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Research Article

Inequalities in Additive N-isometries on Linear N-normed Banach Spaces

Choonkil Park and Themistocles M. Rassias

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We prove the generalized Hyers-Ulam stability of additive N-isometries on linear N-normed Banach spaces.

1. Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies

$$d_Y(f(x), f(y)) = d_X(x, y) \tag{1.1}$$

for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r, that is, for all x, y in X with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$. Then r is called a conservative (or preserved) distance for the mapping f. Aleksandrov [1] posed the following problem.

Aleksandrov problem. Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.

The Aleksandrov problem has been investigated in several papers (see [2, 3, 6–9, 13–15, 20, 23, 26, 28]). Rassias and Šemrl [25] proved the following theorem for mappings satisfying the strong distance one preserving property (SDOPP), that is, for every $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$ and conversely.

Theorem 1.1 [25]. *Let X and Y be real normed linear spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a Lipschitz mapping with Lipschitz constant $\kappa \le 1$. Assume that f is a surjective mapping satisfying SDOPP. Then f is an isometry.*

Definition 1.2 [4]. Let X be a real linear space with $\dim X \geq N$ and $\|\cdot,\ldots,\cdot\| : X^N \to \mathbb{R}$ a function. Then $(X, \|\cdot,\ldots,\cdot\|)$ is called a *linear N-normed space* if

(N_1) $\|x_1,\ldots,x_N\| = 0 \Leftrightarrow x_1,\ldots,x_N$ are linearly dependent;
(N_2) $\|x_1,\ldots,x_N\| = \|x_{j_1},\ldots,x_{j_N}\|$ for every permutation $(j_1,\ldots,j_N)$ of $(1,\ldots,N)$;
(N_3) $\|\alpha x_1,\ldots,x_N\| = |\alpha|\,\|x_1,\ldots,x_N\|$;
(N_4) $\|x+y,x_2,\ldots,x_N\| \leq \|x,x_2,\ldots,x_n\| + \|y,x_2,\ldots,x_N\|$

for all $\alpha \in \mathbb{R}$ and all $x,y,x_1,\ldots,x_N \in X$. The function $\|\cdot,\ldots,\cdot\|$ is called the *N-norm on X*.

Note that the notion of *1-norm* is the same as that of *norm*.

In [18], it was defined the notion of n-isometry and proved the Rassias and Šemrl's theorem in linear N-normed spaces.

Definition 1.3 [18]. $f : X \to Y$ is called an *N-Lipschitz mapping* if there is a $\kappa \geq 0$ such that

$$||f(x_1) - f(y_1),\ldots,f(x_N) - f(y_N)|| \leq \kappa||x_1 - y_1,\ldots,x_N - y_N|| \tag{1.2}$$

for all $x_1,\ldots,x_N, y_1,\ldots,y_N \in X$. The smallest such κ is called the *N-Lipschitz constant*.

Definition 1.4 [18]. Let X and Y be linear N-normed spaces and $f : X \to Y$ a mapping. f is called an *N-isometry* if

$$||x_1 - y_1,\ldots,x_N - y_N|| = ||f(x_1) - f(y_1),\ldots,f(x_N) - f(y_N)|| \tag{1.3}$$

for all $x_1,\ldots,x_N,\ y_1,\ldots,y_N \in X$.

For a mapping $f : X \to Y$, consider the following condition which is called the *N-distance one preserving property*: for $x_1,\ldots,x_N,y_1,\ldots,y_N \in X$ with $\|x_1 - y_1,\ldots,x_N - y_N\| = 1$, $\|f(x_1) - f(y_1),\ldots,f(x_N) - f(y_N)\| = 1$.

Definition 1.5 [5]. The points $x,y,z \in X$ are said to be *colinear* if $x - y$ and $x - z$ are linearly dependent.

Theorem 1.6 [18, Theorem 2.7]. *Let $f : X \to Y$ be an N-Lipschitz mapping with N-Lipschitz constant $\kappa \leq 1$. Assume that if x,y,z are colinear, then $f(x)$, $f(y)$, $f(z)$ are colinear, and that if $x_1 - y_1,\ldots,x_N - y_N$ are linearly dependent, then $f(x_1) - f(y_1),\ldots,f(x_N) - f(y_N)$ are linearly dependent. If f satisfies the N-distance one preserving property, then f is an N-isometry.*

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Rassias [19] introduced the following inequality: assume that there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that

$$||f(x+y) - f(x) - f(y)|| \leq \theta(\|x\|^p + \|y\|^p) \tag{$*$}$$

for all $x, y \in X$. Rassias [19] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$||f(x) - T(x)|| \leq \frac{2\theta}{2 - 2^p}\|x\|^p \tag{1.4}$$

for all $x \in X$. The inequality $(*)$ has provided a lot of influence in the development of what is known as *generalized Hyers–Ulam stability* of functional equations. Beginning around the year 1980, the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians (see [10–12, 16, 21, 22, 24]).

