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Research Article

On the Kneser-Type Solutions for Two-Dimensional Linear Differential Systems with Deviating Arguments

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For the differential system $u_1'(t) = p(t)u_2(\tau(t))$, $u_2'(t) = q(t)u_1(\sigma(t))$, $t \in [0,+\infty)$, where $p,q \in L_{\text{loc}}(\mathbb{R}_+;\mathbb{R}_+)$, $\tau,\sigma \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\lim_{t\to+\infty}\tau(t) = \lim_{t\to+\infty}\sigma(t) = +\infty$, we get necessary and sufficient conditions that this system does not have solutions satisfying the condition $u_1(t)u_2(t) < 0$ for $t \in [t_0,+\infty)$. Note one of our results obtained for this system with constant coefficients and delays ($p(t) \equiv p, q(t) \equiv q, \tau(t) = t - \Delta, \sigma(t) = t - \delta$, where $\delta,\Delta \in \mathbb{R}$ and $\Delta + \delta > 0$). The inequality $(\delta + \Delta)\sqrt{pq} > 2/e$ is necessary and sufficient for nonexistence of solutions satisfying this condition.

1. Introduction

The equation $u''(t) = pu(t)$, $t \in [0,+\infty)$ with positive constant coefficient p, has two linearly independent solutions $u_1 = e^{\sqrt{p}t}$ and $u_2 = e^{-\sqrt{p}t}$. The second solution satisfies the property $u(t)u'(t) < 0$ for $t \in [0,+\infty)$ and it is the Kneser-type solution. The ordinary differential equation with variable coefficient $u''(t) = p(t)u(t)$, $p(t) \geq 0$, $t \in [0,+\infty)$, preserves the solutions of the Kneser-type. The differential equation with deviating argument

$$u''(t) = p(t)u(\tau(t)), \quad p(t) \geq 0,\ t \in [0,+\infty), \tag{1.1}$$

where $u(\xi) = \varphi(\xi)$, for $\xi < 0$, generally speaking, does not inherit this property. The problems of existence/nonexistence of the Kneser-type solutions were studied in [1–4]. Assertions on existence of bounded solutions, their uniqueness, and oscillation were obtained in the monograph by Ladde et al. (see [5, pages 130–139]). Several possible types of the solution's behavior of this equation can be the following:

(a) $|x(t)| \to \infty$ for $t \to \infty$;

(b) $x(t)$ oscillates;

(c) $x(t) \to 0$, $x'(t) \to 0$ for $t \to \infty$.

Existence and uniqueness of solutions of these types were obtained in [4, 6, 7]. Note that in the case of delay differential equations ($\tau(t) \le t$) with the zero initial function φ, the space of solutions is two-dimensional. In this case it was proven in [8] that existence of the Kneser-type solution was equivalent to nonvanishing of the Wronskian $W(t)$ of the fundamental system and positivity of Green's function of the one point problem

$$u''(t) = p(t)u(\tau(t)) + f(t), \quad p(t) \ge 0,\ t \in [0,\omega], \qquad x(\omega) = 0, \quad x'(\omega) = 0, \tag{1.2}$$

where $x(\xi) = 0$ for $\xi < 0$ and ω can be each positive real number. A generalization of this result to nth-order equations became a basis for study of nonoscillation and differential inequalities for nth-order functional differential equations [9, 10]. If $W(t) \neq 0$ for $t \in [0,+\infty)$, then the Sturm separation theorem (between two zeros of each nontrivial solution there is one and only one zero of other solution) is fulfilled for the second-order delay equation. Properties of the Wronskian and their corollaries were discussed in the recent paper [11].

Consider the differential system

$$\begin{aligned} u_1'(t) &= p(t)u_2(\sigma(t)), \\ u_2'(t) &= q(t)u_1(\tau(t)), \end{aligned} \tag{1.3}$$

where $p,q : \mathbb{R}_+ \to \mathbb{R}_+$ are locally summable functions, $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable function. Throughout this paper we will assume that $\sigma'(t) \ge 0$ and $\tau(\sigma(t) \le t$ for $t \in [0,+\infty)$ and τ is a nondecreasing function.

In the present paper, necessary and sufficient conditions for nonexistence of solutions satisfying the condition

$$u_1(t)u_2(t) < 0, \quad \text{for } t \ge t_0, \tag{1.4}$$

are established for the system (1.3). In the recent paper by Kiguradze and Partsvania [12] the existence of the Kneser-type solution was proven in the case of advanced argument ($\sigma(t) \ge t$, $\tau(t) \ge t$).

It is clear that equation $u''(t) = p(t)u(\tau(t))$ can be represented in the form of system (1.3), where $q = 1$, and the property (1.4) is the analog of the inequality $u(t)u'(t) < 0$ for $t \in [0,+\infty)$, for this scalar equation.

In [8], it was obtained that the inequality $\sqrt{p^*}\delta^* \le 2/e$, where $p^* = \operatorname{vrai\,sup}_{t\in[0,+\infty)} p(t)$, $\delta^* = \operatorname{vrai\,sup}_{t\in[0,+\infty)} t - \tau(t)$, implied the existence of the Kneser-type solution for the noted above scalar homogeneous equation of the second order. Note one of our results obtained for the system (1.3) with constant coefficients and delays ($p(t) \equiv p$, $q(t) \equiv q$, $\tau(t) = t - \Delta$, $\sigma(t) = t - \delta$, where $p,q \in (0,+\infty)$, $\delta,\Delta \in \mathbb{R}$ and $\Delta + \delta > 0$). The condition $(\delta + \Delta)\sqrt{pq} > 2/e$ is necessary and sufficient for nonexistence of solutions satisfying the condition (1.4). It is clear that the inequality $\sqrt{p}\delta > 2/e$ is necessary and sufficient for nonexistence of solutions satisfying the inequality $u(t)u'(t) < 0$ for $t \in [0,+\infty)$ for the scalar second-order equation $u''(t) = pu(t-\delta)$ with constant coefficients p and δ.

Definition 1.1. Let $t_0 \in \mathbb{R}_+$ and $t_* = \min(\inf_{t\geq t_0}\tau(t); \inf_{t\geq t_0}\sigma(t))$. A continuous vector function (u_1,u_2) defined on $[t_*,+\infty)$ is said to be solution of system (1.3) in $[t_0,+\infty)$ if it is absolutely continuous on each finite segment contained in $[t_0,+\infty)$ and satisfies (1.3) almost everywhere on $[t_0,+\infty)$.

From this point on we assume that

$$h(t,s) = \int_s^t p(s)ds, \qquad h(t,s) \longrightarrow +\infty \text{ as } t \longrightarrow +\infty. \tag{1.5}$$

2. Some auxiliary lemmas

Lemma 2.1. *Let $t_0 \in \mathbb{R}_+$ and (u_1,u_2) be a solution of the problem (1.3), (1.4). Then*

$$\nu_k(t)\,|\,u_1(\tau(\sigma(t)))\,| \leq \rho_k(t) \quad \text{for } t \geq \eta(t_0)\ (k=0,1), \tag{2.1}$$

where $\eta(t) = \min\{s : \tau(\sigma(s)) \geq t\}$,

$$\rho_k(t) = (1-k)\,|\,u_1(t)\,| + |u_2(\sigma(t))\,|h^{1-k}(t,0) \quad (k=0,1), \tag{2.2$_k$}$$

$$\nu_k(t) = \max\{w_k(t,s,s_1) : s_1 \in [t,\eta(t)],\ s \in [\tau(\sigma(t)),t]\}, \tag{2.3$_k$}$$

$$\begin{aligned} w_k(t,s,s_1) = {} & h^{k-1}(s,0)h(s,\tau(\sigma(s_1)))\int_s^t h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi \\ & \times \int_t^{s_1} h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi \quad (k=0,1). \end{aligned} \tag{2.4$_k$}$$

Proof. Without loss of generality, we suppose that

$$u_1(t) > 0, \quad u_2(t) < 0 \quad \text{for } t \geq t_0. \tag{2.5}$$

Because

$$\begin{aligned} & \int_s^t u_2'(\sigma(\xi))\sigma'(\xi)h^{1-k}(\xi,0)d\xi \\ & \quad = h^{1-k}(t,0)u_2(\sigma(t)) - h^{1-k}(s,0)u_2(\sigma(s)) - (1-k)\int_s^t p(\xi)h^{-k}(\xi,0)u_2(\sigma(\xi))d\xi \\ & \quad = h^{1-k}(t,0)u_2(\sigma(t)) - h^{1-k}(s,0)u_2(\sigma(s)) + (1-k)\int_s^t h^{-k}(\xi,0)\,|\,u_1'(\xi)\,|\,d\xi \\ & \quad \leq h^{1-k}(s,0)\,|\,u_2(\sigma(s))\,| + (1-k)h^{-k}(s,0)u_1(s) - (1-k)h^{-k}(s,0)u_1(t) \\ & \quad = \rho_k(s) - (1-k)h^{-k}(s,0)u_1(t) \quad (k=0,1), \end{aligned} \tag{2.6}$$

therefore from equality

$$\int_s^t u_2'(\sigma(\xi))\sigma'(\xi)h^{1-k}(\xi,0)d\xi = \int_s^t q(\sigma(\xi))h^{1-k}(\xi,0)\sigma'(\xi)u_1(\tau(\sigma(\xi)))d\xi \quad (k=0,1), \tag{2.7}$$

we have

$$\rho_k(s) \geq \int_s^t h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)u_1(\tau(\sigma(\xi)))d\xi \quad \text{for } t \geq s \geq \eta(t_0)\ (k=0,1), \tag{2.8}$$

where the function ρ_k is given by equality (2.2_k).

