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Research Article

Convergece Theorems for Finite Families of Asymptotically Quasi-Nonexpansive Mappings

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Let E be a real Banach space, K a closed convex nonempty subset of E, and $T_1, T_2, \dots, T_m : K \to K$ asymptotically quasi-nonexpansive mappings with sequences (resp.) $\{k_{in}\}_{n=1}^{\infty}$ satisfying $k_{in} \to 1$ as $n \to \infty$, and $\sum_{n=1}^{\infty}(k_{in} - 1) < \infty$, $i = 1,2,\dots,m$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Define a sequence $\{x_n\}$ by $x_1 \in K$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_{n+m-2}$, $y_{n+m-2} = (1-\alpha_n)x_n + \alpha_n T_2^n y_{n+m-3}$, $\dots$, $y_n = (1-\alpha_n)x_n + \alpha_n T_m^n x_n$, $n \ge 1$, $m \ge 2$. Let $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Necessary and sufficient conditions for a strong convergence of the sequence $\{x_n\}$ to a common fixed point of the family $\{T_i\}_{i=1}^{m}$ are proved. Under some appropriate conditions, strong and weak convergence theorems are also proved.

1. Introduction

Let K be a nonempty subset of a real normed space E. A self-mapping $T : K \to K$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in K$, and *quasi-nonexpansive* if $F(T) := \{x \in K : Tx = x\} \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for every $x \in K$ and $p \in F(T)$. The mapping T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \in \mathbb{N}$,

$$||T^n x - T^n y|| \le k_n \|x - y\| \quad \text{for every } x, y \in K. \tag{1.1}$$

If $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that for

$n \in \mathbb{N}$,

$$||T^n x - p|| \le k_n \|x - p\| \quad \text{for every } x \in K, \tag{1.2}$$

and $p \in F(T)$, then T is called *asymptotically quasi-nonexpansive mapping*.

Iterative methods for approximating fixed points of nonexpansive mappings and their generalisations have been studied by numerous authors (see, e.g., [1–9] and the references contained therein).

Petryshyn and Williamson [4] proved necessary and sufficient conditions for the Picard and Mann [10] iterative sequences to strongly converge to a fixed point of a *quasi-nonexpansive* map T in a real Banach space.

Ghosh and Debnath [3] extended the results in [4] and proved necessary and sufficient conditions for strong convergence of Ishikawa-type [11] iteration process to a fixed point of a quasi-nonexpansive mapping T in a real Banach space. Furthermore, they proved strong convergence theorem of the Ishikawa-type iteration process for quasi-nonexpansive mappings in a *uniformly convex Banach space.*

Qihou [5] extended the results of Ghosh and Debnath to *asymptotically quasi-nonexpansive mappings.* In some other papers, Qihou [6, 7] studied the convergence of Ishikawa-type iteration process *with errors* for asymtotically quasi-nonexpansive mappings.

Recently, Sun [12] studied the convergence of an *implicit* iteration process (see [12] for definition) to a *common fixed point of finite family of asymptotically quasi-nonexpansive mappings.* He proved the following theorems.

Theorem 1.1 (see [12]). *Let K be a nonempty closed convex subset of a Banach space E. Let $\{T_i, i \in I\}$ be m asymptotically quasi-nonexpansive self-mappings of K with sequences $\{1 + u_{in}\}_n$, $i = 1,2,\dots,m$, respectively. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and that $x_0 \in K$, $\{\alpha_n\} \subset (s,1-s)$ for some $s \in (0,1)$, $\sum_{n=1}^\infty u_{in} < \infty$ for all $i \in I$. Then the implicit iterative sequence $\{x_n\}$ generated by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n, \quad n \ge 1, \qquad n = (k-1)m + i, \quad i = 1,2,\dots,m, \tag{1.3}$$

converges to a common fixed point in F if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$, where $d(x_n,F) = \inf_{x^* \in F} \|x_n - x^*\|$.