Trif [27] proved that, for vector spaces X and Y, a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation

$$d\,{}_{d-2}C_{l-2}f\left(\frac{x_1 + \cdots + x_d}{d}\right) + {}_{d-2}C_{l-1}\sum_{i=1}^{d} f(x_i) = d\sum_{1\leq i_1<\cdots<i_l\leq d} f\left(\frac{x_{i_1} + \cdots + x_{i_l}}{l}\right) \tag{T}$$

for all $x_1,\ldots,x_d \in X$ if and only if the mapping $f : X \to Y$ satisfies the Cauchy additive equation $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. Here ${}_dC_l := d!/l!(d-l)!$. He proved the stability of the functional equation (T) (see [27, Theorems 3.1 and 3.2]).

In [17], it was proved that, for vector spaces X and Y, a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation

$$\begin{aligned} &mn\,{}_{mn-2}C_{k-2}f\left(\frac{x_1 + \cdots + x_{mn}}{mn}\right) + m\,{}_{mn-2}C_{k-1}\sum_{i=1}^{n} f\left(\frac{x_{mi-m+1} + \cdots + x_{mi}}{m}\right) \\ &\quad = k\sum_{1\leq i_1<\cdots<i_k\leq mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \end{aligned} \tag{P}$$

for all $x_1,\ldots,x_{mn} \in X$ if and only if the mapping $f : X \to Y$ satisfies the Cauchy additive equation $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

In this paper, we introduce the concept of linear N-normed Banach space, and we prove the generalized Hyers-Ulam stability of additive N-isometries on linear N-normed Banach spaces.

2. Generalized Hyers-Ulam stability of additive N-isometries on linear N-normed Banach spaces

We define the notion of linear N-normed Banach space.

Definition 2.1. A linear N-normed and normed space X with N-norm $\|\cdot,\ldots,\cdot\|_X$ and norm $\|\cdot\|$ is called a *linear N-normed Banach space* if $(X, \|\cdot\|)$ is a Banach space.

In this section, assume that X is a linear N-normed Banach space with N-norm $\|\cdot,\ldots,\cdot\|_X$ and norm $\|\cdot\|$, and that Y is a linear N-normed Banach space with N-norm $\|\cdot,\ldots,\cdot\|_Y$ and norm $\|\cdot\|$.

Assume that $1 \leq N \leq d$. Note that the notion of "1-isomery" is the same as that of "isometry."

Let $q = l(d-1)/(d-l)$ and $r = -l/(d-l)$ for positive integers l, d with $2 \leq l \leq d-1$.

THEOREM 2.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^d \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x_1,\dots,x_d) := \sum_{j=0}^{\infty} \frac{1}{q^j}\varphi(q^j x_1,\dots,q^j x_d) < \infty, \tag{2.1}$$

$$\begin{aligned}&\bigg\| d\,{}_{d-2}C_{l-2}\, f\left(\frac{x_1+\cdots+x_d}{d}\right) + {}_{d-2}C_{l-1}\sum_{j=1}^{d} f(x_j)\\ &\quad - l \sum_{1\leq j_1<\cdots<j_l\leq d} f\left(\frac{x_{j_1}+\cdots+x_{j_l}}{l}\right)\bigg\| \leq \varphi(x_1,\dots,x_d),\end{aligned} \tag{2.2}$$

$$\big|\, \|f(x_1),\dots,f(x_N)\|_Y - \|x_1,\dots,x_N\|_X \,\big| \leq \varphi\Bigg(x_1,\dots,x_N, \underbrace{0,\dots,0}_{d-N\ \text{times}}\Bigg) \tag{2.3}$$

for all $x_1,\dots,x_d \in X$. Then there exists a unique additive N-isometry $U : X \to Y$ such that

$$\|f(x) - U(x)\| \leq \frac{1}{l\,{}_{d-1}C_{l-1}}\,\widetilde{\varphi}\Bigg(qx, \underbrace{rx,\dots,rx}_{d-1\ \text{times}}\Bigg) \tag{2.4}$$

for all $x \in X$.

Proof. By the Trif's theorem [27, Theorem 3.1], it follows from (2.1) and (2.2) that there exists a unique additive mapping $U : X \to Y$ satisfying (2.4). The additive mapping $U : X \to Y$ is given by

$$U(x) = \lim_{b\to\infty} \frac{1}{q^b} f(q^b x) \tag{2.5}$$

for all $x \in X$.