Let $t \in [t_0,+\infty)$ and $(s_0,s_*) \in ([\tau(\sigma(t)),t] \times [t,\eta(t)])$ be a maximum point of the function $w(t,\cdot,\cdot)$. Then by (2.8), we obtain

$$\begin{aligned}\rho_k(s_0) &\geq \int_{s_0}^t h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)u_1(\tau(\sigma(\xi)))d\xi\\ &\geq u_1(\tau(\sigma(t)))\int_{s_0}^t h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi,\\ \rho_k(t) &\geq \int_t^{s_*} h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)u_1(\tau(\sigma(\xi)))d\xi\\ &\geq u_1(\tau(\sigma(s_*)))\int_t^{s_*} h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi.\end{aligned} \tag{2.9}$$

On the other hand, in view of the fact that the function $|u_2(t)|$ is nonincreasing, it follows from the first equation of system (1.3) that

$$\begin{aligned}u_1(\tau(\sigma(s_*))) &= u_1(s_0) + \int_{\tau(\sigma(s_*))}^{s_0} p(\xi)\,|\,u_2(\sigma(\xi))\,|\,d\xi\\ &\geq u_1(s_0) + |\,u_2(\sigma(s_0))\,|\,h(s_0,\tau(\sigma(s_*)))\\ &\geq h(s_0,\tau(\sigma(s_*)))h^{-1}(s_0,0)u_1(s_0) + u_2(\sigma(s_0))h(s_0,\tau(\sigma(s_*')))\\ &= h(s_0,\tau(\sigma(s_*)))h^{k-1}(s_0,0)\rho_k(s_0) \quad (k=0,1).\end{aligned} \tag{2.10}$$

Hence, by (2.9), we obtain

$$\begin{aligned}\rho_k(t) &\geq u_1(\tau(\sigma(s_*)))\int_t^{s_*} h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi\\ &\geq h(s_0,\tau(\sigma(s_0)))h^{k-1}(s_0,0)\rho_k(s_0)\int_t^{s_*} h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi\\ &\geq h(s_0,\tau(\sigma(s_0)))h^{k-1}(s_0,0)\int_t^{s_*} h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi\\ &\quad\times\int_{s_0}^t h^{1-k}(\xi,0)q(\sigma(\xi))\sigma'(\xi)d\xi u_1(\tau(\sigma(t))\\ &= \nu_k(t)u_1(\tau(\sigma(t))).\end{aligned} \tag{2.11}$$

Therefore, since t is arbitrary, the last inequality yields (2.1). □

Lemma 2.2. *Let $t_0 \in \mathbb{R}_+$ and (u_1,u_2) be a solution of problem (1.3), (1.4),*

$$\liminf_{t\to+\infty} \int_{\tau(\sigma(t))}^{t} q(\sigma(s))\sigma'(s)ds > 0, \tag{2.12}$$

$$\sup\left(q(\sigma(t))\sigma'(t) : t \in \mathbb{R}_+\right) < +\infty, \qquad \operatorname{vrai\,inf}\left(p(t) : t \in \mathbb{R}_+\right) > 0. \tag{2.13}$$

Then

$$\limsup_{t\to+\infty} \frac{|u_1(\tau(t))|}{|u_2(t)|} < +\infty. \tag{2.14}$$

Proof. By Lemma 2.1, it is sufficient to show that

$$\liminf_{t\to+\infty} v_1(t) > 0, \tag{2.15}$$

where the function v_1 is defined by equalities (2.3_k) and (2.4_k), where $k = 1$. According to (2.12), there exist $c > 0$ and $t_1 \in [t_0,+\infty)$ such that

$$\int_{\tau(\sigma(t))}^{t} q(\sigma(s))\sigma'(s)ds \geq c \quad \text{for } t \geq t_1. \tag{2.16}$$

Let $t \in [t_1,+\infty)$. By (2.16), there exist $t^* \in (t,\eta(t)]$, $\bar{t} \in (t,t^*)$, and $\underline{t} \in (\tau(\sigma(t_*)),t)$ such that

$$\int_{\tau(\sigma(t^*))}^{\underline{t}} q(\sigma(s))\sigma'(s)ds \geq \frac{c}{4}, \qquad \int_{\underline{t}}^{t} q(\sigma(s))\sigma'(s)ds \geq \frac{c}{4}, \tag{2.17}$$

$$\int_{t}^{\bar{t}} q(\sigma(s))\sigma'(s)ds \geq \frac{c}{4}. \tag{2.18}$$

According to (2.3_k), where $k = 1$, and (2.18),

$$v_1(t) \geq \int_{\underline{t}}^{t} q(\sigma(\xi))\sigma'(\xi)ds \int_{t}^{\bar{t}} q(\sigma(\xi))\sigma'(\xi)ds\, h(\underline{t},\tau(\sigma(\bar{t}))) \geq \frac{c^2}{16} h(\underline{t},\tau(\sigma(\bar{t}))). \tag{2.19}$$

By the first condition of (2.13) and (2.18)

$$\underline{t} - \tau(\sigma(t^*)) \geq \frac{c}{4M}, \tag{2.20}$$

where $M = \operatorname{vrai\,sup}(q(\sigma(t)\sigma'(t) : t \in \mathbb{R}_+)$. Therefore by the second condition of (2.13), we have

$$h(\underline{t},\tau(\sigma(\bar{t}))) \geq r(\underline{t} - \tau(\sigma(\bar{t}))) \geq r(\underline{t} - \tau(\sigma(t^*))) \geq \frac{cr}{4M}, \tag{2.21}$$

where $r = \operatorname{vrai\,inf}(p(t) : t \in \mathbb{R}_+) > 0$.

Consequently, from (2.19), we obtain $v_1(t) \geq c^3 r/64M^2$, for $t \geq t_1$, which proves the inequality (2.15). □

Lemma 2.3. *Let $t_0 \in \mathbb{R}_+$ and (u_1,u_2) be a solution of the problem (1.3), (1.4), for some $k \in \{0,1\}$*

$$\liminf_{t\to+\infty} \int_{\tau(\sigma(t))}^{t} h^{1-k}(s,0)q(\sigma(s))\sigma'(s)ds > 0, \tag{2.22$_k$}$$

$$\operatorname{vrai\,sup}\left(h^{2-k}(t,0)q(\sigma(t))\sigma'(t) : t \in \mathbb{R}_+\right) < +\infty, \qquad \operatorname{vrai\,inf}\left(p(t) : t \in \mathbb{R}_+\right) > 0. \tag{2.23$_k$}$$

Then

$$\limsup_{t\to+\infty} \frac{h^k(\tau(\sigma(t)),0)\,|\,u_1(\tau(\sigma(t)))\,|}{\rho_k(t)} < +\infty, \tag{2.24$_k$}$$

where functions h and ρ_k are defined by (1.5) and (2.2$_k$), respectively.

