{aligned}\tag{3.31}$$

which implies that

$$\|x_{n+1}-z_0\|^2 \le (1-\alpha_n)\|x_n-z_0\|^2+2\alpha_n\langle u-z_0, x_{n+1}-z_0\rangle,\tag{3.32}$$

this together with (3.30) and Lemma 2.3, we can obtain the conclusion. This completes the proof. □

We observe that some strong convergence theorems for the iterative scheme (3.1) were established under the assumption that the mapping A is α-inverse strongly monotone in [15].

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone L-Lipschitz continuous mapping of C into H such that $\Omega \neq \emptyset$. For fixed $u \in H$ and given $x_0 \in H$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{aligned} y_n &= P_C(x_n-\lambda_n Ax_n),\\ x_{n+1} &= \alpha_n u+\beta_n x_n+\gamma_n P_C(x_n-\lambda_n Ay_n),\end{aligned}\tag{3.33}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:
(C1) $\alpha_n+\beta_n+\gamma_n=1$,
(C2) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$,
(C3) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$,
(C4) $\lim_{n\to\infty}\lambda_n=0$.
Then $\{x_n\}$ converges strongly to $P_\Omega u$.

4. Applications

A mapping $T : C \to C$ is called strictly pseudocontractive if there exists k with $0 \le k < 1$ such that

$$\|Tx-Ty\|^2 \le \|x-y\|^2+k\|(I-T)x-(I-T)y\|^2,\tag{4.1}$$

for all $x, y \in C$. Put $A = I - T$, then we have

$$\|(I-A)x-(I-A)y\|^2 \le \|x-y\|^2+k\|Ax-Ay\|^2.\tag{4.2}$$

On the other hand,

$$\left\|(I-A)x-(I-A)y\right\|^2 = \|x-y\|^2+\|Ax-Ay\|^2-2\langle x-y, Ax-Ay\rangle. \tag{4.3}$$

Hence, we have

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-k}{2}\|Ax-Ay\|^2 \geq 0. \tag{4.4}$$

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. Let T be a k-strictly pseudocontractive mapping of C into itself, and let S be a nonexpansive mapping of C into itself such that $F(T)\cap F(S)\neq\emptyset$. For fixed $u\in H$ and given $x_0\in H$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{aligned} y_n &= (1-\lambda_n)x_n+\lambda_n Tx_n, \\ x_{n+1} &= \alpha_n u+\beta_n x_n+\gamma_n S((1-\lambda_n)y_n+\lambda_n Ty_n), \end{aligned} \tag{4.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:
(C1) $\alpha_n+\beta_n+\gamma_n=1$,
(C2) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$,
(C3) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$,
(C4) $\lim_{n\to\infty}\lambda_n=0$.
Then $\{x_n\}$ converges strongly to $P_{F(T)\cap F(S)}u$.

Proof. Put $A=I-T$. Then A is monotone. We have $F(T)=\Omega$ and $P_C(x_n-\lambda_n Ax_n)=(1-\lambda_n)x_n+\lambda_n Tx_n$. So, by Theorem 3.1, we can obtain the desired result. This completes the proof. □

Theorem 4.2. *Let H be a real Hilbert space. Let A be a monotone mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $A^{-1}0\cap F(S)\neq\emptyset$. For fixed $u\in H$ and given $x_0\in H$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{aligned} y_n &= x_n-\lambda_n Ax_n, \\ x_{n+1} &= \alpha_n u+\beta_n x_n+\gamma_n S(y_n-\lambda_n Ay_n), \end{aligned} \tag{4.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:
(C1) $\alpha_n+\beta_n+\gamma_n=1$,
(C2) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$,
(C3) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$,
(C4) $\lim_{n\to\infty}\lambda_n=0$.
Then $\{x_n\}$ converges strongly to $P_{A^{-1}0\cap F(S)}u$.

Proof. Since $A^{-1}0=\Omega$, putting $P_H=I$, by Theorem 3.1, we can obtain the conclusion. This completes the proof. □

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