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Research Article

Generalized Vector Equilibrium-Like Problems without Pseudomonotonicity in Banach Spaces

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Let X and Y be real Banach spaces, D a nonempty closed convex subset of X, and $C : D \to 2^Y$ a multifunction such that for each $u \in D$, $C(u)$ is a proper, closed and convex cone with $\operatorname{int} C(u) \neq \varnothing$, where $\operatorname{int} C(u)$ denotes the interior of $C(u)$. Given the mappings $T : D \to 2^{L(X,Y)}$, $A : L(X,Y) \to L(X,Y)$, $f : L(X,Y) \times D \times D \to Y$, and $h : D \to Y$, we study the generalized vector equilibrium-like problem: find $u_0 \in D$ such that $f(As_0, u_0, v) + h(v) - h(u_0) \notin -\operatorname{int} C(u_0)$ for all $v \in D$ for some $s_0 \in Tu_0$. By using the KKM technique and the well-known Nadler result, we prove some existence theorems of solutions for this class of generalized vector equilibrium-like problems. Furthermore, these existence theorems can be applied to derive some existence results of solutions for the generalized vector variational-like inequalities. It is worth pointing out that there are no assumptions of pseudomonotonicity in our existence results.

1. Introduction

In 1980, Giannessi [1] first introduced and studied the vector variational inequality in a finite-dimensional Euclidean space, which is a vector-valued version of the variational inequality of Hartman and Stampacchia. Subsequently, many authors investigated vector variational inequalities in abstract spaces, and extended vector variational inequalities to vector equilibrium problems, which include as special cases various problems, for example, vector complementarity problems, vector optimization problems, abstract economical equilibria, and saddle-point problems (see, e.g., [1–17]).

In 1999, B.-S. Lee and G.-M. Lee [12] first established a vector version of Minty's lemma (see [18]) by using Nadler's result [19]. They considered vector variational-like inequalities for multifunctions under pseudomonotonicity and hemicontinuity conditions. Recently, Khan and Salahuddin [5] also established a vector version of Minty's lemma and applied it to obtain an existence theorem for a class of vector variational-like inequalities for compact-valued multifunctions under similar pseudomonotonicity condition and similar hemicontinuity condition.

On the other hand, as a natural generalization of the vector equilibrium problem, the generalized vector equilibrium problem includes as special cases various problems, for example, generalized vector variational inequality problem, generalized vector variational-like inequality problem, generalized vector complementarity problem and vector equilibrium problem. Inspired by early results in this field, many authors have considered and studied the generalized vector equilibrium problem, that is, the vector equilibrium problem for multifunctions; see, for example, [6, 8, 13–15, 17].

In this paper, let X and Y be two real Banach spaces and D a nonempty closed convex subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $u \in D$, $C(u)$ is a proper, closed, and convex cone with $\operatorname{int} C(u) \neq \emptyset$, where $\operatorname{int} C(u)$ denotes the interior of $C(u)$. For convenience, we let $P = \bigcap_{u\in D} C(u)$. Given the mappings $T : D \to 2^{L(X,Y)}$, $A : L(X,Y) \to L(X,Y)$, $f : L(X,Y) \times D \times D \to Y$, and $h : D \to Y$, we consider the *generalized vector equilibrium-like problem* as follows,

$$\text{find } u_0 \in D \text{ such that } f(As_0, u_0, v) + h(v) - h(u_0) \notin -\operatorname{int} C(u_0), \quad \forall v \in D \text{ for some } s_0 \in Tu_0. \tag{1.1}$$

In particular, if we put $f(z,x,y) = \langle z, \eta(y,x)\rangle$ for all $(z,x,y) \in L(X,Y) \times D \times D$, where $\eta : D \times D \to X$, then the above problem reduces to the following generalized vector variational-like inequality problem:

$$\text{find } u_0 \in D \text{ such that } \langle As_0, \eta(v,u_0)\rangle + h(v) - h(u_0) \notin -\operatorname{int} C(u_0), \quad \forall v \in D \text{ for some } s_0 \in Tu_0. \tag{1.2}$$

By using the KKM technique [20] and the Nadler's result [19], we prove some existence theorems of solutions for this class of generalized vector equilibrium-like problems. Furthermore, these existence theorems can be applied to derive some existence results of solutions for the generalized vector variational-like inequalities. It is worth pointing out that there are no assumptions of pseudomonotonicity in our existence results.

2. Preliminaries

In this section, we recall some notations, definitions and results, which are essential for our main results.