Proof. By Lemma 2.1, in order to prove inequality (2.24$_k$), it is sufficient to show that

$$\liminf_{t\to+\infty} v_k(t)h^{-k}(\tau(\sigma(t)),0) > 0. \tag{2.25}$$

By virtue of (2.22$_k$), we can choose $t_1 \in \mathbb{R}_+$ and $c > 0$ such that

$$\int_{\tau(\sigma(t))}^{t} h^{1-k}(\xi,0)q(\sigma(\xi)\sigma'(\xi)d\xi \geq c \quad \text{for } t \geq t_1. \tag{2.26}$$

Let $t \in [t_1,+\infty)$. According to (2.26),

$$\exists t^* \in (t,\eta(t)), \quad \bar{t} \in (t,t^*), \quad \underline{t} \in (\tau(\sigma(t^*)),t) \tag{2.27}$$

such that

$$\int_{\tau(\sigma(t^*))}^{\underline{t}} h^{1-k}(s,0)q(\sigma(s))\sigma'(s)ds \geq \frac{c}{4}, \qquad \int_{\underline{t}}^{t} h^{1-k}(s,0)q(\sigma(s))\sigma'(s)ds \geq \frac{c}{4}, \tag{2.28}$$

$$\int_{t}^{\bar{t}} h^{1-k}(s,0)q(\sigma(s))\sigma'(s)ds \geq \frac{c}{4}. \tag{2.29}$$

In view (2.3$_k$), (2.4$_k$), (2.27), and (2.29), we have

$$\begin{aligned} v_k(t) \geq{}& \int_{\underline{t}}^{t} h^{1-k}(s,0)q(\sigma(s))\sigma'(s)ds \int_{t}^{\bar{t}} h^{1-k}(s,0)q(\sigma(s))\sigma'(s)ds \\ &\times h^{k-1}(\underline{t},0)h(t,\tau(\sigma(\bar{t}))) \geq \frac{c^2}{16}h^{k-1}(\underline{t},0)h(\underline{t},\tau(\sigma(\bar{t}))). \end{aligned} \tag{2.30}$$

On the other hand by (1.5) and (2.27), taking into account that the function $h(t,0)$ is nondecreasing, we obtain

$$\begin{aligned} h(\underline{t},\tau(\sigma(\bar{t}))) &= \int_{\tau(\sigma(\bar{t}))}^{\underline{t}} p(s)ds = \int_{\tau(\sigma(\bar{t}))}^{\underline{t}} h(s,0)h^{-1}(s,0)p(s)ds \\ &\geq h(\tau(\sigma(\bar{t})),0)\ln\frac{h(\underline{t},0)}{h(\tau(\sigma(\bar{t})),0)}. \end{aligned} \tag{2.31}$$

Therefore, from (2.30)

$$\begin{aligned} v_k(t)h^{-k}(\tau(\sigma(t)),0) &\geq \frac{c^2}{16}h^{k-1}(\underline{t},0)h(\tau(\sigma(\bar{t})),0)h^{-k}(\tau(\sigma(t)),0) \\ &\quad\times\ln\frac{h(\underline{t},0)}{h(\tau(\sigma(\bar{t})),0)} \geq \frac{c^2}{16}\ln\frac{h(\underline{t},0)}{h(\tau(\sigma(\bar{t})),0)}. \end{aligned} \tag{2.32}$$

From the first condition of (2.29) by (2.23_k), we have

$$\begin{aligned} \frac{c}{4} &\leq \int_{\tau(\sigma(t^*))}^{\underline{t}} h^{2-k}(s,0)q(\sigma(s))\sigma'(s)h^{-1}(s,0)ds \\ &\leq \frac{M}{r}\int_{\tau(\sigma(t^*))}^{\underline{t}} \frac{p(s)}{h(s,0)}ds \leq \frac{M}{r}\ln\frac{h(\underline{t},0)}{h(\tau(\sigma(\bar{t})),0)}, \end{aligned} \tag{2.33}$$

where $M = \operatorname{vrai\,sup}(h^{2-k}(t,0)q(\sigma(t))\sigma'(t) : t \in \mathbb{R}_+)$, $r = \operatorname{vrai\,inf}(p(t) : t \in \mathbb{R}_+)$. Therefore, from (2.32)

$$v_k(t)h^{-k}(\tau(\sigma(t)),0) \geq \frac{c^2}{16}\frac{r}{M}. \tag{2.34}$$

Hence, this implies (2.25) for arbitrary t. The lemma is proved. □

Lemma 2.4. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of problem (1.3), (1.4), let (2.12), (2.13) be fulfilled, and*

$$\limsup_{t\to+\infty}\frac{1}{h(t,0)}\int_0^t q(\sigma(s))\sigma'(s)ds < +\infty. \tag{2.35}$$

Then, there exists $\lambda > 0$ such that

$$\lim_{t\to+\infty}|u_1(t)|e^{\lambda h(t,0)} = +\infty. \tag{2.36}$$

Proof. Since every condition of Lemma 2.2 is fulfilled, there exist $t_1 > t_0$ and $M > 0$ such that

$$|u_1(\tau(\sigma(t)))| \leq M|u_2(\sigma(t))| \quad \text{for } t \geq t_1. \tag{2.37}$$

From the second equation of the system (1.3), we have

$$\frac{u_2'(\sigma(t))\sigma'(t)}{|u_2(\sigma(t))|} = q(\sigma(t))\sigma'(t)\frac{u_1(\tau(\sigma(t)))}{|u_2(\sigma(t))|}. \tag{2.38}$$

Integrating the equality from t_1 to t, we obtain

$$|u_2(\sigma(t))| \geq |u_2(\sigma(t_1))| \exp\left(-\int_{t_1}^{t} q(\sigma(s))\sigma'(s)\frac{|u_1(\tau(\sigma(t)))|}{|u_2(\sigma(t))|}ds\right). \tag{2.39}$$

Therefore, according to (2.37)

$$|u_2(\sigma(t))| \geq |u_2(\sigma(t_1))| \exp\left(-M\int_{t_1}^{t} q(\sigma(s))\sigma'(s)ds\right). \tag{2.40}$$

By (2.35), we get

$$|u_2(\sigma(t))| \geq \exp(-M\gamma h(t,0)) \quad \text{for } t \geq t_*, \tag{2.41}$$

where

$$\gamma > \limsup_{t\to+\infty} \frac{1}{h(t,0)}\int_0^t q(\sigma(s))\sigma'(s)ds, \tag{2.42}$$

and t_* sufficiently large. From (2.41) and by the first equation of the system (1.3) we get

$$\left|\int_t^{+\infty} u_1'(s)ds\right| \geq \int_t^{+\infty} p(s)\exp(-M\gamma h(t,0))ds. \tag{2.43}$$

Hence, if we take into account the notation (1.5), we find

$$|u_1(t)| \geq \frac{1}{M\gamma}\exp(-M\gamma h(t,0)) \quad \text{for } t \geq t_*. \tag{2.44}$$

Consequently, if $\lambda > M\gamma$, condition (2.36) is fulfilled. □

Lemma 2.5. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of problem (1.3), (1.4), let conditions (2.22_k), (2.23_k), where $k = 0$, and (2.35) be fulfilled, and*

$$\limsup_{t\to+\infty} \frac{1}{h(t,0)}\int_0^t h^{-1}(\tau(\sigma(s)),0)q(\sigma(s))\sigma'(s)ds < +\infty. \tag{2.45}$$

Then there exists $\lambda > 0$ such that (2.36) is fulfilled.

Lemma 2.5 can be proven analogously to Lemma 2.4.

Lemma 2.6. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of problem (1.3), (1.4), and let conditions (2.22_k), (2.23_k), where $k = 0$, and*

$$\limsup_{t\to+\infty} \frac{1}{h(t,0)}\int_0^t q(\sigma(s))\sigma'(s)h(s,0)ds < +\infty \tag{2.46}$$

hold. Then there exists $\lambda > 0$ such that (2.36) is fulfilled.

