Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 39345, 12 pages
doi:10.1155/2007/39345

Research Article

On Star Duality of Mixed Intersection Bodies

Lu Fenghong, Mao Weihong, and Leng Gangsong

Received 7 July 2006; Revised 22 October 2006; Accepted 30 October 2006

Recommended by Y. Giga

A new kind of duality between intersection bodies and projection bodies is presented. Furthermore, some inequalities for mixed intersection bodies are established. A geometric inequality is derived between the volumes of star duality of star bodies and their associated mixed intersection integral.

1. Introduction and main results

Intersection bodies were first explicitly defined and named by Lutwak [1]. It was here that the duality between intersection bodies and projection bodies was first made clear. Despite considerable ingenuity of earlier attacks on the Busemann-Petty problem, it seems fair to say that the work of Lutwak [1] represents the beginning of its eventual solution. In [1], Lutwak also showed that if a convex body is sufficiently smooth and not an intersection body, then there exists a centered star body such that the conditions of Busemann-Petty problem hold, but the result inequality is reversed. Following Lutwak, the intersection body of order i of a star body is introduced by Zhang [2]. It follows from this definition that every intersection body of order i of a star body is an intersection body of a star body, and vice versa. As Zhang observes, the new definition of intersection body allows a more appealing formulation, namely, the Busemann-Petty problem has a positive answer in n-dimensional Euclidean space if and only if each centered convex body is an intersection body. The intersection body plays an essential role in Busemann's theory [3] of area in Minkowski spaces.

In [4], Moszyńska introduced the notion of the star dual of a star body. Generally, star dual of a convex body is different from its polar dual. For every convex body K, let K^* and K^o denote the polar body and the star dual of K, respectively.

In recent years, some authors including Haberl and Ludwig [5], Kalton and Koldobsky [6], Klain [7, 8], Koldobsky [9], Ludwig [10, 11], and so on have given considerable attention to the intersection bodies and their various properties. The aim of this paper is to establish several inequalities about the star dual version of intersection bodies. We establish the star dual version of the general Busemann intersection inequality.

Theorem 1.1. *Let $K_1,\ldots,K_{n-1}$ be star bodies in $\mathbb{R}^n$. Then*

$$V(K_1)\cdots V(K_{n-1})V(I^o(K_1,\ldots,K_{n-1})) \geq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{1.1}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are dilates of centered balls.

Theorem 1.1 is an analogue of the general Petty projection inequality which was given by Lutwak [12], concerning the polar duality of convex bodies.

Theorem 1.2. *Let $K_1,\ldots,K_{n-1}$ be convex bodies in $\mathbb{R}^n$. Then*

$$V(K_1)\cdots V(K_{n-1})V(\Pi^*(K_1,\ldots,K_{n-1})) \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{1.2}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are homothetic ellipsoids.

For two star bodies K and L, let $K\check{+}L$ denote the radial Blaschke sum of K and L [1]. We establish the dual Brunn-Minkowski inequality for the star duality of mixed intersection bodies concerning the radial Blaschke sum.

Theorem 1.3. *If K, L are star bodies in $\mathbb{R}^n$ and $0 \leq i < n$, then*

$$\widetilde{W}_i(I^o(K\check{+}L))^{-1/(n-i)} \geq \widetilde{W}_i(I^oK)^{-1/(n-i)} + \widetilde{W}_i(I^oL)^{-1/(n-i)} \tag{1.3}$$

with equality if and only if K and L are dilates. For $i > n$, inequality (1.3) is reversed.

Theorem 1.3 is an analogue of the general Brunn-Minkowski inequality for the polar duality of mixed projection bodies concerning the Blaschke sum [1].

Theorem 1.4. *If K,L are convex bodies in $\mathbb{R}^n$ and $0 \leq i < n$, then*

$$\widetilde{W}_i(\Pi^*(K+L))^{-1/(n-i)} \geq \widetilde{W}_i(\Pi^*K)^{-1/(n-i)} + \widetilde{W}_i(\Pi^*L)^{-1/(n-i)} \tag{1.4}$$

with equality if and only if K and L are dilates. For $i > n$, inequality (1.4) is reversed.