Definition 2.1 (see [11]). Let D be a nonempty subset of a vector space X. Then a multifunction $T : D \to 2^X$ is called a KKM-map where 2^X denotes the collection of all nonempty subsets of X, if for each nonempty finite subset $\{u_1, u_2, \dots, u_n\}$ of D, $\operatorname{co}\{u_1, u_2, \dots, u_n\} \subset \bigcup_{i=1}^n Tu_i$, where $\operatorname{co}\{u_1, u_2, \dots, u_n\}$ denotes the convex hull of $\{u_1, u_2, \dots, u_n\}$.

Lemma 2.2 (Fan's lemma [20]). *Let D be an arbitrary set in a Huasdorff topological vector space X. Let $T : D \to 2^X$ be a KKM-map such that Tu is closed for all $u \in D$ and is compact for at least one $u \in D$. Then $\bigcap_{u\in D} Tu \neq \varnothing$.*

Lemma 2.3 (Nadler's theorem [19]). *Let $(X, \|\cdot\|)$ be a normed vector space and H the Hausdorff metric on the collection $CB(X)$ of all closed and bounded subsets of X, induced by a metric d in terms of $d(x,y) = \|x-y\|$, which is defined by*

$$H(A,B) = \max\left(\sup_{u\in A}\inf_{v\in B}\|u-v\|, \sup_{v\in B}\inf_{u\in A}\|u-v\|\right), \tag{2.1}$$

for A and B in $CB(X)$. If A and B are any two members in $CB(X)$, then for each $\varepsilon > 0$ and each $u \in A$, there exists $v \in B$ such that

$$\|u-v\| \leq (1+\varepsilon)H(A,B). \tag{2.2}$$

In particular, if A and B are any two compact subsets in X, then for each $u \in A$, there exists $v \in B$ such that

$$\|u-v\| \leq H(A,B). \tag{2.3}$$

Lemma 2.4 (see [16]). *Let Y be a topological vector space with a pointed, closed and convex cone C such that $\operatorname{int} C \neq \varnothing$. Then for all $x,y,z \in Y$,*

(i) $x - y \in -\operatorname{int} C$ *and* $x \notin -\operatorname{int} C \Rightarrow y \notin -\operatorname{int} C$;
(ii) $x + y \in -C$ *and* $x + z \notin -\operatorname{int} C \Rightarrow z - y \notin -\operatorname{int} C$;
(iii) $x + z - y \notin -\operatorname{int} C$ *and* $-y \in -C \Rightarrow x + z \notin -\operatorname{int} C$;
(iv) $x + y \notin -\operatorname{int} C$ *and* $y - z \in -C \Rightarrow x + z \notin -\operatorname{int} C$.

Definition 2.5 (see [13]). A multifunction $T : D \to 2^Y$ is called P-convex if, for all $u,v \in D$ and $\lambda \in (0,1)$,

$$T(\lambda u + (1-\lambda)v) \subseteq \lambda Tu + (1-\lambda)Tv - P. \tag{2.4}$$

Similarly, one can define the P-convexity of single-valued mappings.

Definition 2.6. Let $T : D \to 2^Y$. The graph of T, denoted by $\operatorname{Gr}(T)$, is the following set:

$$\operatorname{Gr}(T) = \{(x,y) : y \in Tx\}. \tag{2.5}$$

Definition 2.7 (see [16]). Let $f : D \times D \to Y$ be a vector-valued bifunction. Then $f(x,y)$ is said to be hemicontinuous with respect to y if for any given $x \in D$,

$$\lim_{\lambda\to 0^+} f(x, \lambda y_1 + (1-\lambda)y_2) = f(x,y_2) \tag{2.6}$$

for all $y_1, y_2 \in D$.

Throughout the rest of this paper, by "$\to$" and "$\rightharpoonup$" we denote the strong convergence and weak convergence, respectively.

3. Main results

In this section, we will present two theorems for the existence results to the generalized vector equilibrium-like problem.

THEOREM 3.1. *Let X and Y be real Banach spaces, D a nonempty convex subset of X, and $\{C(u) : u \in D\}$ a family of closed proper convex solid cones of Y such that for each $u \in D$, $C(u) \neq Y$. Let $W : D \to 2^Y$ be a multifunction, defined by $W(u) = Y \setminus (-\operatorname{int} C(u))$, such that the graph $\operatorname{Gr}(W)$ is weakly closed in $X \times Y$. Suppose that the following conditions hold:*

(i) *for each $u, v \in D$, $f(At_\lambda, v_\lambda, v_\lambda) \in C(u)$, for all $t_\lambda \in Tv_\lambda$, and $f(At_\lambda, v_\lambda, u) + f(At_\lambda, u, v_\lambda) = 0$, where $v_\lambda := u + \lambda(v - u)$, $\lambda \in (0,1)$;*