Proof. According to Lemma 2.3, condition (2.24_k), where $k = 0$, is valid, where function ρ_0 is given by equality (2.2_k), where $k = 0$. Therefore, from of the second equation of the system (1.3), we have

$$-\frac{\rho_0'(t)}{\rho_0(t)} = q(\sigma(t))\sigma'(t)h(t,0)\frac{|u_1(\tau(\sigma(t)))|}{\rho_0(t)} \le Mq(\sigma(t))\sigma'(t)h(t,0) \quad \text{for } t \ge t_*, \tag{2.47}$$

where $M > \limsup_{t\to+\infty}(|u_1(\tau(\sigma(t)))|/\rho_0(t))$ and t_* is sufficiently large. Therefore, integrating the last inequality from t_* to t, we get

$$\rho_0(t) \ge \rho_0(t_*)\exp\left(-M\int_{t_*}^{t} q(\sigma(s))\sigma'(s)h(s,0)ds\right) \quad \text{for } t \ge t_*. \tag{2.48}$$

On the other hand, by (2.46), there exist $r > 0$ and $t^* > t_*$ such that

$$\int_{t_*}^{t} q(\sigma(s))\sigma'(s)h(s,0)ds \le rh(t,0) \quad \text{for } t \ge t^*. \tag{2.49}$$

Consequently, there exist $r_1 > 0$ and $t_1^* > t^*$ such that

$$\rho_0(t) \ge \exp(-r_1h(t,0)) \quad \text{for } t \ge t_1^*. \tag{2.50}$$

Hence for any $\gamma > 0$, we have

$$\begin{aligned} &|u_1(t)|p(t)\exp(-\gamma h(t,0)) + |u_2(\sigma(t))|p(t)h(t,0)\exp(-\gamma h(t,0)) \\ &\quad \ge \exp(-(r_1+\gamma)h(t,0))p(t) \quad \text{for } t \ge t_1^*. \end{aligned} \tag{2.51}$$

Therefore, by the first equation of the system (1.3)

$$\begin{aligned} &\int_t^{+\infty} |u_1(s)|p(s)\exp(-\gamma h(s,0))ds + \int_t^{+\infty} |u_1'(s)|h(s,0)\exp(-\gamma h(s,0))ds \\ &\quad \ge \frac{1}{r_1+\gamma}\exp(-(r_1+\gamma)h(t,0)) \quad \text{for } t \ge t_1^*. \end{aligned} \tag{2.52}$$

Because, for large t, $h(t,0)\exp(-(\gamma/2)h(t,0)) \le 1$, from the last inequality, we have

$$\begin{aligned} &\int_t^{+\infty} |u_1(s)|p(s)\exp\left(-\frac{\gamma}{2}h(s,0)\right)ds + \int_t^{+\infty} |u_1'(s)|\exp\left(-\frac{\gamma}{2}h(s,0)\right)ds \\ &\quad \ge \frac{1}{r_1+\gamma}\exp(-(r_1+\gamma)h(t,0)) \quad \text{for } t \ge t_2^*, \end{aligned} \tag{2.53}$$

where $t_2^* > t_1^*$—sufficiently large. Hence, taking into account that functions $|u_1(t)|$ and $\exp(-(\gamma/2)h(t,0))$ are nonincreasing, we get

$$\left(1+\frac{2}{\gamma}\right)\exp\left(-\frac{\gamma}{2}h(t,0)\right)|u_1(t)| \ge \frac{1}{\gamma+r_1}\exp(-(r_1+\gamma)h(t,0)) \quad \text{for } t \ge t_1^*. \tag{2.54}$$

Consequently

$$|u_1(t)| \geq \frac{\gamma}{(\gamma+r_1)(2+\gamma)} \exp\left(-\left(r_1+\frac{\gamma}{2}\right)h(t,0)\right) \quad \text{for } t \geq t_2^*. \tag{2.55}$$

Hence, it is obvious that, if $\lambda > r_1 + \gamma/2$, then condition (2.36) holds. □

Lemmas 2.7–2.12 can be proved analogously to Lemmas 2.4–2.6.

Lemma 2.7. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of the problem (1.3), (1.4), let conditions (2.12), (2.13) be fulfilled, and*

$$\limsup_{t\to+\infty} \frac{1}{\ln h(t,0)} \int_0^t q(\sigma(s))\sigma'(s)ds < +\infty. \tag{2.56}$$

Then there exists $\lambda > 0$ such that

$$\lim_{t\to+\infty} |u_1(t)|\left(h(t,0)\right)^{\lambda} = +\infty. \tag{2.57}$$

Lemma 2.8. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of the problem (1.3), (1.4), let conditions (2.22_k), (2.23_k), where $k = 1$, be fulfilled, and*

$$\limsup_{t\to+\infty} \frac{1}{\ln h(t,0)} \int_0^t h^{-1}(\tau(\sigma(s)),0)q(\sigma(s))\sigma'(s)ds < +\infty. \tag{2.58}$$

Then there exists $\lambda > 0$ such that (2.57) holds.

Lemma 2.9. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of the problem (1.3), (1.4), and let conditions (2.22_k), (2.23_k), where $k = 0$, and*

$$\limsup_{t\to+\infty} \frac{1}{\ln h(t,0)} \int_0^t q(\sigma(s))\sigma'(s)h(s,0)ds < +\infty \tag{2.59}$$

be fulfilled. Then there exists $\lambda > 0$ such that (2.57) holds.

Lemma 2.10. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of the problem (1.3), (1.4), let conditions (2.12), (2.13) be fulfilled, and*

$$\limsup_{t\to+\infty} \frac{1}{\ln(\ln h(t,0))} \int_0^t q(\sigma(s))\sigma'(s)ds < +\infty. \tag{2.60}$$

Then there exists $\lambda > 0$ such that

$$\lim_{t\to+\infty} |u_1(t)|\left(\ln h(t,0)\right)^{\lambda} = +\infty. \tag{2.61}$$

Lemma 2.11. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of the problem (1.3), (1.4), let conditions (2.22_k), (2.23_k), where $k = 1$, be fulfilled, and*

$$\limsup_{t\to+\infty} \frac{1}{\ln(\ln h(t,0))} \int_0^t h^{-1}(\tau(\sigma(t)),0)q(\sigma(s))\sigma'(s)ds < +\infty. \tag{2.62}$$

Then there exists $\lambda > 0$ such that (2.61) holds.

Lemma 2.12. *Let $t_0 \in \mathbb{R}_+$, (u_1,u_2) be a solution of the problem (1.3), (1.4), and let conditions (2.22$_k$), (2.23$_k$), where $k=0$, and*

$$\limsup_{t\to+\infty}\frac{1}{\ln(\ln h(t,0))}\int_0^t q(\sigma(s))\sigma'(s)h(s,0)ds<+\infty \tag{2.63}$$

be fulfilled. Then there exists $\lambda>0$ such that (2.61) holds.

3. Basic lemmas

Lemma 3.1. *Let $t_0 \in \mathbb{R}_+$, $\varphi,\psi \in C([t_0,+\infty),(0,+\infty))$, let ψ be a nonincreasing function, and*

$$\lim_{t\to+\infty}\varphi(t)=+\infty, \tag{3.1}$$

$$\liminf_{t\to+\infty}\psi(t)\widetilde{\varphi}(t)=0, \tag{3.2}$$

where $\widetilde{\varphi}(t)=\inf\{\varphi(s): s\geq t\geq t_0\}$. Then there exists a sequence $\{t_k\}$ such that $t_k\uparrow+\infty$ as $k\uparrow+\infty$ and

$$\widetilde{\varphi}(t_k)=\varphi(t_k),\quad \psi(t)\widetilde{\varphi}(t)\geq\psi(t_k)\widetilde{\varphi}(t_k)\quad \text{for } t_0\leq t\leq t_k\ (k=1,2,\dots). \tag{3.3}$$

Proof. Let $t\in[t_0,+\infty)$. Define the sets E_i $(i=1,2)$ by

$$t\in E_1 \Longleftrightarrow \widetilde{\varphi}(t)=\varphi(t),\quad t\in E_2\Longleftrightarrow \widetilde{\varphi}(s)\psi(s)\geq\widetilde{\varphi}(t)\psi(t),\quad \text{for } s\in[t_0,t]. \tag{3.4}$$

It is clear that, by (3.1) and (3.2), $\sup E_i=+\infty$ $(i=1,2)$. We show that

$$\sup E_1\cap E_2=+\infty. \tag{3.5}$$

Indeed, if we assume that $t_*\in E_2$ and $t_*\notin E_1$, by (3.1) there exists $t^*>t_*$ such that $\widetilde{\varphi}(t)=\widetilde{\varphi}(t_*)$ for $t\in[t_*,t^*]$ and $\widetilde{\varphi}(t^*)=\varphi(t^*)$. On the other hand, since ψ is a nonincreasing function, we have $\psi(t)\widetilde{\varphi}(t)\geq\psi(t^*)\widetilde{\varphi}(t^*)$ for $t\in[t_0,t^*]$. Therefore $t^*\in E_1\cap E_2$. By the above reasoning we easily ascertain that (3.5) is fulfilled. Thus there exists a sequence of points $\{t_k\}$ such that $t_k\uparrow+\infty$ for $k\uparrow+\infty$ and (3.3) holds. □

Remark 3.2. Lemma 3.1 was first proven in [4].