Theorem 1.2 (see [12]). *Let K be a nonempty closed convex and bounded subset of a real uniformly convex Banach space E. Let $\{T_i, i \in I\}$ be m uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of K with sequences $\{1 + u_{in}\}_n$, $i = 1,2,\dots,m$, respectively. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and that $x_0 \in K$, $\{\alpha_n\} \subset (s,1-s)$ for some $s \in (0,1)$, $\sum_{n=1}^\infty u_{in} < \infty$ for all $i \in I$. If there exists one member $T \in \{T_i, i \in I\}$ which is semi-compact, then the implicit iterative sequence $\{x_n\}$ generated by (1.3) converges strongly to a common fixed point of the mappings $\{T_i, i \in I\}$.*

Very recently, Shahzad and Udomene [8] proved necessary and sufficient conditions for the strong convergence of the Ishikawa-like iteration process to a common fixed point of *two* uniformly continuous asymptotically quasi-nonexpansive mappings.

Their main results are the following theorems.

Theorem 1.3 (see [8]). *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $S,T : K \to K$ be two asymptotically quasi-nonexpansive mappings (S and T need not be continuous) with sequences $\{u_n\},\{v_n\} \subset [0,\infty)$ such that $\sum u_n < \infty$ and $\sum v_n < \infty$, and $F := F(S) \cap F(T) = \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. From arbitrary $x_1 \in K$ define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n[(1-\beta_n)x_n + \beta_n T^n x_n]. \tag{1.4}$$

Then, $\{x_n\}$ converges strongly to some common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$.

Theorem 1.4 (see [8]). *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $S,T : K \to K$ be two uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{u_n\},\{v_n\} \subset [0,\infty)$ such that $\sum u_n < \infty$, $\sum v_n < \infty$, and $F := F(S) \cap F(T) = \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From arbitrary $x_1 \in K$ define a sequence $\{x_n\}$ by (1.4). Assume, in addition, that either T or S is compact. Then, $\{x_n\}$ converges strongly to some common fixed point of S and T.*

More recently, the authors [2] introduced a scheme defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= P\Big[(1-\alpha_{1n})x_n + \alpha_{1n}T_1(PT_1)^{n-1}y_{n+m-2}\Big], \\ y_{n+m-2} &= P\Big[(1-\alpha_{2n})x_n + \alpha_{2n}T_2(PT_2)^{n-1}y_{n+m-3}\Big], \\ &\vdots \\ y_n &= P\Big[(1-\alpha_{mn})x_n + \alpha_{mn}T_m(PT_m)^{n-1}x_n\Big], \quad n \geq 1, \end{aligned} \tag{1.5}$$

and studied the convergence of this sheme to a common fixed point of finite families of nonself asymptotically nonexpansive mappings.

Let $\{\alpha_n\}$ be a real sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $T_1,T_2,\dots,T_m : K \to K$ be a family of mappings. Define a sequence $\{x_n\}$ by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1^n y_{n+m-2}, \\ y_{n+m-2} &= (1-\alpha_n)x_n + \alpha_n T_2^n y_{n+m-3}, \\ &\vdots \\ y_n &= (1-\alpha_n)x_n + \alpha_n T_m^n x_n, \quad n \geq 1. \end{aligned} \tag{1.6}$$

It is our purpose in this paper to prove necessary and sufficient conditions for the strong convergence of the scheme defined by (1.6) to a common fixed point of finite family $T_1,T_2,\dots,T_m$ of *asymptotically quasi-nonexpansive mappings*. We also prove strong and weak convergence theorems for the family in a uniformly convex Banach spaces. Our results generalize and improve some recent important results (see Remark 3.9).

2. Preliminaries

Let E be a real normed linear space. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \epsilon = \|x-y\|\right\}. \tag{2.1}$$

E is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0\ \forall \epsilon \in (0,2]$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x^* \in D(T)$ and $Tx_n \to p$ then $Tx^* = p$.

A mapping $T : K \to K$ is said to be *semicompact* if, for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K.

A Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \forall y \in E,\ y \neq x. \tag{2.2}$$

We will say that a mapping T satisfies condition (P) if it satisfies the weak version of demiclosedness at origin as defined in [4] (i.e., if $\{x_{n_j}\}$ is any subsequence of a sequence $\{x_n\}$ with $x_{n_j} \rightharpoonup x^*$ and $(I-T)x_{n_j} \to 0$ as $j \to \infty$, then $x^* - Tx^* = 0$).

In what follows we will use the following results.

