Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 48294, 6 pages
doi:10.1155/2007/48294

Research Article

On Subordination Result Associated with Certain Subclass of Analytic Functions Involving Salagean Operator

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Received 3 February 2007; Accepted 15 May 2007

Recommended by Narendra K. Govil

We obtain an interesting subordination relation for Salagean-type certain analytic functions by using subordination theorem.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \le \alpha < 1)$ the class of starlike functions of order α and the class of convex functions of order α, respectively, where

$$\begin{gathered}\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{U} \right\}, \\ \mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{U} \right\}.\end{gathered} \tag{1.2}$$

Note that $f(z) \in \mathcal{K}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha)$.

Sălăgean [1] has introduced the following operator:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ &\vdots \\ D^n f(z) &= D\big(D^{n-1} f(z)\big), \quad n \in \mathbb{N}_0 = \{0\} \cup \{1,2,\dots\}. \end{aligned} \tag{1.3}$$

We note that

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \quad \big(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\big). \tag{1.4}$$

We denote by $S_n(\alpha)$ subclass of the class $\mathcal{A}$ which is defined as follows:

$$S_n(\alpha) = \left\{ f : f \in \mathcal{A},\ \operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \alpha\ z \in \mathbb{U};\ 0 < \alpha \leq 1 \right\}. \tag{1.5}$$

The class $S_n(\alpha)$ was introduced by Kadioğlu [2]. We begin by recalling following coefficient inequality associated with the function class $S_n(\alpha)$.

Theorem 1.1 (Kadioğlu [2]). *If $f(z) \in \mathcal{A}$, defined by (1.1), satisfies the coefficient inequality*

$$\sum_{j=2}^{\infty} \left(j^{n+1} - \alpha j^n\right) |a_j| \leq 1 - \alpha, \quad 0 \leq \alpha < 1, \tag{1.6}$$

then $f(z) \in S_n(\alpha)$.

In view of Theorem 1.1, we now introduce the subclass

$$\widetilde{S}_n(\alpha) \subset S_n(\alpha), \tag{1.7}$$

which consists of functions $f(z) \in \mathcal{A}$ whose Taylor-Maclaurin coefficients satisfy the inequality (1.6).

In this paper, we prove an interesting subordination result for the class $\widetilde{S}_n(\alpha)$. In our proposed investigation of functions in the class $\widetilde{S}_n(\alpha)$, we need the following definitions and results.

Definition 1.2 (Hadamard product or convolution). Given two functions $f, g \in \mathcal{A}$ where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{j=2}^{\infty} b_j z^j. \tag{1.8}$$

The Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), \quad z \in \mathbb{U}. \tag{1.9}$$

Definition 1.3 (subordination principle). For two functions f and g analytic in $\mathbb{U}$, the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$

$$f(z) \prec g(z), \quad z \in \mathbb{U}, \tag{1.10}$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1, \tag{1.11}$$

such that

$$f(z) = g\big(w(z)\big), \quad z \in \mathbb{U}. \tag{1.12}$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.13}$$

Definition 1.4 (subordinating factor sequence). A sequence $\{b_j\}_{j=1}^{\infty}$ of complex numbers is said to be a *subordinating factor sequence* if whenever $f(z)$ of the form (1.1) is analytic, univalent, and convex in $\mathbb{U}$, the subordination is given by

$$\sum_{j=1}^{\infty} a_j b_j z^j \prec f(z); \quad z \in \mathbb{U},\ a_1 = 1. \tag{1.14}$$

Theorem 1.5 (Wilf [3]). *The sequence $\{b_j\}_{j=1}^{\infty}$ is subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{j=1}^{\infty} b_j z^j\right\} > 0 \quad z \in \mathbb{U}. \tag{1.15}$$

2. Main theorem

Theorem 2.1. *Let the function $f(z)$ defined by (1.1) be in the class $\widetilde{S}_n(\alpha)$. Also, let $\mathcal{K}$ denote familiar class of functions $f(z) \in \mathcal{A}$ which are univalent and convex in $\mathbb{U}$. Then*

$$\frac{2^n - \alpha 2^{n-1}}{(1-\alpha) + (2^{n+1} - \alpha 2^n)}(f * g)(z) \prec g(z) \quad (z \in \mathbb{U};\ n \in \mathbb{N}_0;\ g(z) \in \mathcal{K}), \tag{2.1}$$

$$\operatorname{Re} f(z) > -\frac{(1-\alpha) + (2^{n+1} - \alpha 2^n)}{(2^{n+1} - \alpha 2^n)}. \tag{2.2}$$

The following constant factor in the subordination result (2.1):

$$\frac{2^n-\alpha 2^{n-1}}{(1-\alpha)+(2^{n+1}-\alpha 2^n)} \tag{2.3}$$

cannot be replaced by a larger one.