(ii) *$f(z, \cdot, v), h(\cdot) : D \to Y$ are weakly continuous for each $(z, v) \in L(X,Y) \times D$;*

(iii) *$f(z, v, \cdot) + h(\cdot)$ is P-convex on D for each $(z, v) \in L(X,Y) \times D$;*

(iv) *there exists a bifunction $p : D \times D \to Y$ with the following properties:*

 (a) *for each $u, v \in D$, $p(u,v) \notin -\operatorname{int} C(u)$ implies $f(At, u, v) + h(v) - h(u) \notin -\operatorname{int} C(u)$, for all $t \in Tv$,*

 (b) *for each finite subset $\mathcal{A} \subseteq D$ and each $u \in \operatorname{co}\mathcal{A}$, $v \mapsto p(u,v)$ is P-convex,*

 (c) *for each $v \in D$, $p(v,v) \notin \operatorname{int} C(v)$;*

 (d) *there exist a weakly compact convex subset $K \subseteq D$ and $v_0 \in K$ such that $p(u, v_0) \in -\operatorname{int} C(u)$ for all $u \in D \setminus K$.*

Then there exists a solution $u_0 \in D$ such that

$$f(At, v, u_0) + h(u_0) - h(v) \notin \operatorname{int} C(u_0) \tag{3.1}$$

for all $v \in D$ and $t \in Tv$.

Moreover, suppose additionally that $L(X,Y)$ is reflexive and $T : D \to 2^{L(X,Y)}$ is a multifunction which takes bounded, closed, and convex values in $L(X,Y)$ and satisfies the following conditions:

(v) *for each net $\{\lambda\} \subset (0,1)$ such that $\lambda \to 0^+$,*

$$\left.\begin{aligned} t_\lambda &\rightharpoonup s_0, \\ t_\lambda &\in Tv_\lambda \end{aligned}\right\} \Longrightarrow f(At_\lambda, v_\lambda, v) - f(As_0, v_\lambda, v) \rightharpoonup 0, \tag{3.2}$$

where $v_\lambda := u + \lambda(v - u)$ for $(u,v) \in D \times D$;

(vi) *for each $u, v \in D$,*

$$H\big(T(u + \lambda(v - u)), T(u)\big) \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+, \tag{3.3}$$

where H is the Hausdorff metric defined on $CB(L(X,Y))$.

Then there exists a solution $u_0 \in D$ such that for some $s_0 \in Tu_0$,

$$f(As_0, u_0, v) + h(v) - h(u_0) \notin -\operatorname{int} C(u_0) \quad \forall v \in D. \tag{3.4}$$

Proof. For each $v \in D$, we define $G : D \to 2^K$ by

$$G(v) = \{u \in K : f(At, u, v) + h(v) - h(u) \notin -\operatorname{int} C(u),\ \forall t \in Tv\}, \quad \forall v \in D. \tag{3.5}$$

Firstly, we claim that $G(v)$ is weakly closed for each $v \in D$. Indeed, let $\{u_n\} \subseteq G(v)$ be such that $u_n \rightharpoonup u_0 \in K$ as $n \to \infty$. Since $u_n \in G(v)$ for all n, we have that

$$f(At,u_n,v)+h(v)-h(u_n) \notin -\operatorname{int} C(u_n), \quad \forall t \in Tv. \tag{3.6}$$

Since from condition (ii) it follows that $f(At,\cdot,v)-h(\cdot): D \to Y$ is weakly continuous, we have

$$f(At,u_n,v)+h(v)-h(u_n) \rightharpoonup f(At,u_0,v)+h(v)-h(u_0). \tag{3.7}$$

Note that the graph $\operatorname{Gr}(W)$ is weakly closed in $X \times Y$. Hence, we have $f(At,u_0,v)+h(v)-h(u_0) \in Y \setminus (-\operatorname{int} C(u_0))$, that is, $f(At,u_0,v)+h(v)-h(u_0) \notin -\operatorname{int} C(u_0)$. This shows that $u_0 \in G(v)$. Thus, $G(v)$ is weakly closed. Since every element $u_0 \in \bigcap_{v \in D} G(v)$ is a solution of (3.1), we have to prove that

$$\bigcap_{v \in D} G(v) \neq \varnothing. \tag{3.8}$$

Since K is weakly compact, it is sufficient to show that the family $\{G(v)\}_{v \in D}$ has the finite intersection property.

Let $\{v_1,v_2,\dots,v_m\}$ be a finite subset of D. We claim that

$$\bigcap_{j=1}^{m} G(v_j) \neq \varnothing. \tag{3.9}$$

Indeed, note that

$$V := \operatorname{co}\{v_1,v_2,\dots,v_m\} \tag{3.10}$$

is a compact convex subset of D and also a weakly compact convex subset of D. We define a multifunction $F: V \to 2^V$ as

$$F(v) = \{u \in V : p(u,v) \notin -\operatorname{int} C(u)\}, \quad \forall v \in V. \tag{3.11}$$

By (iv)(c), $F(v)$ is nonempty for each $v \in V$.