Lemma 3.3. *Let $t_0\in\mathbb{R}_+$, (u_1,u_2) be a solution of the problem (1.3), (1.4). Besides there exists $\gamma\in C([t_0,+\infty);\mathbb{R}_+)$ and $0<r_1<r_2$ such that*

$$\gamma(t)\uparrow+\infty\quad \text{for } t\uparrow+\infty,\qquad \lim_{t\to+\infty}(\gamma(t))^{r_2}|u_1(t)|=+\infty, \tag{3.6}$$

$$\liminf_{t\to+\infty}(\gamma(t))^{r_1}|u_1(t)|=0,\qquad \limsup_{t\to+\infty}\frac{\gamma(t)}{\gamma(\sigma(\tau(t)))}=c<+\infty. \tag{3.7}$$

Then

$$\liminf_{t\to+\infty}(\gamma(t))^{r_2}\int_t^{+\infty}p(s)\int_{\sigma(s)}^{+\infty}q(\xi)(\gamma(\tau(\xi)))^{-r_2}d\xi\,ds\leq c^{r_2-r_1}. \tag{3.8}$$

Proof. Let (u_1,u_2) be a solution of the problem (1.3), (1.4). Without loss of generality, assume that condition (2.5) is fulfilled. Then from system (1.3), we get

$$u_1(\tau(\sigma(t))) \geq \int_{\tau(\sigma(t))}^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)u_1(\tau(\xi))d\xi\, ds \quad \text{for } t \geq t_1, \tag{3.9}$$

where $t_1 > t_0$—sufficiently large.

Denote

$$\widetilde{\varphi}(t) = \inf\left((\gamma(\tau(s)))^{r_2}u_1(\tau(s)) : s \geq t\right),$$
$$\psi(t) = (\gamma(\tau(t)))^{r_1-r_2}. \tag{3.10}$$

According to (3.6) and (3.7), it is obvious that the functions $\widetilde{\varphi}$ and ψ defined by (3.10) satisfy the conditions of Lemma 3.1. Indeed, by (3.6) it is obvious that condition (3.1) is fulfilled. On the other hand, since the functions γ and τ are nondecreasing, it is clear that the function ψ is nonincreasing. By (3.10), we have

$$\widetilde{\varphi}(t)\psi(t) \leq (\gamma(\tau(t)))^{r_2}(\gamma(\tau(t)))^{r_1-r_2}u_1(\tau(t)) = (\gamma(\tau(t)))^{r_1}u_1(\tau(t)). \tag{3.11}$$

Therefore, according to the first condition of (3.7), (3.2) holds. Consequently, functions φ and ψ satisfied the condition of Lemma 3.1. Therefore there exists a sequence$\{t_k\}$ such that $t_k \uparrow +\infty$ as $k \uparrow +\infty$,

$$\widetilde{\varphi}(\sigma(t_k)) = \varphi(\sigma(t_k)), \tag{3.12}$$

$$(\gamma(\sigma(t_k)))^{r_1-r_2}\widetilde{\varphi}(\sigma(t_k)) \leq (\gamma(\sigma(t)))^{r_1-r_2}\widetilde{\varphi}(\sigma(t)) \quad \text{for } t_* \leq t \leq t_k\ (k = 1,2,\ldots), \tag{3.13}$$

where $t_* > t_1$—sufficiently large. From (3.9), taking into account that $\widetilde{\varphi}(t) \leq (\gamma(\tau(t)))^{r_2}u_1(\tau(t))$, we have

$$\begin{aligned} u_1(\tau(\sigma(t_k))) &\geq \int_{\tau(\sigma(t_k))}^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2}(\gamma(\tau(\xi)))^{r_2}u_1(\tau(\xi))d\xi\, ds \\ &\geq \int_{\tau(\sigma(t_k))}^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2}\widetilde{\varphi}(\xi)d\xi\, ds. \end{aligned} \tag{3.14}$$

Hence, since the functions σ and $\widetilde{\varphi}$ are nondecreasing, we get

$$\begin{aligned} u_1(\tau(\sigma(t_k))) \geq &\int_{\tau(\sigma(t_k))}^{t_k} \widetilde{\varphi}(\sigma(s))p(s)\int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2}d\xi\, ds \\ &+ \widetilde{\varphi}(\sigma(t_k))\int_{t_k}^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2}d\xi\, ds. \end{aligned} \tag{3.15}$$

Therefore, by (3.13)

$$u_1(\tau(\sigma(t_k))) \geq (\gamma(\sigma(t_k)))^{r_1-r_2}\widetilde{\varphi}(\sigma(t_k)) \int_{\tau(\sigma(t_k))}^{t_k} p(s)(\gamma(\sigma(s)))^{r_2-r_1} \times \int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2} d\xi\, ds + \widetilde{\varphi}(\sigma(t_k)) \int_{t_k}^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2} d\xi\, ds. \tag{3.16}$$

On the other hand,

$$\begin{aligned} I(t_k) &= \int_{\tau(\sigma(t_k))}^{t_k} p(s)(\gamma(\sigma(s)))^{r_2-r_1} \int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2} d\xi\, ds \\ &= -(\gamma(\sigma(t_k)))^{r_2-r_1} \int_{t_k}^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2} d\xi\, ds \\ &\quad + (\gamma(\sigma(\tau(\sigma(t_k)))))^{r_2-r_1} \int_{\tau(\sigma(t_k))}^{+\infty} p(\xi) \int_{\sigma(\xi)}^{+\infty} q(\xi_1)(\gamma(\tau(\xi_1)))^{-r_2} d\xi_1\, d\xi \\ &\quad + (r_2 - r_1) \int_{\tau(\sigma(t_k))}^{t_k} (\gamma(\sigma(\xi)))^{r_2-r_1-1} (\gamma(\sigma(\xi)))' \int_{s}^{+\infty} p(\xi) \\ &\quad \times \int_{\sigma(\xi)}^{+\infty} q(\xi_1)(\gamma(\tau(\xi_1)))^{-r_2} d\xi_1\, d\xi\, ds. \end{aligned} \tag{3.17}$$

Since $(\gamma(\sigma(t))' \geq 0$, it follows from the last inequality that

$$\begin{aligned} I(t_k) &\geq -(\gamma(\sigma(t_k)))^{r_2-r_1} \int_{t_k}^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2} d\xi\, ds \\ &\quad + (\gamma(\sigma(\tau(\sigma(t_k)))))^{r_2-r_1} \int_{\tau(\sigma(t_k))}^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi_1)(\gamma(\tau(\xi_1)))^{-r_2} d\xi_1\, ds. \end{aligned} \tag{3.18}$$

Therefore, from (3.16), we get

$$\begin{aligned} u_1(\tau(\sigma(t_k))) &\geq (\gamma(\sigma(t_k)))^{r_1-r_2} (\gamma(\sigma(\tau(\sigma(t_k)))))^{r_2-r_1} \widetilde{\varphi}(\sigma(t_k)) \\ &\quad \times \int_{\tau(\sigma(t_k))}^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2} d\xi\, ds. \end{aligned} \tag{3.19}$$

Hence, by (3.12), we get

$$\begin{aligned} &(\gamma(\tau(\sigma(t_k))))^{r_2} \int_{\tau(\sigma(t_k))}^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2} d\xi\, ds \\ &\quad \leq \left(\frac{\gamma(\sigma(t_k))}{\gamma(\sigma(\tau(\sigma(t_k))))} \right)^{r_2-r_1} \quad (k = 1, 2, \ldots). \end{aligned} \tag{3.20}$$

According to the second condition of (3.7), for any $\varepsilon > 0$, there exists $k_0 \in N$ such that $\gamma(\sigma(t_k))/\gamma(\sigma(\tau(\sigma(t_k)))) \le c+\varepsilon$ for $k \ge k_0$. Therefore by (3.20), we get

$$(\gamma(\tau(\sigma(t_k))))^{r_2}\int_{\tau(\sigma(t_k))}^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2}d\xi\, ds \le (c+\varepsilon)^{r_2-r_1}, \quad k = k_0, k_0+1,\dots,$$

$$\limsup_{k\to+\infty}(\gamma(\tau(\sigma(t_k))))^{r_2}\int_{\tau(\sigma(t_k))}^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)(\gamma(\tau(\xi)))^{-r_2}d\xi\, ds \le (c+\varepsilon)^{r_2-r_1}. \tag{3.21}$$

On the other hand, in view of the arbitrariness of ε, the last inequality implies (3.8). This proves the lemma. □

4. The necessary conditions of the existence of Kneser-type solutions

Let $t_0 \in \mathbb{R}_+$. By $\mathbf{K}_{t_0}$ we denote the set of all solutions of the system (1.3) satisfying the condition (1.4).