Lemma 2.1 (see [9]). *Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \leq \lambda_n + \sigma_n$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.*

Lemma 2.2 (see [13]). *Let $p > 1$ and $r > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|) \tag{2.3}$$

for all $x,y \in B_r(0) = \{z \in E : \|z\| \leq r\}$, $\lambda \in [0,1]$ and $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

3. Main results

In this section, we state and prove the main results of this paper. In the sequel, we designate the set $\{1,2,\dots,m\}$ by I and we always assume $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$.

Lemma 3.1. *Let E be a real normed linear space and let K be a nonempty, closed convex subset of E. Let $T_1, T_2, \dots, T_m : K \to K$ be asymptotically quasi-nonexpansive mappings with sequence $\{k_{in}\}_{n=1}^{\infty}$ satisfying $k_{in} \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(k_{in} - 1) < \infty$, $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be* *a sequences in* $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. *Let* $\{x_n\}$ *be a sequence defined iteratively by*

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1^n y_{n+m-2}, \\ y_{n+m-2} &= (1-\alpha_n)x_n + \alpha_n T_2^n y_{n+m-3}, \\ &\vdots \\ y_n &= (1-\alpha_n)x_n + \alpha_n T_m^n x_n, \quad n \geq 1,\ m \geq 2. \end{aligned} \tag{3.1}$$

Let $x^* \in F$. *Then,* $\{x_n\}$ *is bounded and the limits* $\lim_{n\to\infty} \|x_n - x^*\|$ *and* $\lim_{n\to\infty} d(x_n, F)$ *exist, where* $d(x_n, F) = \inf_{x^*\in F} \|x_n - x^*\|$.

Proof. Set $k_{in} = 1 + u_{in}$ so that $\sum_{n=1}^{\infty} u_{in} < \infty$ for each $i \in I$. Let $w_n := \sum_{i=1}^{m} u_{in}$. Let $x^* \in F$. Then we have, for some positive integer h, $2 \leq h < m$,

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T_1^n y_{n+m-2} - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1+u_{1n})\|y_{n+m-2} - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n(1+u_{1n})\Big[(1-\alpha_n)\|x_n - x^*\| + \alpha_n(1+u_{2n})\|y_{n+m-3} - x^*\|\Big] \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1-\alpha_n)(1+u_{1n})\|x_n - x^*\| \\ &\quad + \cdots + (\alpha_n)^{h-1}(1-\alpha_n)(1+u_{1n})(1+u_{2n})\cdots(1+u_{h-1n})\|x_n - x^*\| \\ &\quad + \cdots + (\alpha_n)^{m}(1+u_{1n})(1+u_{2n})\cdots(1+u_{mn})\|x_n - x^*\| \\ &\leq \|x_n - x^*\|\Big[1 + u_{1n} + u_{2n}(1+u_{1n}) + u_{3n}(1+u_{1n})(1+u_{2n}) + \cdots \\ &\qquad + u_{mn}(1+u_{1n})(1+u_{2n})\cdots(1+u_{m-1n})\Big] \\ &\leq \|x_n - x^*\|\left[1 + \binom{m}{1}w_n + \binom{m}{2}w_n^2 + \cdots + \binom{m}{m}w_n^m\right] \\ &\leq \|x_n - x^*\|(1+\delta_m w_n) \leq \|x_n - x^*\| e^{\delta_m w_n} \\ &\leq \|x_1 - x^*\| e^{\delta_m \sum_{n=1}^{\infty} w_n} < \infty, \end{aligned} \tag{3.2}$$

where δ_m is a positive real number defined by $\delta_m := [\binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m}]$.

This implies that $\{x_n\}$ is bounded and so there exists a positive integer M such that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \delta_m M w_n. \tag{3.3}$$

Since (3.3) is true for each x^* in F, we have

$$d(x_{n+1},F) \le d(x_n,F) + \delta_m M w_n. \tag{3.4}$$

By Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ and $\lim_{n\to\infty} d(x_n,F)$ exist. This completes the proof of Lemma 3.1. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \ldots, T_m : K \to K$ be asymptotically quasi-nonexpansive mappings with sequences $\{k_{in}\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ as in Lemma 3.1. Let $\{x_n\}$ be defined by (3.1). Then, $\{x_n\}$ converges to a common fixed point of the family $T_1, T_2, \ldots, T_m$ if and only if* $\liminf_{n\to\infty} d(x_n,F) = 0$.

