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Research Article

Equivalent Solutions of Nonlinear Equations in a Topological Vector Space with a Wedge

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We obtain efficient conditions under which some or all solutions of a nonlinear equation in a topological vector space preordered by a closed wedge are comparable with respect to the corresponding preordering. Conditions sufficient for the equivalence of comparable solutions are also given. The wedge under consideration is not assumed to be a cone, nor any continuity conditions are imposed on the mappings considered.

1. Introduction

The aim of this note is to establish certain order-theoretical properties of the set of solutions of the equation

$$\lambda x = T(x) + b, \tag{1.1}$$

where $T : X \to X$ is a (generally speaking, nonlinear and discontinuous) operator in a real topological vector space X, λ is a real constant, and b is a given element of X.

The question on the set of all the admissible values of x and λ in (1.1) is sometimes referred to as a nonlinear eigenvalue problem [1]. The present paper is motivated by some results obtained in [2] and, recently, in [3–5]. In the case where the wedge in question is a normal cone, we arrive at statements similar to the uniqueness results from [6]. Note that the algebraic conditions used here, generally speaking, do not guarantee the solvability of (1.1). The study of the question on the existence of a solution, which is not treated in this paper, depends upon further assumptions expressing a certain closer interplay between the partial ordering and the topology in X. For some efficient conditions sufficient for the solvability of abstract second-kind equations of type (1.1) in a Banach space with a normal cone, we refer, for example, to [2].

2. Definitions, notation, and auxiliary statements

Let X be a topological vector space over the field $\mathbb{R}$ (of course, $X \neq \{0\}$). Throughout the paper, we assume that the space X is equipped with the preordering $\leqq_K$ generated by a certain wedge K. According to this preordering, elements x_1 and x_2, by definition, satisfy the relation $x_1 \leqq_K x_2$ if and only if $x_2 - x_1 \in K$ (this fact will also be expressed alternatively as $x_2 \geqq_K x_1$ in the sequel). Recall that, by a *wedge* [7], or a *linear semigroup* [8] in a topological vector space X, a closed set $K \subset X$ is meant such that $\alpha_1 x_1 + \alpha_2 x_2 \in K$ for arbitrary $\{\alpha_1, \alpha_2\} \subset [0, +\infty)$ and $\{x_1, x_2\} \subset K$. It should be noted that the fulfilment of both of the relations $x_1 \leqq_K x_2$ and $x_2 \leqq_K x_1$, generally speaking, does not imply that x_1 and x_2 should coincide with one another. A wedge K is said to be *proper* if $K \neq \{0\}$ and $K \neq X$.

The linear manifold

$$K \cap (-K) =: K^{\diamond} \tag{2.1}$$

consisting of the elements x that satisfy each of the relations $x \geqq_K 0$ and $x \leqq_K 0$ is called the *blade* [7] of the wedge K (here, as usual, $\alpha K := \{\alpha x \mid x \in K\}$ for all real α). The most extensively used class of wedges is constituted by the *cones* [8, 9], that is, the wedges whose blade is trivial.

Definition 2.1. Two elements x_1 and x_2 from X are said to be *K-comparable* if at least one of the relations $x_1 \leqq_K x_2$ and $x_2 \leqq_K x_1$ is satisfied.

The property of x_1 and x_2 being K-comparable will be designated in the sequel by the symbol $\lesseqqgtr_K$: $x_1 \lesseqqgtr_K x_2$. It is clear that $x_1 \lesseqqgtr_K x_2$ means the same as the relation $x_2 \lesseqqgtr_K x_1$. Elements x_1 and x_2 such that $x_2 \not\lesseqqgtr_K x_1$ will be referred to as *K-incomparable.*

In the general case, the relation $x_1 \lesseqqgtr_K x_2$ is satisfied not for all pairs of elements $(x_1, x_2) \in X^2$.

Definition 2.2. The relation $x_1 \backsimeq_K x_2$ holds for two elements x_1 and x_2 of X if and only if $x_1 - x_2 \in K^{\diamond}$.

Clearly, $\backsimeq_K$ is an equivalence relation in X for an arbitrary choice of the wedge K. Two elements x_1 and x_2 satisfying the relation $x_1 \backsimeq_K x_2$ will be referred to as *K-equivalent*. The elements x from X satisfying the relation $x \backsimeq_K 0$ (i.e., those belonging to the set $K^{\diamond}$) will be called *K-negligible.*

Example 2.3. If $X = l_\infty$, the space of bounded real sequences with the usual topology, and K is the wedge defined by the formula

$$K = \{x : \mathbb{N} \longrightarrow \mathbb{R} \mid x \in l_\infty,\, x(k) \geq 0 \ \forall k \in S\} \tag{2.2}$$

with some nonempty set $S \subseteq \mathbb{N}$, then an element x is K-negligible if and only if the equality $x(k) = 0$ is satisfied for all k from S.

The lemma below states some simple properties of the symmetric two-sided inequalities that are often referred to in the sequel.

Lemma 2.4. *Elements x and u from X satisfy the relation*

$$-u \leqq_K x \leqq_K u \tag{2.3}$$

if, and only if

$$-u \leqq_K -x \leqq_K u. \tag{2.4}$$

If relation (2.3) holds for x and u from X, then necessarily $u \geqq_K 0$. If, in addition, $x \not\backsimeq_K 0$, then the relation $u \not\leqq_K 0$ is also satisfied.

Proof. The equivalence of (2.3) and (2.4) is obvious. If x and u satisfy (2.3), then, combining (2.3) and (2.4), we obtain

$$-2u \leqq_K 0 \leqq_K 2u, \tag{2.5}$$

that is, $u \geqq_K 0$. If, moreover, u satisfies the relation $u \backsimeq_K 0$, then inequality (2.3) implies immediately that $0 \leqq_K x \leqq_K 0$, that is, $x \backsimeq_K 0$. □

The following definition, which is a modified version of one introduced in [3], provides a kind of the strict inequality in X.

Definition 2.5. Let H be a linear manifold in the space X. Two elements f_1 and f_2 of X are in the relation

$$f_1 \succ_{K;H} f_2 \tag{2.6}$$

if, for an arbitrary x from H, one can specify a nonnegative real constant β such that

$$-\beta(f_1 - f_2) \leqq_K x \leqq_K \beta(f_1 - f_2). \tag{2.7}$$

In the case where the linear manifold H coincides with the entire space X, the subscript "X" in the expression "$\succ_{K;X}$" will be omitted. Thus, the following definition is introduced.

Definition 2.6. One says that

$$f_1 \succ_K f_2 \tag{2.8}$$

if and only if, for an arbitrary x from X, relation (2.7) is true with some $\beta \in [0, +\infty)$.

In other words, the elements f_1 and f_2 satisfy relation (2.8) whenever (2.6) is true for an arbitrary H.

Proposition 2.7. *If some elements f_1 and f_2 from X satisfy relation (2.6) for a certain linear manifold H such that*

$$H \not\subseteq K^{\diamond}, \tag{2.9}$$

then the relations

$$f_1 \geqq_K f_2, \qquad f_1 \nleqq_K f_2 \tag{2.10}$$

are necessarily satisfied. In the case where condition (2.9) is not satisfied, an arbitrary pair of elements $(f_1, f_2) \in X^2$ *possesses property (2.6).*

Proof. Indeed, according to Definition 2.5, relation (2.6) means that every element x from H satisfies condition (2.7) with a certain constant $\beta \geq 0$. Amidst such x, in view of assumption (2.9), there are some that are not K-negligible, that is,

$$x \not\trianglelefteq_K 0. \tag{2.11}$$

For x satisfying (2.11), the constant β in (2.7) cannot be equal to zero, and therefore Lemma 2.4 implies relations (2.10).