Besides, we establish the following relationship between star duality and intersection operator I.

Theorem 1.5. *If $K_1,\ldots,K_{n-1}$ are star bodies in $\mathbb{R}^n$, then*

$$\omega_{n-1}^2 I^o(K_1,\ldots,K_{n-1}) \subset I(K_1^o,\ldots,K_{n-1}^o) \tag{1.5}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are dilates of centered balls.

In Section 2, some basic definitions and facts are restated. The elementary results (and definitions) are from the theory of convex bodies. The reader may consult the standard works on the subject [13, 14] for reference. Some properties and inequalities of star duality are established in Section 3. A general Busemann intersection inequality and its star dual forms are derived; the Brunn-Minkowski inequalities for the star dual and other inequalities are given in Section 4. By using the inequalities concerning star duality of mixed intersection bodies, a geometric inequality is derived between the volumes of star duality of star bodies and their associated mixed intersection integral in Section 5.

2. Basic definitions and notation

As usual, let B denote the unit ball in Euclidean n-space, $\mathbb{R}^n$. While its boundary is S^{n-1} and the origin is denoted by o, let ω_i denote the volume of the i-dimensional unit ball. If u is a unit vector, that is, an element of S^{n-1}, we denote by $u^\perp$ the $(n-1)$-dimensional linear subspace orthogonal to u.

For a compact subset L of $\mathbb{R}^n$, with $o \in L$, star-shaped with respect to o, the radial function $\rho(L,\cdot): S^{n-1} \to \mathbb{R}$ is defined by

$$\rho(L,u) = \rho_L(u) = \max\{\lambda : \lambda u \in L\}. \tag{2.1}$$

If $\rho(L,\cdot)$ is continuous and positive, L will be called a star body.

Let $\mathcal{S}_o^n$ denote the set of star bodies in $\mathbb{R}^n$. Two star bodies $K, L \in \mathcal{S}_o^n$ are said to be dilatate(of each other) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$.

2.1. Dual-mixed volume. If $x_i \in \mathbb{R}^n$, $1 \le i \le m$, then $x_1 \tilde{+} \cdots \tilde{+} x_m$ is defined to be the usual vector sum of the points x_i, if all of them are contained in a line through origin, and 0 otherwise.

If $K_i \in \mathcal{S}_o^n$ and $t_i \ge 0$, $1 \le i \le m$, then the radial linear combination, $t_1K_1 \tilde{+} \cdots \tilde{+} t_mK_m$, is defined by

$$t_1K_1 \tilde{+} \cdots \tilde{+} t_mK_m = \{t_1x_1 \tilde{+} \cdots \tilde{+} t_mx_m : x_i \in K_i\}. \tag{2.2}$$

Moreover, for each $u \in S^{n-1}$,

$$\rho_{t_1K_1 \tilde{+} t_2K_2}(u) = t_1\rho_{K_1}(u) + t_2\rho_{K_2}(u). \tag{2.3}$$

If $L \in \mathcal{S}_o^n$, then the polar coordinate formula for volume is

$$V(L) = \frac{1}{n}\int_{S^{n-1}} \rho_L(u)^n dS(u). \tag{2.4}$$

Let $L_j \in \mathcal{S}_o^n$ $(1 \le j \le n)$. The dual-mixed volume $\widetilde{V}(L_1,\dots,L_n)$ is defined by Lutwak in [15, 16] by

$$\widetilde{V}(L_1,\dots,L_n) = \frac{1}{n}\int_{S^{n-1}} \rho_{L_1}(u)\cdots\rho_{L_n}(u)dS(u). \tag{2.5}$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual-mixed volumes are written as $\widetilde{V}_i(K,L)$ and the dual-mixed volumes $\widetilde{V}_i(K,B)$ are written as $\widetilde{W}_i(K)$.