Now we assert that F is a KKM-map.

Indeed, suppose to the contrary that there exists a finite subset $\{y_1,y_2,\dots,y_n\} \subseteq V$ and scalars $\alpha_i \geq 0$, $i=1,2,\dots,n$, with $\sum_{i=1}^{n} \alpha_i = 1$, such that

$$\sum_{i=1}^{n} \alpha_i y_i \notin \bigcup_{i=1}^{n} F(y_i). \tag{3.12}$$

Then, we have

$$p\left(\sum_{i=1}^{n} \alpha_i y_i, y_i\right) \in -\operatorname{int} C\left(\sum_{i=1}^{n} \alpha_i y_i\right), \quad \forall i. \tag{3.13}$$

By (iv)(b), we have

$$\begin{aligned} p\left(\sum_{i=1}^{n}\alpha_i y_i,\sum_{i=1}^{n}\alpha_i y_i\right) &\in \sum_{i=1}^{n}\alpha_i p\left(\sum_{i=1}^{n}\alpha_i y_i, y_i\right) - P \\ &\subseteq \sum_{i=1}^{n}\alpha_i\left(-\operatorname{int}C\left(\sum_{i=1}^{n}\alpha_i y_i\right)\right) - C\left(\sum_{i=1}^{n}\alpha_i y_i\right) \\ &\subseteq -\operatorname{int}C\left(\sum_{i=1}^{n}\alpha_i y_i\right) - C\left(\sum_{i=1}^{n}\alpha_i y_i\right) \\ &= -\operatorname{int}C\left(\sum_{i=1}^{n}\alpha_i y_i\right), \end{aligned} \tag{3.14}$$

a contradiction to condition (iv)(c). Hence, F is a KKM-map. From condition (iv)(a), we have that

$$F(v) \subseteq G(v), \quad \forall v \in V. \tag{3.15}$$

Observe that, for each $v \in V$, the closure $\mathrm{cl}_V(F(v))$ of $F(v)$ in V is closed in V, and therefore is compact also. By Lemma 2.2,

$$\bigcap_{v\in V}\mathrm{cl}_V\,(F(v)) \neq \emptyset. \tag{3.16}$$

We can choose

$$\overline{u} \in \bigcap_{v\in V}\mathrm{cl}_V\,(F(v)) \tag{3.17}$$

and note that $v_0 \in K$ and $F(v_0) \subseteq K$ by (iv)(d). Thus,

$$\overline{u} \in \mathrm{cl}_V\,(F(v_0)) \subseteq \mathrm{cl}_D\,(F(v_0)) = \mathrm{cl}_K\,(F(v_0)) \subseteq K. \tag{3.18}$$

Moreover, it is easy to see that for each $v \in V$,

$$\{u \in V : f(At,u,v) + h(v) - h(u) \notin -\operatorname{int}C(u),\ \forall t \in Tv\} \tag{3.19}$$

is weakly closed. Since

$$\overline{u} \in \bigcap_{j=1}^{m}\mathrm{cl}_V\,(F(v_j)) \tag{3.20}$$

and since, for each $j = 1,2,\dots,m$,

$$\begin{aligned} \mathrm{cl}_V\,(F(v_j)) &= \mathrm{cl}_V\,(\{u \in V : p(u,v_j) \notin -\operatorname{int}C(u)\}) \\ &\subseteq \mathrm{cl}_V\,(\{u \in V : f(At,u,v_j) + h(v_j) - h(u) \notin -\operatorname{int}C(u),\ \forall t \in Tv_j\}) \\ &\subseteq \{u \in V : f(At,u,v_j) + h(v_j) - h(u) \notin -\operatorname{int}C(u),\ \forall t \in Tv_j\}, \end{aligned} \tag{3.21}$$

we have

$$f(At,\overline{u},v_j)+h(v_j)-h(\overline{u})\notin -\operatorname{int}C(\overline{u}),\quad \forall t\in Tv_j \tag{3.22}$$

for all $j=1,2,\ldots,m$, and hence,

$$\overline{u}\in\bigcap_{j=1}^{m}G(v_j). \tag{3.23}$$

Therefore, $\{G(v)\}_{v\in D}$ has the finite intersection property and so

$$\bigcap_{v\in D}G(v)\neq\varnothing, \tag{3.24}$$

that is, there exists $u_0\in K\subseteq D$ such that

$$f(At,u_0,v)+h(v)-h(u_0)\notin -\operatorname{int}C(u_0) \tag{3.25}$$

for all $v\in D$ and $t\in Tv$.