Remark 4.1. In the definition of the set $\mathbf{K}_{t_0}$, we assume that if there is no solution satisfying (1.4), then $\mathbf{K}_{t_0} = \varnothing$.

Theorem 4.2. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \ne \varnothing$. Assume that conditions (2.12), (2.13), and (2.35) are fulfilled and*

$$\limsup_{t\to+\infty}(h(t,0) - h(\sigma(\tau(t)),0)) < +\infty. \tag{4.1}$$

Then there exists $\lambda \in \mathbb{R}_+$ such that

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{t\to+\infty}\exp((\lambda+\varepsilon)h(t,0))\int_t^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)\exp(-(\lambda+\varepsilon)h(\tau(\xi),0))d\xi\, ds\right) \le 1. \tag{4.2}$$

Proof. Since $\mathbf{K}_{t_0} \ne \varnothing$, we have that the problem (1.3), (1.4) has a solution (u_1, u_2). According to Lemma 2.4, there exist $\lambda > 0$ such that condition (2.36) is fulfilled. Denote by $\mathbf{\Delta}$ the set of all λ satisfying (2.36) and put $\lambda_0 = \inf\mathbf{\Delta}$. It is obvious that $\lambda_0 \ge 0$. Below we will show that for $\lambda = \lambda_0$ inequality (4.2) holds. By (2.36) for all $\varepsilon > 0$, the function $\gamma(t) = \exp(h(t,0))$ satisfies conditions (3.6) and first condition of (3.7), where $r_2 = \lambda_0+\varepsilon$ and $r_1 = \lambda_0 - \varepsilon$. On the other hand, by (4.1) it is clear that the second condition of (3.7) is fulfilled. Therefore, according to Lemma 3.3, for any $\varepsilon > 0$, we get

$$\liminf_{t\to+\infty}\exp((\lambda_0+\varepsilon)h(t,0))\int_t^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)\exp(-(\lambda_0+\varepsilon)h(\tau(\xi),0))d\xi\, ds \le c^{2\varepsilon}. \tag{4.3}$$

Proceeding to greatest lower bound in the last inequality, for $\varepsilon \to 0+$, we obtain inequality (4.2), when $\lambda = \lambda_0$. □

Theorems 4.3 and 4.4 can be proven analogously to Theorem 4.2 if we take into consideration Lemmas 2.5 and 2.6, respectively.

THEOREM 4.3. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Assume that conditions (2.22$_k$), (2.23$_k$), where $k = 1$, (2.45), and (4.1) are fulfilled. Then there exists $\lambda \in \mathbb{R}_+$ which satisfies the inequality (4.2).*

THEOREM 4.4. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Assume that conditions (2.22$_k$), (2.23$_k$), where $k = 0$, (2.46), and (4.1) are fulfilled. Then there exists $\lambda \in \mathbb{R}_+$ which satisfies the inequality (4.2).*

THEOREM 4.5. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Assume that conditions (2.12), (2.13), and (2.56) are fulfilled and*

$$\limsup_{t\to+\infty} \frac{h(t,0)}{h(\sigma(\tau(t),0))} < +\infty. \tag{4.4}$$

Then there exists $\lambda \in \mathbb{R}_+$ such that

$$\limsup_{\varepsilon\to 0+} \liminf_{t\to+\infty} (h(t,0))^{\lambda+\varepsilon} \int_t^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)(h(\tau(\xi),0))^{-(\lambda+\varepsilon)} d\xi\, ds \leq 1. \tag{4.5}$$

Theorem 4.5 can be proven analogously to Theorem 4.2 if we take into consideration the condition (4.4) and Lemma 2.7.

THEOREM 4.6. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Assume that conditions (2.22$_k$), (2.23$_k$), where $k = 1$, (2.58), and (4.4) are fulfilled. Then there exists $\lambda \in \mathbb{R}_+$ which satisfies the inequality (4.5).*

THEOREM 4.7. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Assume that conditions (2.22$_k$), (2.23$_k$), where $k = 0$, (2.59), and (4.4) are fulfilled. Then there exists $\lambda \in \mathbb{R}_+$ which satisfies the inequality (4.5).*

By Lemma 2.10, similarly to Theorem 4.5, one can prove the following theorem.

THEOREM 4.8. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Assume that conditions (2.12), (2.13), and (2.60) are fulfilled and*

$$\limsup_{t\to+\infty} \frac{\ln h(t,0)}{\ln (h(\sigma(\tau(t)),0))} < +\infty. \tag{4.6}$$

Then there exists $\lambda \in \mathbb{R}_+$ such that

$$\limsup_{\varepsilon\to 0+} \Big(\liminf_{t\to+\infty} (\ln h(t,0)) \Big)^{\lambda+\varepsilon} \int_t^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)(\ln (h(\tau(\xi),0)))^{-(\lambda+\varepsilon)} d\xi\, ds \leq 1. \tag{4.7}$$

THEOREM 4.9. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Assume that conditions (2.22$_k$), (2.23$_k$), where $k = 1$, (2.62), and (4.6) are fulfilled. Then there exists $\lambda \in \mathbb{R}_+$ which satisfies the inequality (4.7).*

This theorem is proven analogously to Theorem 4.8 if we replace Lemma 2.10 by Lemma 2.11.

Theorem 4.10. *Let $t_0 \in \mathbb{R}_+$ and $\mathbf{K}_{t_0} \neq \varnothing$. Besides conditions (2.22$_k$), (2.23$_k$), where $k = 0$, (2.63), and (4.6) are fulfilled. Then there exists $\lambda \in \mathbb{R}_+$ such that the inequality (4.7) holds.*

This theorem is proven analogously to Theorem 4.8 if we replace Lemma 2.10 by Lemma 2.12.

5. The sufficient conditions for the problem (1.3), (1.4) has no solution

In this section, we will produce the sufficient conditions under which for any $t_0 \in \mathbb{R}_+$, we have $\mathbf{K}_{t_0} = \varnothing$.

Theorem 5.1. *Let conditions (2.12), (2.13), (2.35), and (4.1) be fulfilled. Assume that for any $\lambda \in \mathbb{R}_+$*

$$\limsup_{\varepsilon \to 0+} \Big(\liminf_{t \to +\infty} \exp\big((\lambda+\varepsilon)h(t,0)\big) \int_t^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi) \exp\big(-(\lambda+\varepsilon)h(\tau(\xi),0)\big) \Big) d\xi\, ds > 1. \tag{5.1}$$

Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.

Proof. Suppose not. Let there exist $t_0 \in \mathbb{R}_+$ such that $\mathbf{K}_{t_0} \neq \varnothing$. Then there exists a solution (u_1, u_2) of the problem (1.3), (1.4). On the other hand, since the conditions of Theorem 4.2 are fulfilled, there exists $\lambda_0 \in \mathbb{R}_+$, such that when $\lambda = \lambda_0$, inequality (4.2) holds. But this inequality contradicts (5.1). The obtained contradiction proves the theorem. □

Taking into account Theorems 4.3 and 4.4, we can easily ascertain the validity of the following theorems (Theorems 5.2 and 5.3).

Theorem 5.2. *Let conditions (2.22$_k$), (2.23$_k$), where $k = 1$, (2.45), and (4.1) be fulfilled. Assume that for any $\lambda \in \mathbb{R}_+$ (5.1) holds. Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.*

Theorem 5.3. *Let conditions (2.22$_k$), (2.23$_k$), where $k = 0$, (2.46), and (4.1) be fulfilled. Assume that for any $\lambda \in \mathbb{R}_+$ (5.1) holds. Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.*

Corollary 5.4. *Let conditions (2.12), (2.13), (4.1), and (2.35) be fulfilled. Assume there exist $t_1 \in \mathbb{R}_+$ such that*

$$\inf\big(\lambda^{-2} a_p(\lambda) a_q(\lambda) : \lambda > 0\big) > 1, \tag{5.2}$$

where

$$\begin{aligned} a_p(\lambda) &= \inf\Big(e^{\lambda(h(t,0)-h(\sigma(t),0))} : t \geq t_1\Big), \\ a_q(\lambda) &= \inf\Big(\frac{q(t)}{p(t)} e^{\lambda(h(t,0)-h(\tau(t),0))} : t \geq t_1\Big). \end{aligned} \tag{5.3}$$

Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.