Proof. The necessity is trivial. We prove the sufficiency. Let $\liminf_{n\to\infty} d(x_n,F) = 0$. Since $\lim_{n\to\infty} d(x_n,F)$ exists by Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n,F) = 0$. Thus, given $\epsilon > 0$ there exist a positive integer N_0 and $b^* \in F$ such that for all $n \ge N_0$ $\|x_n - b^*\| < \epsilon/2$. Then, for any $k \in \mathbb{N}$, we have for $n \ge N_0$,

$$||x_{n+k} - x_n|| \le ||x_{n+k} - b^*|| + ||b^* - x_n|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{3.5}$$

and so $\{x_n\}$ is Cauchy. Let $\lim_{n\to\infty} x_n = b$. We need to show that $b \in F$. Let $T_i \in \{T_1, T_2, \ldots, T_m\}$. Since $\lim_{n\to\infty} d(x_n,F) = 0$, there exists $N \in \mathbb{N}$ sufficiently large and $b^* \in F$ such that $n \ge N$ implies $\|b - x_n\| < \epsilon/6(1+w_1)$, $\|b^* - x_n\| < \epsilon/6(1+w_1)$. Then, $\|b^* - b\| < \epsilon/3(1+w_1)$. Thus, we have the following estimates, for $n \ge N$ and arbitrary T_i, $i = 1,2,\ldots,m$,

$$\begin{aligned} ||b - T_i b|| &\le ||b - x_n|| + ||x_n - b^*|| + ||b^* - T_i b|| \\ &\le ||b - x_n|| + ||x_n - b^*|| + (1+w_1)||b^* - b|| \\ &< \frac{\epsilon}{3(1+w_1)} + \frac{\epsilon}{3(1+w_1)} + \frac{\epsilon}{3} \le \epsilon. \end{aligned} \tag{3.6}$$

This implies that $b \in \mathrm{Fix}(T_i)$ for all $i = 1,2,\ldots,m$ and thus $b \in F$. This completes the proof. □

Corollary 3.3. *Let K be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \ldots, T_m : K \to K$ be quasi-nonexpansive mappings. Let the sequence $\{\alpha_n\}_{n=1}^{\infty}$ be as in Lemma 3.1. Let $\{x_n\}$ be defined by*

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_{n+m-2}, \\ y_{n+m-2} &= (1-\alpha_n)x_n + \alpha_n T_2 y_{n+m-3}, \\ &\vdots \\ y_n &= (1-\alpha_n)x_n + \alpha_n T_m x_n, \quad n \ge 1. \end{aligned} \tag{3.7}$$

Then, $\{x_n\}$ converges to a common fixed point of the family $T_1, T_2, \ldots, T_m$ if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$.

For our next theorems, we start by proving the following lemma which will be needed in the sequel.

Lemma 3.4. *Let E be a real uniformly convex Banach space and let K be a closed convex nonempty subset of E. Let $T_1, T_2, \dots, T_m : K \to K$ be uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{k_{in}\}_{n=1}^{\infty}$ satisfying $k_{in} \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(k_{in} - 1) < \infty$, $i = 1, 2, \dots, m$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $\{x_n\}$ be a sequence defined iteratively by (3.1). Then,*

$$\lim_{n\to\infty} ||x_n - T_1 x_n|| = \lim_{n\to\infty} ||x_n - T_2 x_n|| = \cdots = \lim_{n\to\infty} ||x_n - T_m x_n|| = 0. \tag{3.8}$$

Proof. Since $\{x_n\}$ is bounded, for some $x^* \in F$, there exists a positive real number γ such that $\|x_n - x^*\|^2 \le \gamma$ for all $n \ge 1$. By using Lemma 2.2 and the recursion formula (3.1), we have

$$\begin{aligned}
||y_n - x^*||^2 &= ||(1-\alpha_n)(x_n - x^*) + \alpha_n (T_m^n x_n - x^*)||^2 \\
&\le (1-\alpha_n)||x_n - x^*||^2 + \alpha_n (1+u_{mn})^2 ||x_n - x^*||^2 - \alpha_n(1-\alpha_n) g(||x_n - T_m^n x_n||) \\
&\le ||x_n - x^*||^2 + \alpha_n (2u_{mn} + u_{mn}^2)||x_n - x^*||^2 - \epsilon^2 g(||x_n - T_m^n x_n||) \\
&\le ||x_n - x^*||^2 + 3w_n\gamma - \epsilon^2 g(||x_n - T_m^n x_n||).
\end{aligned} \tag{3.9}$$