Condition (2.9) is violated if and only if every element from H is K-negligible. Therefore, with $\beta = 0$, relation (2.7) is satisfied in this case for an arbitrary x from H and every f_1, f_2 from X. According to Definition 2.5, this means that (2.6) is true independently of f_1 and f_2. □

As follows from Proposition 2.7, assumption (2.9) allows one to interpret the property

$$f \succ_{K;H} 0 \tag{2.12}$$

as a kind of the strong positivity of an element f. Condition (2.9) is thus quite natural, because in the case where it is violated, all the elements of X prove to be "strongly positive," which circumstance makes the notion useless.

Definition 2.8 [8]. Two elements f_1 and f_2 are said to satisfy the relation $f_1 \gg_K f_2$ if the difference $f_1 - f_2$ is an interior element of the wedge K.

It is well known [8] that if elements f_1 and f_2 satisfy the condition $f_1 \gg_K f_2$, then relation (2.8) is true. The converse statement, generally speaking, is not true (see Example 2.9). Of course, the notion described by Definition 2.8 makes sense only if K has nonempty interior (i.e., is solid [8]).

Example 2.9. In the Banach space $L_\infty[0,1]$ of measurable and essentially bounded scalar functions on the interval $[0,1]$ with the cone K of functions that are nonnegative almost everywhere on $[0,1]$, the corresponding relation $f \succ_K 0$ is satisfied, for example, for the positive-valued constant functions. However, the interior of the abovementioned cone in $L_\infty[0,1]$ is empty.

The proofs of the results of this paper rely upon properties of a certain nonlinear functional associated with the wedge K and a certain suitably chosen element f from X.

Definition 2.10. Given some elements f and x, put

$$n_{K,f}(x) := \inf\{\beta \mid \beta \in [0,+\infty) \text{ is such that } -\beta f \leqq_K x \leqq_K \beta f\} \tag{2.13}$$

if the set in the curly braces is nonempty, and put formally $n_{K,f}(x) := +\infty$ in the contrary case.

Thus, a mapping $n_{K,f} : X \to [0,+\infty]$ is associated with an arbitrary f from X. Besides the properties of this mapping stated in Lemma 2.12 below, we note the equality

$$n_{K,f}(-x) = n_{K,f}(x), \tag{2.14}$$

which is true for any $x \in X$ because the corresponding sets in the right-hand side of (2.13) coincide with one another.

Remark 2.11. In the case where X is a Banach space, K is a solid wedge, and $f \gg_K 0$, the functional determined by formula (2.13) was considered in [8]. Functionals of this kind are quite often used in the literature (see, e.g., [7, 9–12]).

For a suitable f, there is a close interplay between $K^\diamond$ and the set of zeroes of the mapping $n_{K,f} : X \to [0,+\infty]$.

Lemma 2.12. *Let f be an element satisfying relation (2.12) with respect to a certain linear manifold $H \subseteq X$ possessing property (2.9). Then,*

(i) $n_{K,f}(x) = 0$ *if and only if* $x \backsimeq_K 0$.

(ii) *For all* $x \in H \cup K^\diamond$, *the relation*

$$n_{K,f}(x) < +\infty \tag{2.15}$$

is true.

(iii) *If* $x \in X$ *satisfies (2.15), then the relation*

$$-n_{K,f}(x) f \leqq_K x \leqq_K n_{K,f}(x) f \tag{2.16}$$

holds.

Proof. Assertion (i) is established in the same manner as [3, Lemma 2.13] is in the case of a Banach space X. Indeed, if

$$x \backsimeq_K 0, \tag{2.17}$$

then the relation

$$-\beta f \leqq_K x \leqq_K \beta f \tag{2.18}$$

is satisfied with $\beta = 0$, and hence by (2.13), we have

$$n_{K,f}(x) = 0. \tag{2.19}$$

Conversely, let x be an element from X such that equality (2.19) holds. According to Definition 2.10, there exists a sequence $\{\beta_m \mid m \in \mathbb{N}\} \subset [0,+\infty)$ such that

$$\lim_{m \to +\infty} \beta_m = 0, \tag{2.20}$$

$$-\beta_m f \leqq_K x \leqq_K \beta_m f \tag{2.21}$$

for all $m \in \mathbb{N}$. In view of (2.20), $\lim_{m\to+\infty} \beta_m f = 0$ in the topology of X. Since (2.21) can be rewritten in the form of the inclusion

$$\{\beta_m f - x,\ \beta_m f + x\} \subset K, \tag{2.22}$$

taking into account the closedness of K and passing to the limit as $m \to +\infty$ in (2.22), we conclude that $\{-x, x\} \subset K$, that is, relation (2.17) holds.

To obtain assertion (ii), we note that, firstly, the relation $n_{K,f}|_{K^\diamond} = 0$ is true and, secondly, condition (2.12) guarantees the nonemptiness of the set

$$\{\beta \in [0, +\infty) \mid (2.18) \text{ is satisfied}\} \tag{2.23}$$

for an arbitrary x from H. Therefore, the value $n_{K,f}(x)$ is finite for all x belonging to $H \cup K^\diamond$.

Let us now establish assertion (iii). Assume that $x \in X$ possesses property (2.15). According to Definition 2.10, we have

$$n_{K,f}(x) = \lim_{m\to+\infty} \beta_m \tag{2.24}$$

with some sequence $\{\beta_m \mid m \in \mathbb{N}\} \subset [0, +\infty)$ such that (2.21) holds for all $m \in \mathbb{N}$. Passing to the limit as $m \to +\infty$ in (2.21) and using (2.24), we arrive at (2.16). □

In view of Proposition 2.7, there is no much sense to consider relations of type (2.12) with respect to the linear manifold H for which condition (2.9) is not satisfied. This fact explains the presence of assumption (2.9) in Lemma 2.12 and its absence from the formulations of the results of Sections 3 and 4 (see Remark 3.2).

Remark 2.13. The fulfilment of assumption (2.12) in Lemma 2.12 implies, in particular, that the element f satisfies the relations $f \geqq_K 0$ and $f \not\leqq_K 0$.

In the statements established in Sections 3 and 4, certain conditions generalizing the property of linearity of a mapping are used. The corresponding notions are introduced by Definitions 2.14 and 2.16 given below. Note that other similar notions of subadditivity, superadditivity, convexity, and concavity for operators in various partially ordered spaces and their algebraic properties are treated in [9, 13–16].

Definition 2.14. An operator $A : X \to X$ is said to be *positively homogeneous* on a set $S \subseteq X$ if the relation

$$A(\alpha u) = \alpha A(u) \tag{2.25}$$

is satisfied for arbitrary $u \in S$ and $\alpha \in [0, +\infty)$.

Remark 2.15. It is clear that every mapping $A : X \to X$ which is continuous in a neighbourhood of 0 and positively homogeneous on a nonempty set possesses the property $A(0) = 0$.

Definition 2.16. The operator $A : X \to X$ is *K-superadditive* on a set $S \subseteq X$ if the relation

$$A(u_1) + A(u_2) \leqq_K A(u_1 + u_2) \tag{2.26}$$

is true for all u_1 and u_2 from S. Similarly, an operator $A : X \to X$ will be called K-*subadditive* on a set $S \subseteq X$ if

$$A(u_1 + u_2) \leqq_K A(u_1) + A(u_2) \tag{2.27}$$

for all u_1 and u_2 from S.

In the case where relation (2.26) (resp., (2.27)) is satisfied on the entire space X, one will speak simply on the K-*superadditivity* (resp., K-*subadditivity*) of the operator A.

Every linear operator in X is of course positively homogeneous and both K-superadditive and K-subadditive with respect to an arbitrary wedge $K \subseteq X$. A characteristic example of a pair of nonlinear operators possessing the properties indicated is provided by the positive and negative parts of a function.