Let $L \in \mathcal{S}_o^n$, $i \in \mathbb{R}^n$, the dual quermassintegrals $\widetilde{W}_i(L)$ is defined by Lutwak in [15] by

$$\widetilde{W}_i(L) = \frac{1}{n}\int_{S^{n-1}} \rho_L(u)^{n-i}dS(u). \tag{2.6}$$

If K is a star body in $\mathbb{R}^n$ and $u \in S^{n-1}$, then we use $K \cap u^\perp$ to denote the intersection of K with the subspace $u^\perp$ that passes through the origin and is orthogonal to u. If $K_1,\dots,K_{n-1}$ are star bodies in $\mathbb{R}^n$ and $u \in S^{n-1}$, then the $(n-1)$-dimensional dual-mixed volume of $K_1 \cap u^\perp,\dots,K_{n-1} \cap u^\perp$ in $u^\perp$ is written $\widetilde{v}(K_1 \cap u^\perp,\dots,K_{n-1} \cap u^\perp)$. If $K_1 = \cdots = K_{n-i-1} = K$ and $K_{n-i} = \cdots = K_{n-1} = B$, then $\widetilde{v}(K_1 \cap u^\perp,\dots,K_{n-1} \cap u^\perp)$ is just the ith dual quermassintegrals of $K \cap u^\perp$ in $u^\perp$, it will be denoted by $\widetilde{w}_i(K \cap u^\perp)$ and is called the $(n-i-1)$-section of K in the direction u. The $(n-1)$-dimensional volume of $K \cap u^\perp$ will be written $v(K \cap u^\perp)$ rather than $\widetilde{w}_0(K \cap u^\perp)$.

2.2. Mixed intersection bodies. Let $K \in \mathcal{S}_o^n$. The intersection body IK of K is a star body such that [1]

$$\rho_{IK}(u) = v(K \cap u^\perp) = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_K(v)^{n-1}d\lambda_{n-2}(v), \tag{2.7}$$

where λ_i denote the i-dimensional volume.

Let $K_1,\dots,K_{n-1} \in \mathcal{S}_o^n$. The mixed intersection body $I(K_1,\dots,K_{n-1})$ of star bodies $K_1,\dots,K_{n-1}$ is defined by

$$\rho_{I(K_1,\dots,K_{n-1})}(u) = \widetilde{v}(K_1 \cap u^\perp,\dots,K_{n-1} \cap u^\perp) = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)d\lambda_{n-2}(v). \tag{2.8}$$

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, then $I(K_1,\dots,K_{n-1})$ will be denoted as $I_i(K,L)$. If $L = B$, then $I_i(K,B)$ is called the intersection body of order i of K; it will often be written as I_iK. Specially, $I_0K = IK$. This term was introduced by Zhang [2].

Let $K \in \mathcal{S}_o^n$ and $i \in \mathbb{R}$, the intersection body of order i of K is the centered star body I_iK such that [2]

$$\rho_{I_iK}(u) = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_K^{n-1-i}(v)d\lambda_{n-2}(v). \tag{2.9}$$

If $K,L \in \mathcal{S}_o^n$ and $\lambda,\mu \geq 0$ (not both zero), then for each $u \in S^{n-1}$, the radial Blaschke linear combination, $\lambda \cdot K \check{+} \mu \cdot L$, is the star body whose radial function is given by [15]

$$\rho(\lambda \cdot K \check{+} \mu \cdot L,u)^{n-1} = \lambda\rho(K,u)^{n-1} + \mu\rho(L,u)^{n-1}. \tag{2.10}$$

It is easy to verify the following relation between radial Blaschke and radial Minkowski scalar multiplication: if $K \in \mathcal{S}_o^n$ and $\lambda \geq 0$, then $\lambda \cdot K = \lambda^{1/(n-1)}K$.