Also

$$\begin{aligned}
||y_{n+1} - x^*||^2 &= ||(1-\alpha_n)(x_n - x^*) + \alpha_n (T_{m-1}^n y_n - x^*)||^2 \\
&\le (1-\alpha_n)||x_n - x^*||^2 + \alpha_n (1+u_{m-1n})^2 ||y_n - x^*||^2 \\
&\quad - \alpha_n(1-\alpha_n) g(||x_n - T_{m-1}^n y_n||) \\
&\le (1-\alpha_n)||x_n - x^*||^2 + \alpha_n (1 + 2u_{m-1n} + u_{m-1n}^2)||y_n - x^*||^2 \\
&\quad - \epsilon^2 g(||x_n - T_{m-1}^n y_n||) \le (1-\alpha_n)||x_n - x^*||^2 \\
&\quad + \alpha_n (1+3u_{m-1n})[||x_n - x^*||^2 + 3w_n\gamma - \epsilon^2 g(||x_n - T_m^n x_n||)] \\
&\quad - \epsilon^2 g(||x_n - T_{m-1}^n y_n||) \\
&\le ||x_n - x^*||^2 + 3w_n\gamma - \epsilon^3 g(||x_n - T_m^n x_n||) + 3w_n\gamma + (3w_n)^2\gamma \\
&\quad - 3w_n\epsilon^3 g(||x_n - T_m^n x_n||) - \epsilon^2 g(||x_n - T_{m-1}^n y_n||) \\
&\le ||x_n - x^*||^2 + 3^3 w_n\gamma - \epsilon^3 [g(||x_n - T_m^n x_n||) + g(||x_n - T_{m-1}^n y_n||)].
\end{aligned} \tag{3.10}$$

Continuing in this fashion we get, using $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_{n+m-2}$, that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\le ||x_n - x^*||^2 + 3^{2m-1} w_n\gamma \\
&\quad - \epsilon^{m+1}\left(g(||x_n - T_m^n x_n||) + \sum_{k=1}^{m-1} g(||x_n - T_{m-k}^n y_{n+k-1}||)\right),
\end{aligned} \tag{3.11}$$

so that

$$\epsilon^{m+1}\left(g(||x_n - T_m^n x_n||) + \sum_{k=1}^{m-1} g(||x_n - T_{m-k}^n y_{n+k-1}||)\right) \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + 3^{2m-1} w_n \gamma. \tag{3.12}$$

This implies that

$$\epsilon^{m+1}\sum_{n=1}^{\infty}\left(g(||x_n - T_m^n x_n||) + \sum_{k=1}^{m-1} g(||x_n - T_{m-k}^n y_{n+k-1}||)\right) < \infty, \tag{3.13}$$

and by the property of g, we have

$$\begin{aligned}\lim_{n\to\infty}||x_n - T_m^n x_n|| &= \lim_{n\to\infty}||x_n - T_{m-1}^n y_n|| \\ &\ \ \vdots \\ &= \lim_{n\to\infty}||x_n - T_h^n y_{n+m-h-1}|| \\ &\ \ \vdots \\ &= \lim_{n\to\infty}||x_n - T_1^n y_{n+m-2}|| = 0\end{aligned} \tag{3.14}$$

for $2 \leq h < m$.

Now,

$$||x_n - T_h x_n|| \leq ||x_n - T_h^n y_{n+m-h-1}|| + ||T_h^n y_{n+m-h-1} - T_h x_n||, \tag{3.15}$$

but $(T_h^{n-1} y_{n+m-h-1} - x_n) \to 0$ as $n \to \infty$, and since T_h is uniformly continuous we have that $(T_h^n y_{n+m-1} - T_h x_n) \to 0$ as $n \to \infty$. So, from inequality (3.15), we get $\lim_{n\to\infty}\|x_n - T_h x_n\| = 0$. Also for $h = m$, from (3.14) we have

$$\lim_{n\longrightarrow\infty}||x_n - T_m^n x_n|| = 0. \tag{3.16}$$

Moreover,

$$||x_n - T_m x_n|| \leq ||x_n - T_m^n x_n|| + ||T_m^n x_n - T_m x_n||. \tag{3.17}$$