Example 2.17. Let $X := C([0,1],\mathbb{R})$ be the space of the continuous scalar-valued functions on the interval $[0,1]$, let $K := C([0,1],\mathbb{R}_+)$ be the cone of nonnegative functions from $C([0,1],\mathbb{R})$, and let $A_\epsilon : X \to X$, where $\epsilon \in \{-1,1\}$, be the operator defined by the formula

$$(A_\epsilon x)(t) := \epsilon \max\{\epsilon x(t), 0\}, \quad t \in [0,1]. \tag{2.28}$$

Then, A_ϵ is K-subadditive (resp., K-superadditive) on the entire space X if $\epsilon = 1$ (resp., $\epsilon = -1$). In both cases, operator (2.28) is positively homogeneous.

In some cases, the K-superadditivy and K-subadditivity conditions are satisfied simultaneously without implying the linearity of the mapping.

Example 2.18. The operator $A : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by formula

$$(Ax)(t) := \int_0^1 p(t,s)\big(x(s) + |x(s)|\big)ds - \big(|x(t)| - x(t)\big)^\gamma, \quad t \in [0,1], \tag{2.29}$$

where $\gamma \in (0,+\infty)$, $p(t,\cdot) \in L_1([0,1],\mathbb{R})$ for all $t \in [0,1]$, and $p(\cdot,s) \in C([0,1],\mathbb{R})$ for a.e. $s \in [0,1]$, is positively homogeneous and both K-superadditive and K-subadditive on the cone $K := C([0,1],\mathbb{R}_+)$. Note that operator (2.29) is nonlinear.

3. Mutual comparability of solutions of (1.1)

The aim of this section is to establish certain conditions under which each two solutions of (1.1) lying in a certain linear manifold are K-comparable with one another.

3.1. Main theorems. The theorem below claims that, under fairly general assumptions, a certain two-sided condition imposed on the nonlinear mapping T guarantees the mutual comparability of some or all solutions of (1.1) for $|\lambda|$ large enough.

Theorem 3.1. *Assume that, for the mapping $T : X \to X$, there exist a linear manifold $\Pi \subseteq X$ and an operator $A : X \to X$ which is positively homogeneous and K-subadditive on the set Π and satisfies the condition*

$$A(y_2 - y_1) \leqq_K T(y_1) - T(y_2) \leqq_K A(y_1 - y_2) \tag{3.1}$$

for arbitrary $\{y_1, y_2\} \subset \Pi$ *such that* $y_1 \geqq_K y_2$ *and* $y_1 \ngeqq_K y_2$. *Let, moreover, the relation*

$$A(f) \leqq_K \alpha f \tag{3.2}$$

be true with some $\alpha \in [0, +\infty)$ *and* $f \in \Pi$ *for which (2.12) holds, where* $H \subseteq X$ *is a certain linear manifold satisfying the inclusion*

$$H \supseteq T(\Pi). \tag{3.3}$$

Then, for an arbitrary real λ *satisfying the estimate*

$$|\lambda| > \alpha, \tag{3.4}$$

and an arbitrary element $b \in X$*, all the solutions of (1.1) belonging to the set* Π *are* K*-comparable to one another.*

In (3.3) and similar relations, the symbol $T(M)$ stands for the image of a set M under the mapping T. Prior to the proof of Theorem 3.1, we give some comments on the choice of the linear manifold H appearing in relation (3.3).

Remark 3.2. Let the mapping $T : X \to X$ satisfy relation (3.3) with some linear manifolds $H \subseteq X$ and $\Pi \subseteq X$. If condition (2.9) does not hold, then for any nonzero λ, all the solutions of (1.1) that belong to Π are K-equivalent to one another. Indeed, any two solutions $\{x_1, x_2\} \subset \Pi$ of (1.1) obviously satisfy the relation

$$\lambda(x_1 - x_2) = T(x_1) - T(x_2), \tag{3.5}$$

and therefore the difference $x_1 - x_2$ belongs to H because $\lambda \neq 0$. If (2.9) does not hold, then $H \subseteq K^{\diamond}$, and hence $x_1 - x_2 \simeq_K 0$.

The consideration above shows that the assertions of the statements of Sections 3 and 4 involving the linear manifold H become trivial when (2.9) is violated, and we thus do not deal with this case in the proofs.

Proof of Theorem 3.1. In view of Remark 3.2, it will suffice to consider the case where the linear manifold H satisfies condition (2.9).

Let x_1 and x_2 be two distinct solutions of (1.1) lying in the set Π. Then (3.5) is true. Condition (3.3) and the linearity of the set H guarantee that

$$T(\Pi) - T(\Pi) \subseteq H, \tag{3.6}$$

and hence relation (3.5) yields $\lambda(x_1 - x_2) \in H$. In view of estimate (3.4), λ is nonzero, and therefore, again by the linearity of H, the last relation implies that

$$x_1 - x_2 \in H. \tag{3.7}$$

Relations (2.12), (3.7), property (2.9) of the linear manifold H, and assertions (ii) and (iii) of Lemma 2.12 then guarantee that the value $n_{K,f}(x_1 - x_2)$ is a finite number, and the order inequality

$$-n_{K,f}(x_1 - x_2) f \leqq_K x_1 - x_2 \leqq_K n_{K,f}(x_1 - x_2) f \tag{3.8}$$

is true.

We need to prove the mutual K-comparability of the solutions x_1 and x_2. Assume that, on the contrary, x_1 and x_2 are K-incomparable, that is, the relation

$$x_1 \not\lesseqgtr_K x_2 \tag{3.9}$$

holds.

It is easy to verify that the relations [6, the proof of Theorem 49.3]

$$x_1 \geqq_K \frac{1}{2}(x_1 + x_2 - u), \tag{3.10}$$

$$x_2 \geqq_K \frac{1}{2}(x_1 + x_2 - u) \tag{3.11}$$

are true for an arbitrary u satisfying the inequality

$$-u \leqq_K x_1 - x_2 \leqq_K u. \tag{3.12}$$

In view of (3.8), inequalities (3.10) and (3.11) are satisfied, in particular, with

$$u := n_{K,f}(x_1 - x_2) f. \tag{3.13}$$

Assumption (3.9) ensures that

$$x_1 \not\leqq_K \frac{1}{2}(x_1 + x_2 - u). \tag{3.14}$$

Indeed, in the contrary case, we have

$$x_1 \leqq_K \frac{1}{2}(x_1 + x_2 - u), \tag{3.15}$$

which relation, in view of (3.11), implies that $x_1 \leqq_K x_2$, contrary to assumption (3.9). Similarly, assuming that

$$x_2 \leqq_K \frac{1}{2}(x_1 + x_2 - u) \tag{3.16}$$

and using (3.10), we conclude that $x_2 \leqq_K x_1$, which contradicts (3.9). Therefore, in addition to (3.14), the relation

$$x_2 \not\leqq_K \frac{1}{2}(x_1 + x_2 - u) \tag{3.17}$$

is true.

Let us put

$$y_1 := x_1, \qquad y_2 := \frac{1}{2}(x_1 + x_2 - u) \tag{3.18}$$

with u given by formula (3.13). Then

$$y_1 \geqq_K y_2, \qquad y_1 \nleqq_K y_2 \tag{3.19}$$

because relations (3.10) and (3.14) are satisfied.