Proof. It is sufficient to show that for any $\lambda \in \mathbb{R}_+$ inequality (5.1) is satisfied. By (5.2), we have that for any $\lambda \in (0,+\infty)$, there exist $\varepsilon_0 > 0$ such that

$$\lambda^{-2}a_p(\lambda)a_q(\lambda) \geq 1+\varepsilon_0 \quad \text{for } \lambda \in (0,+\infty). \tag{5.4}$$

Let $\lambda \in \mathbb{R}_+$ and let ε be an arbitrary positive number. Then by (1.5), (5.3), and (5.4), we have that for any $\varepsilon > 0$

$$\begin{aligned}
&\exp\big((\lambda+\varepsilon)h(t,0)\big)\int_t^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)\exp\big(-(\lambda+\varepsilon)h(\tau(\xi),0)\big)d\xi\,ds \\
&\quad\geq \exp\big((\lambda+\varepsilon)h(t,0)\big)a_q(\lambda+\varepsilon)\int_t^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} p(\xi)\exp\big(-(\lambda+\varepsilon)h(\xi,0)\big)d\xi\,ds \\
&\quad\geq \frac{a_q(\lambda+\varepsilon)a_p(\lambda+\varepsilon)}{\lambda+\varepsilon}\exp\big((\lambda+\varepsilon)h(t,0)\big)\int_t^{+\infty} p(s)\exp\big(-(\lambda+\varepsilon)h(s,0)\big)ds \\
&\quad= \frac{a_q(\lambda+\varepsilon)a_p(\lambda+\varepsilon)}{(\lambda+\varepsilon)^2} \geq 1+\varepsilon_0 \quad \text{for } t \geq t_1^*,
\end{aligned} \tag{5.5}$$

where $t_1^* > t_1$—sufficiently large. Consequently, from the last inequality (5.1) follows. □

Corollary 5.5. *Let conditions (2.12), (2.13), (2.35), and (4.1) be fulfilled. Assume that*

$$\begin{gathered}
\sigma(t) \leq t, \quad \tau(t) \leq t \quad \text{for } t \in \mathbb{R}_+, \\
\inf\big(\big(h(t,0)-h(\sigma(t),0)\big) : t \geq t_1\big)\inf\left(\frac{q(t)}{p(t)}\big(h(t,0)-h(\tau(t),0)\big) : t \geq t_1\right) > \frac{1}{e^2}.
\end{gathered} \tag{5.6}$$

Then $\mathbf{K}_{t_0} = \varnothing$ *for any* $t_0 \in \mathbb{R}_+$.

Proof. If we apply the inequality $e^x \geq ex$, it will be clear that (5.1) follows from (5.6). □

Theorem 5.6. *Let* $p(t) \equiv p$, $q(t) \equiv q$, $\tau(t) = t-\Delta$, $\sigma(t) = t-\delta$, *where* $p,q \in (0,+\infty)$, $\delta,\Delta \in \mathbb{R}$, *and* $\Delta+\delta > 0$. *Then the condition*

$$(\delta+\Delta)\sqrt{pq} > \frac{2}{e} \tag{5.7}$$

is necessary and sufficient for $\mathbf{K}_{t_0} = \varnothing$ *for any* $t_0 \in \mathbb{R}_+$.

Proof. Sufficiency. By (5.7) it is obvious that condition (5.2) is satisfied. Therefore sufficiency follows from Corollary 5.4.

Necessity. Let for any $t_0 \in \mathbb{R}_+$, $\mathbf{K}_{t_0} = \varnothing$ and

$$(\delta+\Delta)\sqrt{pq} \leq \frac{2}{e}. \tag{5.8}$$

Then it is obvious that the equation

$$qe^{\lambda p(\delta+\Delta)} = p\lambda^2 \tag{5.9}$$

has a solution $\lambda = \lambda_0 > 0$. Therefore the system

$$c_1\lambda_0 + c_2 e^{\lambda_0 p\Delta} = 0, \qquad c_1 q e^{\lambda_0 p\delta} + c_2 p\lambda_0 = 0 \tag{5.10}$$

has a solution c_1 and c_2, such that $c_1c_2 < 0$. It is clear that vector function $(c_1e^{-\lambda_0 t}, c_2e^{-\lambda_0 t})$ is a solution of the problem (1.3)-(1.4). But this contradicts the fact that $\mathbf{K}_{t_0} = \varnothing$. □

Remark 5.7. If the function τ satisfies condition (4.1), then the strong inequality (5.1) cannot be changed by nonstrong one. Otherwise, the problem (1.3), (1.4) has a solution as the proof of necessity in Theorem 5.6 demonstrates: actually in this case the left-hand side of (5.1) is one.

Theorem 5.8. *Let conditions (2.12), (2.13), (2.56), and (4.4) be fulfilled. Assume that for any $\lambda \in \mathbb{R}_+$*

$$\limsup_{\varepsilon\to 0+}\liminf_{t\to+\infty}\,(h(t,0))^{\lambda+\varepsilon}\int_t^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)\big(h(\tau(\xi),0)\big)^{-(\lambda+\varepsilon)}d\xi\,ds > 1. \tag{5.11}$$

Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.

Taking into account Theorem 4.5, we can prove the following assertion analogously to Theorem 4.2.

Theorem 5.9. *Let conditions (2.22$_k$), (2.23$_k$), where $k = 0$, (2.59), and (4.4) be fulfilled and for any $\lambda \in \mathbb{R}_+$ let inequality (5.11) be satisfied. Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.*

By Theorem 4.6, we can easily ascertain the validity of the following assertion.

Theorem 5.10. *Let conditions (2.22$_k$), (2.23$_k$), where $k = 1$, (2.58), and (4.4) be fulfilled and for any $\lambda \in \mathbb{R}_+$ let inequality (5.11) be satisfied. Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.*

Corollary 5.11. *Let conditions (2.12), (2.13), (2.56), and (4.4) be satisfied. Assume there exist $t_1 \in \mathbb{R}_+$ such that*

$$\inf\left(\frac{1}{\lambda(\lambda+1)}a_p(\lambda)a_q(\lambda) : \lambda > 0\right) > 1, \tag{5.12}$$

where

$$a_p(\lambda) = \inf\left(\left(\frac{h(t,0)}{h(\sigma(t),0)}\right)^{1+\lambda} : t \geq t_1\right),$$
$$a_q(\lambda) = \inf\left(\frac{q(t)}{p(t)}h^2(t,0)\Big(\frac{h(t,0)}{h(\tau(t),0)}\Big)^{\lambda} : t \geq t_1\right). \tag{5.13}$$

Then $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.

Proof. Let us demonstrate that for any $\lambda \in (0,+\infty)$ inequalities (5.12) and (5.13) imply (5.11). Indeed, for any $\lambda \in \mathbb{R}_+$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&(h(t,0))^{\lambda+\varepsilon}\int_t^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} q(\xi)\big(h(\tau(\xi),0)\big)^{-(\lambda+\varepsilon)}d\xi\,ds \\
&\quad\geq a_q(\lambda+\varepsilon)(h(t,0))^{\lambda+\varepsilon}\int_t^{+\infty} p(s)\int_{\sigma(s)}^{+\infty} p(\xi)\big(h(\xi,0)\big)^{-2-\lambda-\varepsilon}d\xi\,ds \\
&\quad= \frac{a_q(\lambda+\varepsilon)}{1+\lambda+\varepsilon}(h(t,0))^{\lambda+\varepsilon}\int_t^{+\infty} p(s)\big(h(\sigma(s),0)\big)^{-1-\lambda-\varepsilon}ds \\
&\quad\geq \frac{a_q(\lambda+\varepsilon)a_p(\lambda+\varepsilon)}{1+\lambda+\varepsilon}(h(t,0))^{\lambda+\varepsilon}\int_t^{+\infty} p(s)\big(h(s,0)\big)^{-1-\lambda-\varepsilon}ds \\
&\quad= \frac{a_q(\lambda+\varepsilon)a_p(\lambda+\varepsilon)}{(1+\lambda+\varepsilon)(\lambda+\varepsilon)} \\
&\quad\geq 1+\varepsilon_0,
\end{aligned} \tag{5.14}$$

where $\varepsilon_0 > 0$, which proves the corollary. □

THEOREM 5.12. *Let $p(t) \equiv p$, $q(t) = q/t^2$, $\sigma(t) = \alpha t$, and $\tau(t) = \beta t$, where $p,q \in (0,+\infty)$, $\alpha,\beta \in (0,+\infty)$ and $\alpha\beta < 1$. Then the condition*

$$\inf\left(\frac{1}{\lambda(1+\lambda)}\alpha^{-\lambda-1}\beta^{-\lambda} : \lambda \in (0,+\infty)\right) > \frac{1}{pq} \tag{5.15}$$

is necessary and sufficient for $\mathbf{K}_{t_0} = \varnothing$ for any $t_0 \in \mathbb{R}_+$.