On the other hand, for any arbitrary $v\in D$, letting $v_\lambda=\lambda v+(1-\lambda)u_0$, $0<\lambda<1$, we have $v_\lambda\in D$ by the convexity of D. Hence, for all $t_\lambda\in Tv_\lambda$

$$f(At_\lambda,u_0,v_\lambda)+h(v_\lambda)-h(u_0)\notin -\operatorname{int}C(u_0). \tag{3.26}$$

Since the operator

$$u\longmapsto f(z,u,v)-h(u) \tag{3.27}$$

is P-convex for each $(z,v)\in L(X,Y)\times D$, so from condition (i) we have

$$\begin{aligned}
f(At_\lambda,v_\lambda,v_\lambda)+h(v_\lambda)-h(v_\lambda) &= f(At_\lambda,v_\lambda,\lambda v+(1-\lambda)u_0)+h(\lambda v+(1-\lambda)u_0)-h(v_\lambda)\\
&\in \lambda[f(At_\lambda,v_\lambda,v)+h(v)-h(v_\lambda)]\\
&\quad +(1-\lambda)[f(At_\lambda,v_\lambda,u_0)+h(u_0)-h(v_\lambda)]-P\\
&\subseteq \lambda[f(At_\lambda,v_\lambda,v)+h(v)-h(v_\lambda)]\\
&\quad +(1-\lambda)[f(At_\lambda,v_\lambda,u_0)+h(u_0)-h(v_\lambda)]-C(u_0)\\
&\subseteq \lambda[f(At_\lambda,v_\lambda,v)+h(v)-h(v_\lambda)]\\
&\quad -(1-\lambda)[f(At_\lambda,u_0,v_\lambda)+h(v_\lambda)-h(u_0)]-C(u_0).
\end{aligned} \tag{3.28}$$

Hence,

$$f(At_\lambda,v_\lambda,v)+h(v)-h(v_\lambda)\notin -\operatorname{int}C(u_0). \tag{3.29}$$

Indeed, suppose to the contrary that

$$f(At_\lambda, v_\lambda, v) + h(v) - h(v_\lambda) \in -\operatorname{int} C(u_0). \tag{3.30}$$

Since $-\operatorname{int} C(u_0)$ is a convex cone,

$$\lambda[f(At_\lambda, v_\lambda, v) + h(v) - h(v_\lambda)] \in -\operatorname{int} C(u_0). \tag{3.31}$$

Since condition (i) implies that

$$f(At_\lambda, v_\lambda, v_\lambda) \in C(u_0), \tag{3.32}$$

so from (3.28) we derive

$$\begin{aligned}
(1-\lambda)&[f(At_\lambda, u_0, v_\lambda) + h(v_\lambda) - h(u_0)] \\
&\in \lambda[f(At_\lambda, v_\lambda, v) + h(v) - h(v_\lambda)] - f(At_\lambda, v_\lambda, v_\lambda) - C(u_0) \\
&\subseteq -\operatorname{int} C(u_0) - C(u_0) - C(u_0) \\
&\subseteq -\operatorname{int} C(u_0) - C(u_0) \\
&= -\operatorname{int} C(u_0).
\end{aligned} \tag{3.33}$$

Thus,

$$f(At_\lambda, u_0, v_\lambda) + h(v_\lambda) - h(u_0) \in -\operatorname{int} C(u_0), \tag{3.34}$$

which contradicts (3.26). Consequently

$$f(At_\lambda, v_\lambda, v) + h(v) - h(v_\lambda) \notin -\operatorname{int} C(u_0). \tag{3.35}$$

Since Tv_λ and Tu_0 are bounded closed subsets in $L(X,Y)$, by Lemma 2.3 for each $t_\lambda \in Tv_\lambda$ we can find an $s_\lambda \in Tu_0$ such that

$$||t_\lambda - s_\lambda|| \le (1+\lambda)H(Tv_\lambda, Tu_0). \tag{3.36}$$

Since $L(X,Y)$ is reflexive and Tu_0 is a bounded, closed, and convex subset in $L(X,Y)$, Tu_0 is a weakly compact subset in $L(X,Y)$. Hence, without loss of generality we may assume that $s_\lambda \rightharpoonup s_0 \in Tu_0$ as $\lambda \to 0^+$. Moreover, for each $\phi \in (L(X,Y))^*$ we have

$$\begin{aligned}
|\phi(t_\lambda - s_0)| &\le |\phi(t_\lambda - s_\lambda)| + |\phi(s_\lambda - s_0)| \\
&\le \|\phi\| ||t_\lambda - s_\lambda|| + |\phi(s_\lambda - s_0)| \\
&\le \|\phi\|(1+\lambda)H(Tv_\lambda, Tu_0) + |\phi(s_\lambda - s_0)|.
\end{aligned} \tag{3.37}$$