In addition, both y_1 and y_2 lie in Π because, by assumption, the set mentioned is a linear manifold containing the element f. Therefore, in view of the assumption (3.1) and the equalities

$$y_1 - y_2 = \frac{1}{2}(x_1 - x_2 + u), \qquad y_2 - y_1 = \frac{1}{2}(x_2 - x_1 - u), \tag{3.20}$$

we have

$$A\left(\frac{1}{2}(x_2 - x_1 - u)\right) \leqq_K T(x_1) - T\left(\frac{1}{2}(x_1 + x_2 - u)\right) \leqq_K A\left(\frac{1}{2}(x_1 - x_2 + u)\right). \tag{3.21}$$

Similarly, by putting

$$y_1 := x_2, \qquad y_2 := \frac{1}{2}(x_1 + x_2 - u), \tag{3.22}$$

in view of relations (3.11) and (3.17), we get relation (3.19) and the inclusion $\{y_1, y_2\} \subset \Pi$. By virtue of condition (3.1), we obtain

$$A\left(\frac{1}{2}(x_1 - x_2 - u)\right) \leqq_K T(x_2) - T\left(\frac{1}{2}(x_1 + x_2 - u)\right) \leqq_K A\left(\frac{1}{2}(x_2 - x_1 + u)\right). \tag{3.23}$$

Using the positive homogenity and K-subadditivity of A on the set Π in relations (3.21) and (3.23), we obtain

$$\begin{gathered} T(x_1) - T\left(\frac{1}{2}(x_1 + x_2 - u)\right) \leqq_K \frac{1}{2}A(x_1 - x_2 + u) \leqq_K \frac{1}{2}A(x_1 - x_2) + \frac{1}{2}A(u), \\ -T(x_2) + T\left(\frac{1}{2}(x_1 + x_2 - u)\right) \leqq_K -\frac{1}{2}A(x_1 - x_2 - u) \leqq_K -\frac{1}{2}A(x_1 - x_2) + \frac{1}{2}A(u), \end{gathered} \tag{3.24}$$

that is,

$$T(x_1) - T(x_2) \leqq_K A(u). \tag{3.25}$$

Analogously, we get

$$T(x_2) - T\left(\frac{1}{2}(x_1 + x_2 - u)\right) \leqq_K \frac{1}{2}A(x_2 - x_1 + u) \leqq_K \frac{1}{2}A(x_2 - x_1) + \frac{1}{2}A(u),$$
$$-T(x_1) + T\left(\frac{1}{2}(x_1 + x_2 - u)\right) \leqq_K -\frac{1}{2}A(x_2 - x_1 - u) \leqq_K -\frac{1}{2}A(x_2 - x_1) + \frac{1}{2}A(u), \tag{3.26}$$

and thus

$$T(x_2) - T(x_1) \leqq_K A(u). \tag{3.27}$$

Now, relations (3.25) and (3.27) imply that

$$-A(u) \leqq_K T(x_1) - T(x_2) \leqq_K A(u) \tag{3.28}$$

or, which is the same,

$$-n_{K,f}(x_1 - x_2)A(f) \leqq_K T(x_1) - T(x_2) \leqq_K n_{K,f}(x_1 - x_2)A(f), \tag{3.29}$$

because u is given by formula (3.13) and the operator A is positively homogeneous.

The element f is assumed to satisfy condition (3.2), and therefore the last inequality yields

$$-\alpha n_{K,f}(x_1 - x_2) f \leqq_K T(x_1) - T(x_2) \leqq_K \alpha n_{K,f}(x_1 - x_2) f. \tag{3.30}$$

Taking (3.5) and (3.30) into account, we conclude that

$$-\alpha n_{K,f}(x_1 - x_2) f \leqq_K \lambda(x_1 - x_2) \leqq_K \alpha n_{K,f}(x_1 - x_2) f, \tag{3.31}$$

and hence by virtue of (3.4), the relation

$$-\frac{\alpha n_{K,f}(x_1 - x_2)}{|\lambda|} f \leqq_K x_1 - x_2 \leqq_K \frac{\alpha n_{K,f}(x_1 - x_2)}{|\lambda|} f \tag{3.32}$$

holds. However, according to Definition 2.10, the number $n_{K,f}(x_1 - x_2)$ is equal to the greatest lower bound of all those $\beta \in [0, +\infty)$ for which the relation

$$-\beta f \leqq_K x_1 - x_2 \leqq_K \beta f \tag{3.33}$$

is satisfied. Therefore, in view of relation (3.32), we have

$$0 \leq n_{K,f}(x_1 - x_2) \leq \frac{\alpha n_{K,f}(x_1 - x_2)}{|\lambda|}. \tag{3.34}$$

Since the constant λ is supposed to satisfy estimate (3.4), it follows from inequality (3.34) that

$$n_{K,f}(x_1 - x_2) = 0. \tag{3.35}$$

By virtue of assertion (i) of Lemma 2.12, equality (3.35) yields $x_1 \simeq_K x_2$. However, assumption (3.9) implies, in particular, that

$$x_1 \not\simeq_K x_2, \tag{3.36}$$

which leads us to a contradiction. Thus, we have shown that x_1 and x_2 satisfy the desired relation $x_1 \lesseqqgtr_K x_2$. □

Condition (3.1), as the following theorem shows, can also be assumed in the cases where the auxiliary operator A is not K-subadditive but K-superadditive.

Theorem 3.3. *Assume that, for the mapping $T : X \to X$, there exist a linear manifold $\Pi \subseteq X$ and an operator $A : X \to X$ which is positively homogeneous and K-superadditive on the set Π and satisfies condition (3.1) for arbitrary $\{y_1, y_2\} \subset \Pi$ such that $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$. Let, moreover, the relation*

$$A(-f) \geqq_K -\alpha f \tag{3.37}$$

be true with some $\alpha \in [0,+\infty)$ and $f \in \Pi$ for which (2.12) holds, where $H \subseteq X$ is a certain linear manifold possessing property (3.3).

Then, for an arbitrary real λ satisfying estimate (3.4) and an arbitrary element $b \in X$, all the solutions of (1.1) belonging to the set Π are K-comparable to one another.

Proof. It is easy to see that x is a solution of (1.1) if and only if the element $w := -x$ is a solution of the equation

$$\mu w = \hat{T}(w) + b, \tag{3.38}$$

where $\mu = -\lambda$, and the mappings $\hat{T} : X \to X$ is defined by the formula

$$\hat{T}(z) := T(-z), \quad z \in X. \tag{3.39}$$

Let $z_1, z_2 \in \Pi$ be such that

$$z_1 \geqq_K z_2, \qquad z_1 \nleqq_K z_2. \tag{3.40}$$

We will show that the relation

$$\hat{A}(z_2 - z_1) \leqq_K \hat{T}(z_1) - \hat{T}(z_2) \leqq_K \hat{A}(z_1 - z_2) \tag{3.41}$$

holds, where $\hat{A}$ is given by the formula

$$\hat{A}(z) := -A(-z), \quad z \in X. \tag{3.42}$$

Since Π is a linear manifold, together with z_1 and z_2, it contains the vectors $y_1 := -z_2$ and $y_2 := -z_1$. Note that

$$y_1 \geqq_K y_2, \qquad y_1 \nleqq_K y_2. \tag{3.43}$$

By assumption, T satisfies condition (3.1), and thus we have

$$A(y_2 - y_1) \leqq_K T(y_1) - T(y_2) \leqq_K A(y_1 - y_2), \tag{3.44}$$

that is,

$$-A(y_1 - y_2) \leqq_K T(y_2) - T(y_1) \leqq_K -A(y_2 - y_1). \tag{3.45}$$

Using (3.39) and (3.42), we can bring the last relation to form (3.41).

The operator $\widehat{A}$ is K-subadditive on the set Π. Indeed, if $u_1, u_2 \in \Pi$, then, by virtue of the K-superadditivity of A on the set Π, we have

$$\widehat{A}(u_1 + u_2) = -A(-u_1 - u_2) \leqq_K -A(-u_1) - A(-u_2) = \widehat{A}(u_1) + \widehat{A}(u_2). \tag{3.46}$$

Moreover, it is clear that the operator $\widehat{A}$ is also positively homogeneous on Π.

Since Π is a linear manifold, we have $-\Pi = \Pi$, and hence (3.39) yields

$$\widehat{T}(\Pi) = T(-\Pi) = T(\Pi). \tag{3.47}$$

Assumption (3.3) then implies that $\widehat{T}(\Pi) \subseteq H$.

Finally, relation (3.37), in view of (3.42), can be rewritten as

$$\widehat{A}(f) \leqq_K \alpha f. \tag{3.48}$$

Consequently, Theorem 3.1 can be applied to (3.38). □

Since every linear operator in X is of course positively homogeneous and both K-superadditive and K-subadditive, Theorems 3.1 and 3.3 immediately yield.