Proof. Let $f(z)\in\widetilde{S}_n(\alpha)$ and suppose that

$$g(z)=z+\sum_{j=2}^{\infty}c_j z^j\in\mathcal{K}. \tag{2.4}$$

Then

$$\frac{2^n-\alpha 2^{n-1}}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}(f*g)(z)=\frac{2^n-\alpha 2^{n-1}}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}\left(z+\sum_{j=2}^{\infty}a_jc_jz^j\right). \tag{2.5}$$

Thus, by Definition 1.4, the subordination result (2.1) will hold true if

$$\left\{\frac{2^n-\alpha 2^{n-1}}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}a_j\right\}_{j=1}^{\infty} \tag{2.6}$$

is a subordinating factor sequences, with $a_1=1$. In view of Theorem 1.5, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+2\sum_{j=1}^{\infty}\frac{2^n-\alpha 2^{n-1}}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}a_jz^j\right\}>0\quad z\in\mathbb{U}. \tag{2.7}$$

Now, since $j^{n+1}-j^n$ $(j\geq 2,\ n\in\mathbb{N}_0)$ is an increasing function of j, we have

$$\begin{aligned}
&\operatorname{Re}\left\{1+2\sum_{j=1}^{\infty}\frac{2^n-\alpha 2^{n-1}}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}a_jz^j\right\}\\
&\quad=\operatorname{Re}\left\{1+\sum_{j=1}^{\infty}\frac{2^{n+1}-\alpha 2^n}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}a_jz^j\right\}\\
&\quad=\operatorname{Re}\left\{1+\frac{2^{n+1}-\alpha 2^n}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}a_1z+\frac{1}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}\sum_{j=2}^{\infty}(2^{n+1}-\alpha 2^n)a_jz^j\right\}\\
&\quad\geq 1-\frac{2^{n+1}-\alpha 2^n}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}r-\frac{1}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}\sum_{j=2}^{\infty}(j^{n+1}-\alpha j^n)\,|a_j|\,r^j\\
&\quad>1-\frac{2^{n+1}-\alpha 2^n}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}r-\frac{1-\alpha}{(1-\alpha)+(2^{n+1}-\alpha 2^n)}r>0\quad(|z|=r<1),
\end{aligned} \tag{2.8}$$

where we have also made use of the assertion (1.6) of Theorem 1.1. This evidently proves the inequality (2.7), and hence also the subordination result (2.1) asserted by our theorem. The inequality (2.2) follows from (2.1) upon setting

$$g(z) = \frac{z}{1-z} = \sum_{j=1}^{\infty} z^j \in \mathcal{K}. \tag{2.9}$$

Now, consider the function

$$f_0(z) = z - \frac{1-\alpha}{2^{n+1} - \alpha 2^n} z^2 \quad (n \in \mathbb{N}_0,\ 0 \leq \alpha < 1), \tag{2.10}$$

which is a member of the class $\widetilde{S}_n(\alpha)$. Then by using (2.1), we have

$$\frac{2^n - \alpha 2^{n-1}}{(1-\alpha) + (2^{n+1} - \alpha 2^n)} (f_0 * g)(z) \prec \frac{z}{1-z}. \tag{2.11}$$

It can be easily verified for the function $f_0(z)$ defined by (2.10) that

$$\min \operatorname{Re} \left\{ \frac{2^n - \alpha 2^{n-1}}{(1-\alpha) + (2^{n+1} - \alpha 2^n)} (f_0 * g)(z) \right\} = -\frac{1}{2}, \quad z \in \mathbb{U}, \tag{2.12}$$

which completes the proof of theorem. □

If we take $n = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}^*(\alpha)$ and $g(z) \in \mathcal{K}$, then*

$$\frac{2-\alpha}{2(3-2\alpha)} (f * g)(z) \prec g(z), \tag{2.13}$$

$$\operatorname{Re} f(z) > -\frac{3-2\alpha}{2-\alpha} \quad (z \in \mathbb{U}). \tag{2.14}$$

The constant factor

$$\frac{2-\alpha}{2(3-2\alpha)} \tag{2.15}$$

in the subordination result (2.13) cannot be replaced by a larger one.

If we take $n = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}(\alpha)$ and $g(z) \in \mathcal{K}$, then*

$$\frac{2-\alpha}{5-3\alpha}(f * g)(z) \prec g(z), \tag{2.16}$$

$$\operatorname{Re} f(z) > -\frac{5-3\alpha}{2(2-\alpha)} \quad (z \in \mathbb{U}). \tag{2.17}$$

The constant factor

$$\frac{2-\alpha}{5-3\alpha} \tag{2.18}$$

in the subordination result (2.16) cannot be replaced by a larger one.

References

[1] G. S. Sălăgean, "Subclasses of univalent functions," in *Complex Analysis—Proceedings of 5th Romanian-Finnish Seminar—Part 1 (Bucharest, 1981)*, vol. 1013 of *Lecture Notes in Math.*, pp. 362–372, Springer, Berlin, Germany, 1983.

[2] E. Kadioğlu, "On subclass of univalent functions with negative coefficients," *Applied Mathematics and Computation*, vol. 146, no. 2-3, pp. 351–358, 2003.

[3] H. S. Wilf, "Subordinating factor sequences for convex maps of the unit circle," *Proceedings of the American Mathematical Society*, vol. 12, pp. 689–693, 1961.

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