Since $H(Tv_\lambda, Tu_0) \to 0$ as $\lambda \to 0^+$, so $t_\lambda \rightharpoonup s_0$. Thus, according to condition (v) we have

$$||f(At_\lambda, v_\lambda, v) - f(As_0, v_\lambda, v)|| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+. \tag{3.38}$$

Since $h : D \to Y$ is weakly continuous and $f(z,\cdot,v) : D \to Y$ is continuous for each $(z,v) \in L(X,Y) \times D$, we deduce from (v) that

$$\begin{aligned} & f(At_\lambda, v_\lambda, v) + h(v) - h(v_\lambda) - f(As_0, u_0, v) - h(v) + h(u_0) \\ &\quad = f(At_\lambda, v_\lambda, v) - f(As_0, u_0, v) - (h(v_\lambda) - h(u_0)) \\ &\quad = f(At_\lambda, v_\lambda, v) - f(As_0, v_\lambda, v) + f(As_0, v_\lambda, v) - f(As_0, u_0, v) \\ &\qquad - (h(v_\lambda) - h(u_0)) \rightharpoonup 0 \quad \text{as } \lambda \longrightarrow 0^+, \end{aligned} \tag{3.39}$$

that is,

$$f(At_\lambda, v_\lambda, v) + h(v) - h(v_\lambda) \rightharpoonup f(As_0, u_0, v) + h(v) - h(u_0) \quad \text{as } \lambda \longrightarrow 0^+. \tag{3.40}$$

Therefore, it follows from (3.29) and the weak closedness of $Y \setminus (-\operatorname{int} C(u_0))$ that

$$f(As_0, u_0, v) + h(v) - h(u_0) \notin -\operatorname{int} C(u_0) \tag{3.41}$$

for all $v \in D$.

This completes the proof. □

Theorem 3.2. *Let X and Y be real Banach spaces, let D be a nonempty convex subset of X, and $\{C(u) : u \in D\}$ a family of closed proper convex solid cones of Y such that for each $u \in D$, $C(u) \neq Y$. Let $W : D \to 2^Y$ be a multifunction, defined by $W(u) = Y \setminus (-\operatorname{int} C(u))$, such that the graph $\operatorname{Gr}(W)$ is weakly closed in $X \times Y$. Suppose that the following conditions hold:*

(i) *for each $u, v \in D$, $f(At_\lambda, v_\lambda, v_\lambda) \in C(u)$, for all $t_\lambda \in Tv_\lambda$, and $f(At_\lambda, v_\lambda, u) + f(At_\lambda, u, v_\lambda) = 0$, where $v_\lambda := u + \lambda(v - u)$, $\lambda \in (0,1)$;*

(ii) *$f(z,\cdot,v), h(\cdot) : D \to Y$ are weakly continuous for each $(z,v) \in L(X,Y) \times D$;*

(iii) *$f(z,v,\cdot) + h(\cdot)$ is P-convex on D for each $(z,v) \in L(X,Y) \times D$;*

(iv) *there exists a bifunction $q : D \times D \to Y$ such that*

  (a) *$q(u,u) \notin -\operatorname{int} C(u)$, for all $u \in D$,*

  (b) *$q(u,v) - f(At,u,v) \in -C(u)$, for all $u, v \in D$, $t \in Tv$,*

  (c) *$\{v \in D : q(u,v) + h(v) - h(u) \in -\operatorname{int} C(u)\}$ is convex for each $u \in D$;*

(v) *there exists a weakly compact convex subset $K \subseteq D$ such that for each $u \in D \setminus K$ there exists $v_0 \in D$ satisfying*

$$f(At, u, v) + h(v) - h(u) \in -\operatorname{int} C(u), \quad \forall t \in Tv. \tag{3.42}$$

Then there exists a solution $u_0 \in D$ such that

$$f(At, v, u_0) + h(u_0) - h(v) \notin \operatorname{int} C(u_0) \tag{3.43}$$

for all $v \in D$ and $t \in Tv$.