Corollary 3.4. *Assume that, for the given mapping $T : X \to X$, there exist a linear manifold $\Pi \subseteq X$ and a linear operator $A : X \to X$ such that the condition*

$$-A(y_1 - y_2) \leqq_K T(y_1) - T(y_2) \leqq_K A(y_1 - y_2) \tag{3.49}$$

holds for arbitrary $\{y_1, y_2\} \subset \Pi$ such that $y_1 \geqq_K y_2$ and $y_1 \not\leqq_K y_2$. Let, moreover, relation (3.2) be true with some $\alpha \in [0, +\infty)$ and $f \in \Pi$ for which (2.12) holds, where $H \subseteq X$ is a certain linear manifold possessing property (3.3).

Then, for an arbitrary real λ satisfying estimate (3.4) and an arbitrary element b from X, all the solutions of (1.1) that belong to the set Π are K-comparable to one another.

Remark 3.5. The assertion of Corollary 3.4 can also be proved in the case where A is only assumed to be positively homogeneous and K-superadditive on the wedge K. The resulting theorem is somewhat strange due to the fact that the K-superadditive operators themselves are not typical representatives of the class of mappings T satisfying the symmetric conditions of form (3.49). We do not dwell on this here in more detail.

Remark 3.6. We note that abstract Lipschitz-type conditions of form (3.49) in the case where X is a Banach space, $K \cap (-K) = \{0\}$, and $\Pi = X$ are used by some fixed point theorems (e.g., [6, Theorem 49.3] and [2, Theorem 2]). Certain assumptions on nonlinear functions arising in the theory of differential inequalities also have similar form (see, e.g., [17]).

Remark 3.7. Under the conditions assumed in Corollary 3.4, its assertion is also true for the equation

$$\lambda x = -T(x) + b. \tag{3.50}$$

This fact is an immediate consequence of Corollary 3.4 with λ replaced by $-\lambda$.

Now we will show that, for an operator $T : X \to X$ that is either K-subadditive or K-superadditive, the Lipschitz-type condition (3.1) is satisfied automatically provided a certain additional monotonicity condition is assumed. Therefore, in the cases indicated, the main role in the assumptions of the results obtained is played by conditions of the form (3.2) or (3.37).

Proposition 3.8. *Let the operator $T : X \to X$ be K-subadditive on a linear manifold $\Pi \subseteq X$ and let the relation*

$$T(\Pi \cap (-K)) \subseteq -K \tag{3.51}$$

be true. Then, for arbitrary $y_1, y_2 \in \Pi$ such that $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$, condition (3.1) is satisfied with $A = T$.

Proof. Since the operator T is K-subadditive, for any $y_1, y_2 \in \Pi$, we have

$$-T(y_2 - y_1) \leqq_K T(y_1) - T(y_2) \leqq_K T(y_1 - y_2). \tag{3.52}$$

Let $y_1, y_2 \in \Pi$ be such that $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$. Then $y_2 - y_1 \in \Pi \cap (-K)$ and thus, in view of (3.51), the relations

$$T(y_2 - y_1) \in -K, \qquad -T(y_2 - y_1) \in K \tag{3.53}$$

are true. Therefore, the left-hand side inequality of (3.52) yields

$$T(y_1) - T(y_2) \geqq_K -T(y_2 - y_1) \geqq_K 0 \geqq_K T(y_2 - y_1), \tag{3.54}$$

which, together with the inequality in the right-hand side of (3.57), guarantees the validity of (3.1) with $A = T$. □

Proposition 3.9. *Let the operator $T : X \to X$ be K-superadditive on a linear manifold $\Pi \subseteq X$ and let the relation*

$$T(\Pi \cap (-K)) \subseteq K \tag{3.55}$$

be true. Then, for arbitrary $y_1, y_2 \in \Pi$ such that $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$, condition (3.1) is satisfied with A given by the formula

$$A(z) = T(-z), \quad z \in X. \tag{3.56}$$

Proof. Since the operator T is K-superadditive, for any $y_1, y_2 \in \Pi$, we get the estimate

$$T(y_1 - y_2) \leqq_K T(y_1) - T(y_2) \leqq_K -T(y_2 - y_1). \tag{3.57}$$

Let $y_1, y_2 \in \Pi$ be such that $y_1 \geqq_K y_2$ and $y_1 \ngeqq_K y_2$. Then $y_2 - y_1 \in \Pi \cap (-K)$ and thus, in view of (3.51), the relations

$$T(y_2 - y_1) \in K, \qquad -T(y_2 - y_1) \in -K \tag{3.58}$$

are true. Therefore, the second inequality in (3.57) implies that

$$T(y_1) - T(y_2) \leqq_K -T(y_2 - y_1) \leqq_K 0 \leqq_K T(y_2 - y_1), \tag{3.59}$$

which, together with the first inequality in (3.57), yields estimate (3.1) with A given by (3.56). □

Example 3.10. Let $X := C([0,1], \mathbb{R})$ be the space of the continuous scalar-valued functions on the interval $[0,1]$, let $K := C([0,1], \mathbb{R}_+)$ be the cone of nonnegative functions from $C([0,1], \mathbb{R})$, and let $T_\epsilon : X \to X$, where $\epsilon \in \{-1, 1\}$, be the operator defined by the formula

$$(T_\epsilon x)(t) := \epsilon \int_0^1 p(t,s) \max\{x(\xi) : \tau_1(s) \leq \xi \leq \tau_2(s)\} ds, \quad t \in [0,1], \tag{3.60}$$

where $p(t,\cdot) \in L_1([0,1], \mathbb{R}_+)$ for all $t \in [0,1]$, $p(\cdot, s) \in C([0,1], \mathbb{R}_+)$ for a.e. $s \in [0,1]$, and $\tau_1, \tau_2 : [0,1] \to [0,1]$ are such that $\tau_1(t) \leq \tau_2(t)$ for a.e. $t \in [0,1]$.

Then the operator T_1 (resp., T_{-1}) is positively homogeneous, K-subadditive (resp., K-superadditive), and satisfies the condition

$$T_1(-K) \subseteq -K \qquad (\text{resp., } T_{-1}(-K) \subseteq K). \tag{3.61}$$

3.2. Simpler cases. The best possible choice of H in Theorems 3.1 and 3.3 is, clearly, the minimal linear manifold containing the image of Π under the mapping T. At the same time, the linear manifold Π, to which the solutions in question belong, should be as rich as possible, the case where Π coincides with the entire space X being the most desirable one. It is of course natural to assume that

$$\Pi \nsubseteq K^\diamond, \tag{3.62}$$

because otherwise the assertion of theorems becomes obvious. These considerations lead one to the following corollaries.

Corollary 3.11. *Assume that the mapping $T : X \to X$ satisfies condition (3.1) for arbitrary y_1 and y_2 from X possessing the properties $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$, where $A : X \to X$ is an operator which is positively homogeneous and K-subadditive (resp., K-superadditive). Let, moreover, relation (3.2) (resp., (3.37)) be true with some $\alpha \in [0, +\infty)$ and $f \in X$ satisfying the relation*

$$f \succ_{K; L(T(X))} 0, \tag{3.63}$$

where $L(T(X))$ denotes the minimal linear manifold containing $T(X)$.

Then, for all λ satisfying estimate (3.4) and all $b \in X$, any two solutions of (1.1) are K-comparable to one another.

Proof. It is sufficient to set $\Pi := X$ and $H := L(T(X))$ and apply Theorem 3.1 if A is K-subadditive, or Theorem 3.3 if A is K-superadditive. □

3.3. Equations with f-bounded operators. The condition

$$f \succ_K 0 \tag{3.64}$$

is the strongest one in the entire class of conditions of form (2.12). However, in certain cases where the restrictions of this kind can be removed completely.