The following properties will be used later: if $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \geq 0$, then

$$I(\lambda \cdot K \check{+} \mu \cdot L) = \lambda IK \tilde{+} \mu IL. \tag{2.11}$$

3. Inequalities for star duality of star body

Also associated with a star body $L \in \mathcal{S}_o^n$ is its star duality L^o, which was introduced by Moszyńska [4](and was improved in [17]). Let i be the inversion of $\mathbb{R}^n \backslash \{0\}$, with respect to S^{n-1},

$$i(x) := \frac{x}{\|x\|^2}. \tag{3.1}$$

Then the star duality L^o of a star body $L \in \mathcal{S}_o^n$ is defined by

$$L^o = \mathrm{cl}\big(\mathbb{R}^n \backslash i(L)\big). \tag{3.2}$$

It is easy to verify that for every $u \in S^{n-1}$ [4],

$$\rho(L^o, u) = \frac{1}{\rho(L, u)}. \tag{3.3}$$

By applying the conception of star duality, we establish the following properties for star body and its star duality.

Theorem 3.1. *If $K \in \mathcal{S}_o^n$ and $i \in \mathbb{R}$, then*

$$\widetilde{W}_{2n-i}(K^o) = \widetilde{W}_i(K). \tag{3.4}$$

Proof. From definition (2.6), equality (3.3), and definition (2.6) again, we have

$$\widetilde{W}_{2n-i}(K^o) = \frac{1}{n}\int_{S^{n-1}} \rho(K^o, u)^{n-(2n-i)} dS(u) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n-i} dS(u) = \widetilde{W}_i(K). \tag{3.5}$$

□

In particular, for $i = 0$ in Theorem 3.1, we have

$$\widetilde{W}_{2n}(K^o) = V(K). \tag{3.6}$$

The following statement is an analogue of the Blaschke-Santaló inequality [3] for dual quermassintegrals of star bodies.

Theorem 3.2. *If $K \in \mathcal{S}_o^n$ and $i \in \mathbb{R}$, then*

$$\widetilde{W}_i(K)\widetilde{W}_i(K^o) \geq \omega_n^2 \tag{3.7}$$

with equality if and only if K is a centered ball.

Proof. From equality (3.3), Hölder inequality [18], and definition (2.6), we have

$$\begin{aligned}
\omega_n &= \frac{1}{n}\int_{S^{n-1}} 1\,dS(u) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{(n-i)/2}\rho(K,u)^{-(n-i)/2}dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{(n-i)/2}\rho(K^o,u)^{(n-i)/2}dS(u) \\
&\le \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}dS(u)\right)^{1/2}\left(\frac{1}{n}\int_{S^{n-1}} \rho(K^o,u)^{n-i}dS(u)\right)^{1/2} \\
&= \widetilde{W}_i(K)^{1/2}\widetilde{W}_i(K^o)^{1/2}.
\end{aligned} \tag{3.8}$$

According to the equality condition of Hölder inequality, we know that equality in inequality (3.7) holds if and only if K is a centered ball. □

In particular, for $i = 0$ in Theorem 3.2, we have the following.

Corollary 3.3. *If $K \in \mathcal{S}_o^n$, then*

$$V(K)V(K^o) \ge \omega_n^2 \tag{3.9}$$

with equality if and only if K is a centered ball.

Inequality (3.9) just is an analogue of the Blaschke-Santaló inequality [3] of convex bodies.

Corollary 3.4. *If K is a convex body in $\mathbb{R}^n$, then*

$$V(K)V(K^*) \le \omega_n^2 \tag{3.10}$$

with equality if and only if K is an ellipsoid.

4. Star dual intersection inequalities

The following theorem is the general Busemann intersection inequality involving the volume of a convex body and that of its associated intersection bodies.

Theorem 4.1 (general Busemann intersection inequality). *Let $K_1,\dots,K_{n-1}$ be star bodies in $\mathbb{R}^n$. Then,*

$$V(I(K_1,\dots,K_{n-1})) \le \frac{\omega_{n-1}^n}{\omega_n^{n-2}}V(K_1)\cdots V(K_{n-1}) \tag{4.1}$$

with equality if and only if all $K_1,\dots,K_{n-1}$ are dilates of centered ellipsoids.

Let $K_1 = \cdots = K_{n-1} = L$ in Theorem 4.1, we get the Busemann intersection inequality, which was established by Busemann [19].

Corollary 4.2. *If K is a star body in $\mathbb{R}^n$, then*

$$V(IK) \leq \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K)^{n-1} \tag{4.2}$$

with equality if and only if K is a centered ellipsoid.