Similarly, since $\|T_m^{n-1} x_n - x_n\| \to 0$ as $n \to \infty$ and T_m is uniformly continuous, we have $(T_m^n x_n - T_m x_n) \to 0$ as $n \to \infty$ hence from (3.17) we get $\lim_{n\to\infty}\|x_n - T_m x_n\| = 0$, and this completes the proof. □

Theorem 3.5. *Let E be a real uniformly convex Banach space and let K be a closed convex nonempty subset of E. Let $T_1, T_2, \ldots, T_m : K \to K$ be uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{k_{in}\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ as in Lemma 3.4. If at* *least one member of $\{T_i\}_{i=1}^m$ is semicompact, then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^m$.*

Proof. Assume $T_d \in \{T_i\}_{i=1}^m$ is semicompact. Since $\{x_n\}$ is bounded and by Lemma 3.4 $\|x_n - T_d x_n\| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ converging strongly to say $x \in K$. By the uniform continuity of T_d, $x = T_d x$. Using $x_{n_j} \to x$, $\|x_{n_j} - T_i x_{n_j}\| \to 0$ as $j \to \infty$, and the continuity of T_i for each $i \in \{1,2,\dots,m\}$, we have that $x \in \bigcap_{i=1}^m \operatorname{Fix}(T_i)$. By Lemma 3.1, $\lim \|x_n - x\|$ exists, hence, $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^m$. □

Corollary 3.6. *Let E be a real uniformly convex Banach space and let K be a closed convex nonempty subset of E. Let $T_1, T_2, \dots, T_m : K \to K$ be uniformly continuous quasi-nonexpansive mappings. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence as in Corollary 3.3. If one of $\{T_i\}_{i=1}^m$ is semicompact, then $\{x_n\}$ defined by (3.7) converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^m$.*

We now prove weak convergence theorems.

Theorem 3.7. *Let E be a real uniformly convex Banach space and let K be a closed convex nonempty subset of E. Let $T_1, T_2, \dots, T_m : K \to K$ be uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{k_{in}\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ as in Lemma 3.4. If E satisfies Opial's condition and each T_i, $i \in I$, satisfies condition P, then the sequence $\{x_n\}$ defined by (3.1) converges weakly to a common fixed point of $\{T_i\}_{i=1}^m$.*

Proof. Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$, converging weakly to some point say $p \in K$. By Lemma 3.4, $\|x_{n_k} - T_i x_{n_k}\| \to 0$ as $k \to \infty$. Condition (P) of each T_i guarantees that $p \in \omega(\{x_n\}) \bigcap \bigcap_{i=1}^m \operatorname{Fix}(T_i)$. If we have another subsequence of $\{x_n\}$ converging to another point say $x' \in K$, by similar argument we can easily show that $x' \in \omega(\{x_n\}) \bigcap \bigcap_{i=1}^m \operatorname{Fix}(T_i)$. Since E satisfies Opial's condition, using standard argument we get that $x' = p$, completing the proof. □

The following corollary follows from Theorem 3.7.

Corollary 3.8. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2, \dots, T_m : K \to K$ be uniformly continuous quasi-nonexpansive mappings. Let the sequence $\{\alpha_n\}_{n=1}^\infty$ be as in Corollary 3.3. If E satisfies Opial's condition and at least one of the T_i's $i \in I$ satisfies condition P, then the sequence $\{x_n\}$ defined by (3.7) converges weakly to a common fixed point of $\{T_i\}_{i=1}^m$.*

Remark 3.9. Theorem 3.2 extends [8, Theorem 3.2]. In the same way, Theorem 3.5 extends [8, Theorem 3.4] to finite family of asymptotically quasi-nonexpansive mappings, and includes as a special case [8, Theorem 3.7]. In addition, the condition of compactness on the operators imposed in [8, Theorem 3.4] is weaken, replacing it by semicompactness in Theorem 3.5. It is clear that if T is compact, then it is semicompact and satisfies condition P. The scheme studied in [12] is implicit and *not* iterative. Our scheme is iterative.

Remark 3.10. Addition of bounded error terms to any of the recurrence relations in our iteration methods leads to no further generalization.

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