Definition 3.12 [3]. Let K be a wedge in X and let f be an element from X. An operator $T : X \to X$ is said to be f-*bounded* along K on a set $\Pi \subseteq X$ if, for every $x \in \Pi$, there exists a constant $\beta \in [0,+\infty)$ such that

$$-\beta f \leqq_K T(x) \leqq_K \beta f. \tag{3.65}$$

In the case where $\Pi = X$, one will speak simply that T is f-bounded along K.

Remark 3.13. Definition 3.12 differs from that adopted, for example, in [7, 9]. More precisely, in [7, Section 9.4], an operator $A : X \to X$ in a Banach space X with a cone K is called f-bounded for some element $f \geqq_K 0$ if there exist some functions $\alpha : K \to (0,+\infty)$ and $\beta : K \to (0,+\infty)$ such that

$$\alpha(x) f \leqq_K A(x) \leqq_K \beta(x) f \tag{3.66}$$

for every $x \geqq_K 0$. It is, in general, not true that an operator f-bounded along K on the set K in the sense of Definition 3.12 should possess the property described above.

Example 3.14. Every operator $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is 1-bounded along the cone $C([0,1],\mathbb{R}_+)$ of nonnegative continuous functions (here, 1 stands for the constant function equal identically to 1 on $[0,1]$). The statement indicated in Remark 3.13 is true, in particular, for the operator in $C([0,1],\mathbb{R})$ given by the formula

$$(Tx)(t) := \int_0^1 p(t,s)(x(s))^q ds, \quad t \in [0,1], \tag{3.67}$$

where $q \in [1,+\infty)$, $p(t,\cdot) \in L_1([0,1],\mathbb{R})$ for all $t \in [0,1]$ and $p(\cdot,s) \in C([0,1],\mathbb{R})$ for a.e. $s \in [0,1]$. Operator (3.67) is not 1-bounded in the sense of [7] unless p is nonnegative.

According to Definition 3.12, the operator given by (3.67) is f-bounded along the cone $C([0,1],\mathbb{R}_+)$, with some $\tau \in [0,1]$ and the function $f : [0,1] \to \mathbb{R}$ defined by the formula

$$f(t) := |t - \tau|, \quad t \in [0,1], \tag{3.68}$$

if there exists a nonnegative constant γ such that

$$\int_0^1 |p(t,s)|\,ds \leq \gamma|t-\tau| \tag{3.69}$$

for all t from $[0,1]$.

In the case where the operator $T : X \to X$ in (1.1) possesses the property described by Definition 3.12, the following statements are true.

Corollary 3.15. *Let the mapping $T : X \to X$ be f-bounded along K on a linear manifold Π with a certain element $f \in \Pi \cap K$. Let, in addition, there exist an operator $A : X \to X$ which is positively homogeneous and K-subadditive (resp., K-superadditive) on the set Π, satisfies condition (3.2) (resp., (3.37)) with some $\alpha \in [0,+\infty)$, and, moreover, is such that relation (3.1) is true for arbitrary y_1 and y_2 from Π possessing the properties $y_1 \geqq_K y_2$ and $y_1 \not\leqq_K y_2$.*

Then, for all λ satisfying estimate (3.4) and all $b \in X$, any two solutions of (1.1) belonging to the set Π are K-comparable to one another.

Proof. Indeed, let us put

$$H := \{x \in X \mid \exists \beta \in [0,+\infty) : -\beta f \leqq_K x \leqq_K \beta f\}. \tag{3.70}$$

Set (3.70) is obviously a linear manifold in X. According to Definition 3.12, inclusion (3.3) is true with H given by (3.70). Moreover, recalling Definition 2.5, we see that, due to (3.70), relation (2.12) is satisfied for the element f. Thus, Theorem 3.1 (resp., Theorem 3.3) can be applied with H given by equality (3.70). □

In Corollary 3.15, it is of course natural to exclude the exceptional case where $f \not\approx_K 0$ because otherwise the corresponding assertion becomes trivial.

4. Absence of nonequivalent solutions

It turns out that imposing a natural additional restriction on the operator A in Theorems 3.1 and 3.3, one can prove that solutions of (1.1) are not only comparable but also equivalent to one another.

4.1. General theorems.

The following theorems are true.

Theorem 4.1. *Suppose that, in addition to the assumptions of Theorem 3.1, the relation*

$$A(\Pi \cap (-K)) \subseteq -K \tag{4.1}$$

is true. Then, all the solutions of (1.1) lying in Π are mutually K-equivalent.

Theorem 4.2. *Suppose that, in addition to the assumptions of Theorem 3.3, the relation*

$$A(\Pi \cap K) \subseteq K \tag{4.2}$$

is true. Then, all the solutions of (1.1) lying in Π are mutually K-equivalent.

Remark 4.3. It should be noted that, in the case where K is a cone (i.e., its blade $K^\diamond$ is zero dimensional), the K-equivalence of elements means their coincidence, and thus Theorems 4.1 and 4.2 guarantee that (1.1) has at most one solution. Note that the conditions presented above, generally speaking, do not guarantee the solvability of (1.1).

Proof of Theorem 4.1. Without loss of generality, we may suppose that relation (4.18) is true, for otherwise, in view of Theorem 3.1, the assertion of the theorem becomes obvious.

We first note that, in view of Remark 2.15 and the K-subaditivity of A on the set Π, we get $-A(x) \leqq_K A(-x)$ for $x \in \Pi$, and thus (4.1) implies the relation (4.2).

It follows from Theorem 3.1 that, under the conditions assumed, every two solutions x_1 and x_2 of (1.1) belonging to Π satisfy the relation

$$x_1 \lesseqqgtr_K x_2, \tag{4.3}$$

that is, at least one of the relations $x_1 \leqq_K x_2$ and

$$x_1 \geqq_K x_2 \tag{4.4}$$

is true. Suppose for definiteness that (4.4) holds. We need to prove the K-equivalence of x_1 and x_2. Assume that, on the contrary, (3.36) is true, and hence in view of (4.4), we have

$$x_1 \nleqq_K x_2. \tag{4.5}$$

Then estimate (3.1) yields

$$A(x_2 - x_1) \leqq_K \lambda(x_1 - x_2) \leqq_K A(x_1 - x_2). \tag{4.6}$$

Just as in the proof of Corollary 3.4, by using condition (3.3) and Lemma 2.12, one can show that the number $n_{K,f}(x_1 - x_2)$ is finite and relation (3.8) is satisfied for the difference $x_1 - x_2$. Since Π is a linear manifold and the relation (3.8) holds, it is clear that $x_1 - x_2 - n_{K,f}(x_1 - x_2)f \in \Pi \cap (-K)$. Therefore, using (4.1) and the K-subadditivity of A, we get

$$A(x_1 - x_2) \leqq_K A(x_1 - x_2 - n_{K,f}(x_1 - x_2)f) + A(n_{K,f}(x_1 - x_2)f) \leqq_K A(n_{K,f}(x_1 - x_2)f). \tag{4.7}$$

On the other hand, we have $x_2 - x_1 + n_{K,f}(x_1 - x_2)f \in \Pi \cap K$, and thus in view of (4.2) and the K-subadditivity of A, we obtain

$$-A(x_2 - x_1) \leqq_K A(x_2 - x_1 + n_{K,f}(x_1 - x_2)f) - A(x_2 - x_1) \leqq_K A(n_{K,f}(x_1 - x_2)f). \tag{4.8}$$

By virtue of (4.7) and (4.8), relation (4.6) implies that

$$-A(n_{K,f}(x_1 - x_2)f) \leqq_K \lambda(x_1 - x_2) \leqq_K A(n_{K,f}(x_1 - x_2)f). \tag{4.9}$$

Since the operator A is positively homogeneous on Π and satisfies relation (3.2), the last relation yields

$$-\alpha n_{K,f}(x_1 - x_2) f \leqq_K \lambda(x_1 - x_2) \leqq_K \alpha n_{K,f}(x_1 - x_2) f. \tag{4.10}$$