Proof of Theorem 4.1. From definition (2.4), definition (2.8), Hölder inequality [18], definition (2.7), Hölder inequality, definition (2.4) again, and inequality (4.2), it follows that

$$\begin{aligned}
V(I(K_1,\dots,K_{n-1})) &= \frac{1}{n}\int_{S^{n-1}} \rho(I(K_1,\dots,K_{n-1}),u)^n dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{n-1}\int_{S^{n-1}\cap u\perp} \rho(K_1,v)\cdots\rho(K_{n-1},v)\,d\lambda_{n-2}(v)\right)^n dS(u) \\
&\leq \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{n-1}\int_{S^{n-1}\cap u\perp} \rho(K_1,v)^{n-1} d\lambda_{n-2}(v)\right)^{n/(n-1)} \\
&\quad \times\cdots\left(\frac{1}{n-1}\int_{S^{n-1}\cap u\perp} \rho(K_{n-1},v)^{n-1} d\lambda_{n-2}(v)\right)^{n/(n-1)} dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\rho(IK_1,u)\cdots\rho(IK_{n-1},u)\right)^{n/(n-1)} dS(u) \\
&\leq \left(\frac{1}{n}\int_{S^{n-1}} \rho(IK_1,u)^n dS(u)\right)^{1/(n-1)} \\
&\quad \times\cdots\left(\frac{1}{n}\int_{S^{n-1}} \rho(IK_{n-1},u)^n dS(u)\right)^{1/(n-1)} \\
&= V(IK_1)^{1/(n-1)}\cdots V(IK_{n-1})^{1/(n-1)} \leq \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K_1)\cdots V(K_{n-1}).
\end{aligned} \tag{4.3}$$

According to the equality conditions of Hölder inequality and inequality (4.2), equality holds in inequality (4.1) if and only if K_i are dilates of centered ellipsoids. □

The following statement is the star duality of the general Busemann intersection inequality.

Theorem 4.3. *Let $L_1,\dots,L_{n-1}$ be star bodies in $\mathbb{R}^n$. Then,*

$$V(L_1)\cdots V(L_{n-1})V(I^\circ(L_1,\dots,L_{n-1})) \geq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{4.4}$$

with equality if and only if all L_i $(i = 0,1,\dots,n-1)$ are dilates of centered balls.

Proof. Combing inequality (3.9) with inequality (4.1), we have

$$V(K_1)\cdots V(K_{n-1})V(I^o(K_1,\ldots,K_{n-1})) \geq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n. \tag{4.5}$$

According to the equality conditions of inequality (3.9) and inequality (4.1), equality holds if and only if K_i $(i = 0,1,\ldots,n-1)$ are dilates of centered balls. □

Theorem 4.3 is an analogue of the general Petty projection inequality which was given by Lutwak [12] concerning the polar duality of convex bodies.

In particular, let $L_1 = \cdots = L_{n-1} = L$ in Theorem 4.3, we get the following.

Corollary 4.4. *Let $L \in \mathcal{S}_o^n$. Then,*

$$V(L)^{n-1}V(I^\circ L) \geq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{4.6}$$

with equality if and only if L is a centered ball.

This is just an analogue of the Petty projection inequality concerning the polar duality of convex bodies, which was given by Petty [20].

Corollary 4.5. *Let K be a convex body in $\mathbb{R}^n$. Then,*

$$V(K)^{n-1}V(\Pi^* K) \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{4.7}$$

with equality if and only if K is an ellipsoid.

There is a relationship between star duality and the operator I.

Theorem 4.6. *If $K_1,\ldots,K_{n-1} \in \mathcal{S}_o^n$, then*

$$\omega_{n-1}^2 I^o(K_1,\ldots,K_{n-1}) \subset I(K_1^o,\ldots,K_{n-1}^o) \tag{4.8}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are dilates of centered balls.