Proof. Sufficiency. It follows from Corollary 5.11.

Necessity. Let for any $t_0 \in \mathbb{R}_+$, $\mathbf{K}_{t_0} = \varnothing$ and

$$\inf\left(\frac{1}{\lambda(1+\lambda)}\alpha^{-1-\lambda}\beta^{-\lambda} : \lambda \in (0,+\infty)\right) \leq \frac{1}{pq}. \tag{5.16}$$

Then it is obvious that the equation

$$pq\alpha^{-1-\lambda}\beta^{-\lambda} = \lambda(1+\lambda) \tag{5.17}$$

has a solution $\lambda = \lambda_0 > 0$. Therefore the system

$$c_1\lambda_0 + c_2 p\alpha^{-1-\lambda} = 0, \qquad c_1 q\beta^{-\lambda} + c_2(1+\lambda_0) = 0 \tag{5.18}$$

has a solution c_1 and c_2, such that $c_1c_2 < 0$. On the other hand, it is obvious that the vector function $(c_1t^{-\lambda_0}, c_2t^{-\lambda_0-1})$ is a solution of the problem (1.3), (1.4). But this contradicts the fact that $\mathbf{K}_{t_0} = \varnothing$. □

We can prove Theorems 5.13–5.15 analogously to the proofs of Theorems 5.1–5.3.

Theorem 5.13. *Let conditions (2.12), (2.13), (2.60), and (4.6) be fulfilled and for any* $\lambda \in \mathbb{R}_+$

$$\limsup_{\varepsilon\to 0+}\liminf_{t\to+\infty}\big(\ln h(t,0)\big)^{\lambda+\varepsilon}\int_t^{+\infty}p(s)\int_{\sigma(s)}^{+\infty}q(\xi)\big(\ln h(\tau(\xi),0)\big)^{-(\lambda+\varepsilon)}d\xi\,ds>1. \tag{5.19}$$

Then $\mathbf{K}_{t_0}=\varnothing$ *for any* $t_0\in\mathbb{R}_+$.

Theorem 5.14. *Let conditions* (2.22_k), (2.23_k), *where* $k=1$, *(2.62), and (4.6) be fulfilled and for any* $\lambda\in(0,+\infty)$ *let the inequality (5.19) hold. Then* $\mathbf{K}_{t_0}=\varnothing$ *for any* $t_0\in\mathbb{R}_+$.

Theorem 5.15. *Let conditions* (2.22_k), (2.23_k), *where* $k=0$, *(2.63), and (4.6) be fulfilled and for any* $\lambda\in\mathbb{R}_+$ *let the inequality (5.19) hold. Then* $\mathbf{K}_{t_0}=\varnothing$ *for any* $t_0\in\mathbb{R}_+$.

Corollary 5.16. *Let conditions* (2.22_k), (2.23_k), *where* $k=0$, *(2.60), and (4.6) be fulfilled and for any* $\lambda\in(0,+\infty)$ *there exist* $\varepsilon_0>0$ *such that*

$$\liminf_{t\to+\infty}\big(\ln h(t,0)\big)^{\lambda+1}h(t,0)\int_{\sigma(t)}^{+\infty}q(\xi)\big(\ln h(\tau(\xi),0)\big)^{-\lambda}d\xi\ge(1+\varepsilon_0)\lambda. \tag{5.20}$$

Then $\mathbf{K}_{t_0}=\varnothing$ *for any* $t_0\in\mathbb{R}_+$.

Proof. It suffices to note that (5.20) implies (5.19). □

Corollary 5.17. *Let conditions* (2.22_k), (2.23_k), *where* $k=0$, *(2.60), and (4.6) be fulfilled and there exist* $t_1\in\mathbb{R}_+$ *such that*

$$\inf\left\{\frac{a_q(\lambda)\cdot a_p(\lambda)}{\lambda}:\lambda>0\right\}>1, \tag{5.21}$$

$$\limsup_{t\to+\infty}\frac{h(t,0)}{h(\sigma(t),0)}<+\infty, \tag{5.22}$$

where

$$a_q(\lambda)=\inf\left\{\frac{q(t)h^2(t,0)\ln h(t,0)}{p(t)}\left(\frac{\ln h(t,0)}{\ln h(\tau(t),0)}\right)^{\lambda}:t\ge t_1\right\}, \tag{5.23}$$

$$a_p(\lambda)=\inf\left\{\frac{h(t,0)}{h(\sigma(t),0)}\left(\frac{\ln h(t,0)}{\ln h(\sigma(t),0)}\right)^{\lambda+1}:t\ge t_1\right\}. \tag{5.24}$$

Then $\mathbf{K}_{t_0}=\varnothing$ *for any* $t_0\in\mathbb{R}_+$.

Proof. It is sufficient to show that condition (5.20) is fulfilled. According to (5.21), there exists $\varepsilon_0>0$ such that

$$a_q(\lambda)a_p(\lambda)\ge\lambda(1+\varepsilon_0)\quad\text{for }\lambda\in(0,+\infty). \tag{5.25}$$

Therefore in view of (5.23)

$$\begin{aligned}
&\left(\ln h(t,0)\right)^{1+\lambda}h(t,0)\int_{\sigma(t)}^{+\infty}q(\xi)\left(\ln h(\tau(s),0)\right)^{-\lambda}ds\\
&\quad\geq a_q(\lambda)\left(\ln h(t,0)\right)^{1+\lambda}h(t,0)\int_{\sigma(t)}^{+\infty}p(s)h^{-2}(s,0)\left(\ln h(s,0)\right)^{-1-\lambda}ds\\
&\quad= a_q(\lambda)\left(\ln h(t,0)\right)^{1+\lambda}h(t,0)\int_{\sigma(t)}^{+\infty}\Big[-\left(h^{-1}(s,0)\left(\ln h(s,0)\right)^{-\lambda-1}\right)'\\
&\qquad\qquad -(1+\lambda)h^{-2}(s,0)\left(\ln h(s,0)\right)^{-\lambda-2}p(s)\Big]ds.
\end{aligned}\tag{5.26}$$

On the other hand, according to (5.22), we have

$$\begin{aligned}
&\left(\ln h(t,0)\right)^{1+\lambda}h(t,0)\int_{\sigma(t)}^{+\infty}\left(h(s,0)\right)^{-2}\left(\ln h(s,0)\right)^{-\lambda-2}p(s)ds\\
&\quad\leq\left(\frac{\ln h(t,0)}{\ln h(\sigma(t),0)}\right)^{1+\lambda}\left(\ln h(\sigma(t),0)\right)^{-1}h(t,0)\int_{\sigma(t)}^{+\infty}h^{-2}(s,0)p(s)ds\\
&\quad=\left(\frac{\ln h(t,0)}{\ln h(\sigma(t),0)}\right)^{1+\lambda}\frac{h(t,0)}{h(\sigma(t),0)}\left(\ln h(\sigma(t),0)\right)^{-1}\longrightarrow 0\quad\text{for } t\longrightarrow+\infty.
\end{aligned}\tag{5.27}$$

Therefore, in view of (5.25) and (5.26), we have

$$\liminf_{t\to+\infty}\left(\ln h(t,0)\right)^{1+\lambda}h(t,0)\int_{\sigma(t)}^{+\infty}q(s)\left(\ln h(\tau(s),0)\right)^{-\lambda}ds\geq a_q(\lambda)\cdot a_p(\lambda)\geq(1+\varepsilon_0)\lambda.\tag{5.28}$$

The condition (5.20) is fulfilled. This proves the corollary. □

Remark 5.18. The condition (5.21) ((5.19)) cannot be changed by the nonstrong inequality. Otherwise, Corollary 5.16 (Theorem 5.15) will not be true.

Example 5.19. Let $\beta\in(0,1)$, $p(t)=1$, $\sigma(t)=t$, $\tau(t)=t^\beta$, $q(t)=(1/e|\ln\beta|t^2\ln t)(1+(1+|\ln\beta|)/|\ln\beta|\ln t)$. All the conditions of Corollary 5.17 are fulfilled except (5.21). Furthermore we can easily show that

$$\inf\left\{\frac{a_q(\lambda)\cdot a_p(\lambda)}{\lambda}:\lambda>0\right\}=1.\tag{5.29}$$

And the vector-function $((\ln t)^{1/\ln\beta},(\ln t)^{-1+1/\ln\beta}/t\cdot\ln\beta)$ is the solution of (1.3) satisfying the condition (1.4), while t_0 is the sufficiently large number.

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