Moreover, suppose additionally that $L(X,Y)$ is reflexive and $T : D \to 2^{L(X,Y)}$ is a multifunction which takes bounded, closed, and convex values in $L(X,Y)$ and satisfies the following conditions:

(vi) *for each net* $\{\lambda\} \subset (0,1)$ *such that* $\lambda \to 0^+$,

$$\left.\begin{array}{l} t_\lambda \rightharpoonup s_0, \\ t_\lambda \in Tv_\lambda \end{array}\right\} \Longrightarrow f(At_\lambda, v_\lambda, v) - f(As_0, v_\lambda, v) \rightharpoonup 0, \tag{3.44}$$

where $v_\lambda := u + \lambda(v-u)$ *for* $(u,v) \in D \times D$;

(vii) *for each* $u, v \in D$,

$$H(T(u+\lambda(v-u)), T(u)) \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+, \tag{3.45}$$

where H *is the Hausdorff metric defined on* $CB(L(X,Y))$.

Then there exists a solution $u_0 \in D$ *such that for some* $s_0 \in Tu_0$,

$$f(As_0, u_0, v) + h(v) - h(u_0) \notin -\operatorname{int} C(u_0) \quad \forall v \in D. \tag{3.46}$$

Proof. Define

$$G(v) = \{u \in K : f(At, u, v) + h(v) - h(u) \notin -\operatorname{int} C(u),\ \forall t \in Tv\}, \quad \forall v \in D. \tag{3.47}$$

Following the same proof as in Theorem 3.1, we can prove that $G(v)$ is weakly closed for each $v \in D$. We now claim that $\bigcap_{v \in D} G(v) \neq \emptyset$. Indeed, since K is weakly compact, it is sufficient to show that the family $\{G(v)\}_{v \in D}$ has the finite intersection property. Let $\{v_1, v_2, \ldots, v_n\}$ be a finite subset of D and set $B = \operatorname{co}\{K \cup \{v_1, v_2, \ldots, v_n\}\}$. Then B is a weakly compact and convex subset of D.

We define two vector multifunctions $F_1, F_2 : B \to 2^B$ as follows:

$$\begin{gathered} F_1(v) = \{u \in B : f(At, u, v) + h(v) - h(u) \notin -\operatorname{int} C(u),\ \forall t \in Tv\}, \quad \forall v \in B, \\ F_2(v) = \{u \in B : q(u, v) + h(v) - h(u) \notin -\operatorname{int} C(u)\}, \quad \forall v \in B. \end{gathered} \tag{3.48}$$

From condition (iv)(a), (b), we have

$$\begin{gathered} q(v, v) + h(v) - h(v) \notin -\operatorname{int} C(v), \quad \forall v \in B, \\ q(v, v) - f(At, v, v) \in -C(v), \quad \forall t \in Tv. \end{gathered} \tag{3.49}$$

Now Lemma 2.4(ii) guarantees that

$$f(At, v, v) + h(v) - h(v) \notin -\operatorname{int} C(v), \quad \forall t \in Tv, \tag{3.50}$$

and so $F_1(v)$ is nonempty. Since $F_1(v)$ is a weakly closed subset of the weakly compact subset B, we know that $F_1(v)$ is weakly compact.

Next we claim that F_2 is a KKM-map. Indeed, suppose that there exists a finite subset $\{u_1, u_2, \ldots, u_n\}$ of B and $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$, such that

$$\hat{u} = \sum_{i=1}^n \alpha_i u_i \in \bigcup_{j=1}^n F_2(u_j). \tag{3.51}$$

Then

$$q(\hat{u},u_j)+h(u_j)-h(\hat{u})\in-\operatorname{int}C(\hat{u}),\quad j=1,2,\dots,n. \tag{3.52}$$

From condition (iv)(c), we derive

$$q(\hat{u},\hat{u})=q(\hat{u},\hat{u})+h(\hat{u})-h(\hat{u})\in-\operatorname{int}C(\hat{u}), \tag{3.53}$$

which contradicts condition (iv)(a). Thus, F_2 is a KKM-map. From condition (iv)(b) and Lemma 2.4(ii), we have $F_2v\subseteq F_1(v)$, for all $v\in B$. Indeed, if $u\in F_2(v)$, then $q(u,v)+h(v)-h(u)\notin-\operatorname{int}C(u)$. By condition (iv)(b), we have

$$q(u,v)-f(At,u,v)\in-C(u),\quad\forall t\in Tv. \tag{3.54}$$

Consequently, it follows from Lemma 2.4(ii) that

$$f(At,u,v)+h(v)-h(u)\notin-\operatorname{int}C(u),\quad\forall t\in Tv, \tag{3.55}$$

that is, $u\in F_1(v)$. This shows that F_1 is also a KKM-map. According to Lemma 2.2, there exists $\overline{u}\in B$ such that $\overline{u}\in F_1(v)$ for all $v\in B$; that is, there exists $\overline{u}\in B$ such that

$$f(At,\overline{u},v)+h(v)-h(\overline{u})\notin-\operatorname{int}C(\overline{u}),\quad\forall v\in B,\ t\in Tv. \tag{3.56}$$

By condition (v), we get $\overline{u}\in K$ and $\overline{u}\in G(v_i)$, $i=1,2,\dots,n$. Hence, $\{G(v)\}_{v\in D}$ has the finite intersection property and moreover,

$$\bigcap_{v\in D}G(v)\neq\varnothing, \tag{3.57}$$

that is, there exists $u_0\in K\subseteq D$ such that

$$f(At,u_0,v)+h(v)-h(u_0)\notin-\operatorname{int}C(u_0) \tag{3.58}$$

for all $v\in D$ and $t\in Tv$.