Proof. From equality (3.3), definition (2.8), and Hölder inequality [18], we obtain

$$\begin{aligned}
&\rho(I^o(K_1,\ldots,K_{n-1}),u)^{-1}\rho(I(K_1^o,\ldots,K_{n-1}^o),u)\\
&\quad= \rho(I(K_1,\ldots,K_{n-1}),u)\rho(I(K_1^o,\ldots,K_{n-1}^o),u)\\
&\quad= \frac{1}{(n-1)^2}\int_{S^{n-1}\cap u^\perp}\rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)d\lambda_{n-2}(v)\int_{S^{n-1}\cap u^\perp}\rho_{K_1^o}(v)\cdots\rho_{K_{n-1}^o}(v)d\lambda_{n-2}(v)\\
&\quad\geq \frac{1}{(n-1)^2}\left(\int_{S^{n-1}\cap u^\perp}1d\lambda_{n-2}(v)\right)^2 = \omega_{n-1}^2.
\end{aligned} \tag{4.9}$$

Thus we get the inequality (4.8).

According to the equality conditions of Hölder inequality, equality in inequality (4.8) holds if and only if K_i are dilates of centered balls. □

In particular, for $K_1 = \cdots = K_{n-1-i} = K$, $K_{n-i} = \cdots = K_{n-1} = B$ in Theorem 4.6, we have the following statement which is a result of [4].

Corollary 4.7. *If $K \in \mathcal{S}_o^n$ and $0 \leq j < n-1$, then*

$$\omega_{n-1}^2 I_j^o K \subset I_j K^o \tag{4.10}$$

with equality if and only if K is a centered ball.

By using Theorem 4.6, we obtain the following theorem.

Theorem 4.8. *If $K_1,\ldots,K_{n-1} \in \mathcal{S}_o^n$, then*

(i)

$$V(I(K_1,\ldots,K_{n-1}))\,V(I(K_1^o,\ldots,K_{n-1}^o)) \geq \omega_n^2 \omega_{n-1}^{2n}, \tag{4.11}$$

(ii)

$$V(K_1)\cdots V(K_{n-1})\,V(I(K_1^o,\ldots,K_{n-1}^o)) \geq (\omega_n \omega_{n-1})^n \tag{4.12}$$

with equality in each inequality if and only if $K_1,\ldots,K_{n-1}$ are dilates of centered ball.

Proof. (i) From inequality (4.8) and inequality (3.9), we have

$$\begin{aligned} &V(I(K_1,\ldots,K_{n-1}))\,V(I(K_1^o,\ldots,K_{n-1}^o)) \\ &\quad \geq \omega_{n-1}^{2n} V(I(K_1,\ldots,K_{n-1}))\,V(I(K_1^o,\ldots,K_{n-1}^o)) \geq \omega_n^2 \omega_{n-1}^{2n}. \end{aligned} \tag{4.13}$$

According to the equality conditions of inequality (4.8) and inequality (3.9), equality in inequality (4.11) holds if and only if K_i are dilates of centered balls.

(ii) From inequality (4.1) and inequality (4.11), we get

$$\begin{aligned} \omega_n^2 \omega_{n-1}^{2n} &\leq V(I(K_1,\ldots,K_{n-1}))\,V(I(K_1^o,\ldots,K_{n-1}^o)) \\ &\leq \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K_1^o)\cdots V(K_{n-1}^o)\,V(I(K_1,\ldots,K_{n-1})). \end{aligned} \tag{4.14}$$

Therefore, we obtain inequality (4.12).

According to the equality conditions of inequality (4.1) and inequality (4.11), equality in inequality (4.12) holds if and only if K_i are dilates of centered balls. □

Proof of Theorem 1.3. From definition (2.6), equality (3.3), equality (2.11), Minkowski integral inequality [18], and definition (2.6) again, it follows that for $0 \le i < n$,

$$\begin{aligned}
\widetilde{W}_i(I^o(K\check{+}L))^{-1/(n-i)} &= \left(\frac{1}{n}\int_{S^{n-1}} \rho(I^o(K\check{+}L),u)^{n-i}dS(u)\right)^{-1/(n-i)} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} \rho(I(K\widetilde{+}L),u)^{-(n-i)}dS(u)\right)^{-1/(n-i)} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} (\rho(IK,u)+\rho(IK,u))^{-(n-i)}dS(u)\right)^{-1/(n-i)} \\
&\ge \left(\frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^{-(n-i)}dS(u)\right)^{-1/(n-i)} \\
&\quad + \left(\frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^{-(n-i)}dS(u)\right)^{-1/(n-i)} \\
&= \widetilde{W}_i(I^oK)^{-1/(n-i)} + \widetilde{W}_i(I^oL)^{-1/(n-i)}.
\end{aligned} \tag{4.15}$$