In view of condition (3.4), relation (4.10) yields (3.32), whence by virtue of inequality (3.8) and Definition 2.10, estimate (3.34) follows. Therefore, equality (3.35) is true, and by Lemma 2.12(i), we conclude that

$$x_1 \backsimeq_K x_2, \tag{4.11}$$

contrary to (4.5). □

Proof of Theorem 4.2. Let the operators $\widehat{T}, \widehat{A} : X \to X$ be defined by formulae (3.39) and (3.42), respectively. It is clear that x is a solution of (1.1) if and only if $w := -x$ is a solution of (3.38) with $\mu = -\lambda$. Just as in the proof of Theorem 3.3, one can show that, under the assumptions of this theorem, Theorem 3.1 can be applied to (3.38). Moreover, in view of (3.42), relation (4.2) implies that

$$\widehat{A}(\Pi \cap (-K)) \subseteq -K. \tag{4.12}$$

Consequently, we can apply Theorem 4.1 to (3.38), and thus in view of the equivalence mentioned above, the assertion of the theorem is proved. □

If the operator A appearing in Theorems 4.1 and 4.2 is linear, then the fulfilment of conditions (4.1) and (4.2) is guaranteed, for example, by assumptions (a) and (b) of Corollary 4.5 given below. For the sake of convenience in formulating the result, we introduce a definition.

Definition 4.4. A mapping $B : X \to X$ *preserves the K-negligibility* of elements of a set $\Pi \subseteq X$ if the relation $B(x) \backsimeq_K 0$ is satisfied for all x from Π such that $x \backsimeq_K 0$.

Corollary 4.5. *Suppose that, in addition to the assumptions of Corollary 3.4, one of the following two conditions is true:*

(a) *A preserves the K-negligibility of elements of the set Π;*

(b) *A is continuous in the topology of X.*

Then all the solutions of (1.1) lying in Π are K-equivalent to one another.

Prior to the proof of Corollary 4.5, we establish two lemmas. Since the assertion of the theorems is obvious in the exceptional case where condition (3.62) does not hold, till the end of this section, we assume implicitly that (3.62) is satisfied.

Lemma 4.6. *Let K be a wedge in X and let $\Pi \subseteq X$ be a linear manifold satisfying the condition*

$$\Pi \cap K^{\diamond} \subseteq \mathrm{Cl}(\Pi \cap K \setminus K^{\diamond}). \tag{4.13}$$

If $A : X \to X$ is a continuous operator such that

$$Ay \geqq_K 0 \quad \forall y \in \Pi \text{ such that } 0 \ngeqq_K y \geqq_K 0, \tag{4.14}$$

then

$$Ay \geqq_K 0 \quad \textit{for an arbitrary } y \in \Pi \textit{ such that } y \geqq_K 0. \tag{4.15}$$

Proof. Let w be an arbitrary element from $\Pi \cap K^\diamond$. In view of assumption (4.13), one can specify a sequence $\{v_m \mid m \in \mathbb{N}\} \subset \Pi$ satisfying the conditions

$$v_m \geqq_K 0, \quad v_m \not\leqq_K 0 \quad (\forall m \in \mathbb{N}) \tag{4.16}$$

and the relation

$$\lim_{m \to +\infty} v_m = w \tag{4.17}$$

in the topology of X. Assumption (4.14) guarantees that $Av_m \geqq_K 0$ for all $m = 1,2,\ldots$, whence in view of (4.17), we obtain that $Aw \geqq_K 0$ because the wedge K is closed and the mapping A is continuous. □

Lemma 4.7. *For any proper wedge K and any linear manifold Π such that the relation*

$$\Pi \cap K \not\subseteq K^\diamond \tag{4.18}$$

holds, condition (4.13) is satisfied.

Proof. Assume that, on the contrary, condition (4.13) is violated. Then there exists an element w such that $w \in \Pi \cap K^\diamond$ and

$$w \notin \mathrm{Cl}\,(\Pi \cap K \setminus K^\diamond). \tag{4.19}$$

Therefore,

$$0 \notin \mathrm{Cl}\,(\Pi \cap K \setminus K^\diamond). \tag{4.20}$$

Indeed, if (4.20) does not hold, then there exists a sequence $\{v_m \mid m \in \mathbb{N}\} \subset \Pi \cap K \setminus K^\diamond$ for which $\lim_{m\to+\infty} v_m = 0$. Since K is a wedge and Π and $K^\diamond$ are linear manifolds, it follows that $\{v_m + w \mid m \in \mathbb{N}\} \subset \Pi \cap K \setminus K^\diamond$. However, $\lim_{m\to+\infty}(v_m + w) = w$, contrary to (4.19). Thus, (4.20) is true.

Property (4.20) means the existence of a neighborhood $\mathbb{O}$ of zero such that the following implication is true:

$$(u \in K \cap \Pi \wedge u \in \mathbb{O}) \Longrightarrow u \in K^\diamond. \tag{4.21}$$

Let x be an arbitrary element from $\Pi \cap K$. Since the singleton $\{x\}$ is bounded in X, there exists some $\alpha \in (0,+\infty)$ such that $x \in \alpha\mathbb{O}$. However, this guarantees the existence of an element $x_0 \in \mathbb{O}$ for which the relation

$$x = \alpha x_0 \tag{4.22}$$

is satisfied. Hence, the element $x_0 = \alpha^{-1}x$ belongs to $\Pi \cap K$ because K is a wedge and Π is a linear manifold. Therefore, implication (4.21) yields $x_0 \in K^\diamond$. Since $K^\diamond$ is a linear manifold, it follows from relation (4.22) that $x \in K^\diamond$. We have thus established the inclusion

$$\Pi \cap K \subseteq K^{\diamond}, \tag{4.23}$$

which contradicts assumption (4.18). □

Now we are able to prove Corollary 4.5.

Proof of Corollary 4.5. Without loss of generality, we may suppose that relation (4.18) is true, for otherwise in view of Corollary 3.4, the assertion of the theorem becomes obvious.

It follows from assumption (3.49) and Lemma 2.4 that the operator A possesses property (4.14).

Assume condition (a). In this case, along with (4.14), the stronger condition (4.15) is satisfied, and thus relation (4.1) holds.

Let now assumption (b) be true. Then condition (4.14), in view of Lemmas 4.6 and 4.7, implies that the operator A, in fact, has the stronger property (4.15), and thus condition (4.1) holds as well.

Consequently, in both cases (a) and (b), all the assumptions of Theorem 4.1 are satisfied. □

4.2. Corollaries. The following corollaries allow one, in particular, to prove the uniqueness of a solution of certain boundary value problems for functional differential equations determined by subadditive and superadditive operators (see also [18–20] for some related results).

For the operators $T : X \to X$ that are K-subadditive or K-superadditive and satisfy certain monotonicity conditions, restrictions of type (3.1) are satisfied automatically (see Propositions 3.8 and 3.9). The main role in the assumptions of the results obtained here is thus played by conditions of the form

$$T(f) \leqq_K \alpha f \qquad (\text{resp., } -T(f) \leqq_K \alpha f) \tag{4.24}$$

assumed with a suitable element $f \geqq_K 0$. More precisely, the following statements hold.

Corollary 4.8. *Assume that $T : X \to X$ is a positively homogeneous mapping which is K-subadditive (resp., K-superadditive) on a certain linear manifold $\Pi \subseteq X$ and satisfies the condition*

$$T(\Pi \cap (-K)) \subseteq -K \quad (\textit{resp., } T(\Pi \cap (-K)) \subseteq K). \tag{4.25}$$

Let there exist some constant $\alpha \in [0, +\infty)$ and element $f \in \Pi$ for which the inequality (4.24) is satisfied, and moreover relation (2.12) holds with a certain linear manifold $H \subseteq X$ possessing property (3.3).