For the remainder of the proof, we can derive the conclusion of Theorem 3.2 by following the same proof as in Theorem 3.1. □

Remark 3.3. The above existence theorems can be applied to deriving some existence results of solutions for the generalized vector variational-like inequalities. Here we omit them. It is worth pointing out that there are no assumptions of pseudomonotonicity in our existence results.

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References

[1] F. Giannessi, "Theorems of alternative, quadratic programs and complementarity problems," in *Variational Inequalities and Complementarity Problems (Proc. Internat. School, Erice, 1978)*, R. W. Cottle, F. Giannessi, and J.-L. Lions, Eds., pp. 151–186, John Wiley & Sons, Chichester, UK, 1980.

[2] Q. H. Ansari, A. H. Siddiqi, and J.-C. Yao, "Generalized vector variational-like inequalities and their scalarizations," in *Vector Variational Inequalities and Vector Equilibria*, F. Giannessi, Ed., vol. 38 of *Nonconvex Optim. Appl.*, pp. 17–37, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[3] G.-Y. Chen, C. J. Goh, and X. Q. Yang, "Existence of a solution for generalized vector variational inequalities," *Optimization*, vol. 50, no. 1-2, pp. 1–15, 2001.

[4] O. Chadli, X. Q. Yang, and J.-C. Yao, "On generalized vector pre-variational and prequasivariational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 2, pp. 392–403, 2004.

[5] M. F. Khan and Salahuddin, "On generalized vector variational-like inequalities," *Nonlinear Analysis*, vol. 59, no. 6, pp. 879–889, 2004.

[6] L.-C. Ceng and J.-C. Yao, "Generalized Minty's lemma for generalized vector equilibrium problems," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 32–37, 2007.

[7] I. V. Konnov and J.-C. Yao, "On the generalized vector variational inequality problem," *Journal of Mathematical Analysis and Applications*, vol. 206, no. 1, pp. 42–58, 1997.

[8] I. V. Konnov and J.-C. Yao, "Existence of solutions for generalized vector equilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 1, pp. 328–335, 1999.

[9] K. L. Lin, D.-P. Yang, and J.-C. Yao, "Generalized vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 92, no. 1, pp. 117–125, 1997.

[10] I. V. Konnov and S. Schaible, "Duality for equilibrium problems under generalized monotonicity," *Journal of Optimization Theory and Applications*, vol. 104, no. 2, pp. 395–408, 2000.

[11] Q. H. Ansari, I. V. Konnov, and J.-C. Yao, "Existence of a solution and variational principles for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 110, no. 3, pp. 481–492, 2001.

[12] B.-S. Lee and G.-M. Lee, "A vector version of Minty's lemma and application," *Applied Mathematics Letters*, vol. 12, no. 5, pp. 43–50, 1999.

[13] Q. H. Ansari, I. V. Konnov, and J.-C. Yao, "On generalized vector equilibrium problems," *Nonlinear Analysis*, vol. 47, no. 1, pp. 543–554, 2001.

[14] Q. H. Ansari, W. Oettli, and D. Schläger, "A generalization of vectorial equilibria," *Mathematical Methods of Operations Research*, vol. 46, no. 2, pp. 147–152, 1997.

[15] Q. H. Ansari and J.-C. Yao, "An existence result for the generalized vector equilibrium problem," *Applied Mathematics Letters*, vol. 12, no. 8, pp. 53–56, 1999.

[16] J. Li, N.-J. Huang, and J. K. Kim, "On implicit vector equilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 283, no. 2, pp. 501–512, 2003.

[17] L.-C. Ceng and J.-C. Yao, "An existence result for generalized vector equilibrium problems without pseudomonotonicity," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1320–1326, 2006.

[18] F. Giannessi, "On Minty variational principle," in *New Trends in Mathematical Programming*, vol. 13 of *Appl. Optim.*, pp. 93–99, Kluwer Academic Publishers, Boston, Mass, USA, 1998.

[19] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.
[20] K. Fan, "A generalization of Tychonoff's fixed point theorem," *Mathematische Annalen*, vol. 142, no. 3, pp. 305–310, 1961.

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