According to the equality conditions of Minkowski inequality, equality in inequality (1.5) holds if and only if K and L are dilates. For $i > n$, inequality (1.3) is reversed. □

5. Mixed intersection integrals

For star bodies $K_1,\dots,K_n$ in $\mathbb{R}^n$ and a fixed integer r with $0 \le r < n$, we define the rth mixed intersection integral of $K_1,\dots,K_n$ by

$$J_r(K_1,\dots,K_n) = \frac{\omega^{n-r-2}}{n\omega_{n-1}^n}\int_{S^{n-1}} \widetilde{w}_r(K_1\cap u^\perp)\cdots\widetilde{w}_r(K_n\cap u^\perp)dS(u). \tag{5.1}$$

For $K_1 = \cdots = K_n = B$, a trivial computation shows that

$$J_r(B,\dots,B) = \omega_n^{n-r-1}. \tag{5.2}$$

The following lemma will be used later.

Lemma 5.1. *$K,L \in \mathcal{S}_o^n$ and $0 \le r < n-1$, then*

$$V(I_r^oK) = \frac{1}{n}\int_{S^{n-1}} \widetilde{w}_r(K\cap u^\perp)^{-n}dS(u). \tag{5.3}$$

From definition (2.4), definition (2.9), Lemma 5.1 easily follows.

For $0 \leq r < n-1$, apply the Hölder inequality, use equality (5.3), and we obtain

$$\left(\frac{1}{n}\int_{S^{n-1}}\left(\widetilde{w}_r(K_1\cap u^{\perp})\cdots\widetilde{w}_r(K_n\cap u^{\perp})\right)^{-1}dS(u)\right)^n \leq V(I_r^o K_1)\cdots V(I_r^o K_n) \tag{5.4}$$

with equality if and only if (for all i, j) the $(n-1-r)$-sections of K_i and K_j are proportional.

From Jensen's inequality [18], we get

$$n\omega_n^{n-r} \leq \omega_{n-1}^n J_r(K_1,\ldots,K_n)\frac{1}{n}\int_{S^{n-1}}\left(\widetilde{w}_r(K_1\cap u^{\perp})\cdots\widetilde{w}_r(K_n\cap u^{\perp})\right)^{-1}dS(u). \tag{5.5}$$

If we combine inequalities (4.2), (4.10), (5.4), and (5.5), we obtain the following.

Theorem 5.2. *If $K_1,\ldots,K_{n-1}$ are star bodies in $\mathbb{R}^n$ and $0 \leq r < n-1$, then*

$$\omega_n^{2n(n-r-1)} \leq J_r(K_1,\ldots,K_n)^n\left(V(K_1^o,\ldots,K_n^o)\right)^{n-r-1} \tag{5.6}$$

with equality if and only if $K_1,\ldots,K_n$ are centered balls.

Theorem 5.2 is just an analogue of the geometric inequality between the volumes of convex bodies and their associated mixed projection integrals which was given by Lutwak [12].

Acknowledgments

This work was supported in part by National Natural Sciences Foundation of China (10671117). The authors wish to thank the referees for their very helpful comments and suggestions on the original version of this paper.

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Lu Fenghong: Department of Mathematics, Shanghai University, Shanghai 200444, China;
Department of Mathematics, Shanghai University of Electric Power, Shanghai 200090, China
Email address: lulufh@163.com

Mao Weihong: Department of Mathematics, Jiangsu University, Jiangsu 212013, China
Email address: maoweihong1029@sina.com

Leng Gangsong: Department of Mathematics, Shanghai University, Shanghai 200444, China
Email address: gleng@staff.shu.edu.cn