Then, for any $b \in X$ and any real λ satisfying estimate (3.4), all the solutions of (1.1) lying in Π are K-equivalent to one another.

Proof. One should apply Theorem 4.1 (resp., Theorem 4.2) and Proposition 3.8 (resp., Proposition 3.9). □

Corollary 4.9. *Let K, a proper wedge in X, $A: X \to X$, be a linear bounded operator, and let $T: X \to X$ be a mapping possessing property (3.49) for arbitrary elements y_1 and y_2 such that $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$. Assume, in addition, that A satisfies condition (3.2) with some $\alpha \in [0, +\infty)$ and $f \in X$ for which (3.64) is true, where $L(T(X))$ denotes the minimal linear manifold containing $T(X)$.*

Then, for any real λ satisfying estimate (3.4) and arbitrary element $b \in X$, all the solutions of (1.1) are K-equivalent to one another.

Proof. It is sufficient to set $H := X$ and $\Pi := X$ in Corollary 4.5 (case (b)). □

In the case where the mappings $T: X \to X$ determining (1.1) is f-bounded, in the sense of Definition 3.12, along a wedge K, we have the following.

Corollary 4.10. *Let the mapping $T: X \to X$ be f-bounded along K on a linear manifold Π with a certain element $f \in \Pi \cap K$. Let, in addition, there exist an operator $A: X \to X$ which is positively homogeneous and K-subadditive (resp., K-superadditive) on the set Π, satisfies (4.1) (resp., (4.2)) and condition (3.2) (resp., (3.37)) with some $\alpha \in [0, +\infty)$, and moreover, is such that relation (3.1) is true for arbitrary y_1 and y_2 from Π possessing the properties $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$.*

Then, for all λ satisfying estimate (3.4) and all $b \in X$, any two solutions of (1.1) belonging to the set Π are K-comparable to one another.

Proof. Analogously to the proof of Corollary 3.15, one should apply Theorem 4.1 (resp., Theorem 4.2) in the case where the linear manifold H is defined by formula (3.70). □

Corollary 4.11. *Let the mapping $T: X \to X$ be f-bounded along K on a linear manifold Π with a certain element $f \in \Pi \cap K$. Let, in addition, there exist a linear bounded operator $A: X \to X$ which satisfies condition (3.2) with some $\alpha \in [0, +\infty)$, and moreover is such that relation (3.49) is true for arbitrary y_1 and y_2 from Π possessing the properties $y_1 \geqq_K y_2$ and $y_1 \nleqq_K y_2$.*

Then, for all λ satisfying estimate (3.4) and all $b \in X$, any two solutions of (1.1) belonging to the set Π are K-equivalent to one another.

Proof. Just as in the proof of Corollary 3.15, one can show that the assumptions of Corollary 3.4 are satisfied for the linear manifold H defined by formula (3.70). Consequently, Corollary 4.5(b) can be applied. □

As it was said above (see Remark 4.3), in the case where K is a cone, Theorems 4.1, 4.2, and their corollaries guarantee that (1.1) has at most one solution. Note again that the conditions presented above, generally speaking, do not imply the solvability of (1.1). The existence of a solution is guaranteed by its uniqueness property, for instance, in the linear case dealt with in the following corollary.

Corollary 4.12. *Assume that X is a Banach space, let K be a cone in X, and $T: X \to X$ is a linear bounded operator which is Fredholm of index 0 and satisfies the condition*

$$T(K) \subseteq K \quad (\textit{resp., } T(K) \subseteq -K). \tag{4.26}$$

Let there exist some constant $\alpha \in [0,+\infty)$ and element $f \succ_K 0$ for which inequality (4.24) is true.

Then, for any $b \in X$ and any real λ satisfying estimate (3.4), (1.1) has a unique solution.

Proof. The result is an immediate consequence of Corollary 4.8 with $\Pi = X$ and $H = X$. Let us note that the validity of the corollary follows also from Theorem 4.5(b) and Remark 3.7. □

4.3. Example. As an example, we consider the Cauchy problem for the differential equation with a maximum

$$u'(t) = p(t)\max\{u(s) : \tau_1(t) \leq s \leq \tau_2(t)\} + q(t), \tag{4.27}$$

$$u(a) = c, \tag{4.28}$$

where $-\infty < a < b < \infty$, $c \in \mathbb{R}$, $\tau_1, \tau_2 : [a,b] \to [a,b]$ are measurable functions such that $\tau_1(t) \leq \tau_2(t)$ for a.e. $t \in [a,b]$, and $p, q \in L_1([a,b],\mathbb{R})$. By a solution of the problem (4.27), as usual, we mean an absolutely continuous function $u : [a,b] \to \mathbb{R}$ possessing property (4.28) and satisfying equality (4.27) almost everywhere on the interval $[a,b]$.

The following statement is true.

Corollary 4.13. *Let $p(t) \geq 0$ for a.e. $t \in [a,b]$ and*

$$\int_t^{\tau_2(t)} p(s)ds \leq \frac{1}{e} \quad \text{for a.e. } t \in [a,b]. \tag{4.29}$$

Then, for arbitrary $q \in L_1([a,b],\mathbb{R})$ and $c \in \mathbb{R}$, problem (4.27), (4.28) has at most one solution.

Remark 4.14. Note that some existence results for problem (4.27), (4.28) are established in [19]. It follows from [19, Theorem 1.3] that, under assumptions of Corollary 4.13, problem (4.27), (4.28) has at least one nonnegative solution for arbitrary $q \in L_1([a,b],\mathbb{R}_+)$ and $c \in \mathbb{R}_+$, and at least one nonpositive solution for arbitrary $q \in L_1([a,b],\mathbb{R}_-)$ and $c \in \mathbb{R}_-$.

Proof of Corollary 4.13. Let $X := C([a,b],\mathbb{R})$ be the space of the continuous scalar-valued functions on the interval $[a,b]$, let $K := C([0,1],\mathbb{R}_+)$ be the cone of nonnegative functions from $C([a,b],\mathbb{R})$, and let $T : X \to X$ be the operator defined by the formula

$$(Tx)(t) := \int_a^t p(s)\max\{x(\xi) : \tau_1(s) \leq \xi \leq \tau_2(s)\}ds, \quad t \in [a,b]. \tag{4.30}$$

It is obvious that the set of solutions of problem (4.27), (4.28) coincides with that of the continuous solutions of (1.1), where $\lambda = 1$ and

$$b(t) := c + \int_a^t q(s)ds, \quad t \in [a,b]. \tag{4.31}$$

Moreover, it is clear that the operator T is positively homogeneous, K-subadditive, and satisfies the condition $T(-K) \subseteq -K$. Now we put

$$f(t) := e^{e\int_a^t p(s)ds}, \quad t \in [a,b], \qquad \alpha := 1 - e^{-e\int_a^b p(s)ds}. \tag{4.32}$$

It is obvious that the element f satisfies condition (3.64) and that $\alpha \in [0,1)$. Let us show that the condition $T(f) \leqq_K \alpha f$ holds. Indeed, using assumption (4.29), we get

$$\begin{aligned}(Tf)(t) &= \int_a^t p(s)\max\{f(\xi) : \tau_1(s) \leq \xi \leq \tau_2(s)\}ds \\ &= \int_a^t p(s)e^{e\int_a^{\tau_2(s)} p(\xi)d\xi}ds \leq e\int_a^t p(s)e^{e\int_a^s p(\xi)d\xi}ds = e^{e\int_a^t p(s)ds} - 1 \\ &= f(t) - 1 \leq f(t)\left(1 - e^{-e\int_a^b p(s)ds}\right) = \alpha f(t), \quad t \in [a,b],\end{aligned} \tag{4.33}$$

and thus the relation desired is true.

Consequently, in order to establish Corollary 4.13, it will suffice to apply Corollary 4.8 with $\Pi := X$, $H := X$, and $\lambda = 1$. □

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