It follows from (2.3) that

$$\begin{aligned}&\left|\, \left\|\frac{1}{q^b} f(q^b x_1),\dots,\frac{1}{q^b} f(q^b x_N)\right\|_Y - \|x_1,\dots,x_N\|_X \,\right|\\ &\quad = \frac{1}{q^{bN}}\,\big|\, \|f(q^b x_1),\dots,f(q^b x_N)\|_Y - \|q^b x_1,\dots,q^b x_N\|_X \,\big|\\ &\quad \leq \frac{1}{q^{bN}}\varphi\Bigg(q^b x_1,\dots,q^b x_N, \underbrace{0,\dots,0}_{d-N\ \text{times}}\Bigg)\\ &\quad \leq \frac{1}{q^{b}}\varphi\Bigg(q^b x_1,\dots,q^b x_N, \underbrace{0,\dots,0}_{d-N\ \text{times}}\Bigg),\end{aligned} \tag{2.6}$$

which tends to zero as $b \to \infty$ for all $x_1,\ldots,x_N \in X$ by (2.1). By (2.5),

$$\|U(x_1),\ldots,U(x_N)\|_Y = \lim_{b\longrightarrow\infty}\left\|\frac{1}{q^b}f(q^bx_1),\ldots,\frac{1}{q^b}f(q^bx_N)\right\|_Y = \|x_1,\ldots,x_N\|_X \tag{2.7}$$

for all $x_1,\ldots,x_N \in X$. Since $U : X \to Y$ is additive,

$$\begin{aligned}&\|U(x_1)-U(y_1),\ldots,U(x_N)-U(y_N)\|_Y\\ &\quad = \|U(x_1-y_1),\ldots,U(x_N-y_N)\|_Y = \|x_1-y_1,\ldots,x_N-y_N\|_X\end{aligned} \tag{2.8}$$

for all $x_1,y_1,\ldots,x_N,y_N \in X$. So the additive mapping $U : X \to Y$ is an N-isometry, as desired. □

Corollary 2.3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0,1)$ such that*

$$\begin{aligned}&\Bigg\|d\,{}_{d-2}C_{l-2}f\left(\frac{x_1+\cdots+x_d}{d}\right)+{}_{d-2}C_{l-1}\sum_{j=1}^{d}f(x_j)\\ &\quad -l\sum_{1\le j_1<\cdots<j_l\le d}f\left(\frac{x_{j_1}+\cdots+x_{j_l}}{l}\right)\Bigg\| \le \theta\sum_{j=1}^{d}\|x_j\|^p,\end{aligned} \tag{2.9}$$

$$\big|\,\|f(x_1),\ldots,f(x_N)\|_Y - \|x_1,\ldots,x_N\|_X\big| \le \theta\sum_{j=1}^{N}\|x_j\|^p$$

for all $x_1,\ldots,x_d \in X$. Then there exists a unique additive N-isometry $U : X \to Y$ such that

$$\|f(x)-U(x)\| \le \frac{q^{1-p}(q^p+(d-1)r^p)\theta}{l\,{}_{d-1}C_{l-1}(q^{1-p}-1)}\|x\|^p \tag{2.10}$$

for all $x \in X$.

Proof. Define $\varphi(x_1,\ldots,x_d) = \theta\sum_{j=1}^{d}\|x_j\|^p$, and apply Theorem 2.2. □

From now on, let $q = l(d-1)/(d-l)$ and $r = -1/(d-1)$ for positive integers l, d with $2 \le l \le d-1$.

Theorem 2.4. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^d \to [0,\infty)$ satisfying (2.2) and (2.3) such that*

$$\sum_{j=0}^{\infty} q^{Nj}\varphi\left(\frac{x_1}{q^j},\ldots,\frac{x_d}{q^j}\right) < \infty \tag{2.11}$$

for all $x_1,\ldots,x_d \in X$. Then there exists a unique additive N-isometry $U : X \to Y$ such that

$$\|f(x)-U(x)\| \le \frac{1}{{}_{d-2}C_{l-1}}\widetilde{\varphi}\Bigg(x,\underbrace{rx,\ldots,rx}_{d-1\ \text{times}}\Bigg) \tag{2.12}$$

for all $x \in X$, where

$$\widetilde{\varphi}(x_1,\dots,x_d) := \sum_{j=0}^{\infty} q^j \varphi\left(\frac{x_1}{q^j},\dots,\frac{x_d}{q^j}\right) \tag{2.13}$$

for all $x_1,\dots,x_d \in X$.

Proof. Note that

$$q^j \varphi\left(\frac{x_1}{q^j},\dots,\frac{x_d}{q^j}\right) \le q^{Nj} \varphi\left(\frac{x_1}{q^j},\dots,\frac{x_d}{q^j}\right) \tag{2.14}$$

for all $x_1,\dots,x_d \in X$ and all positive integers j. By the Trif's theorem [27, Theorem 3.2], it follows from (2.2), (2.11), and (2.14) that there exists a unique additive mapping $U : X \to Y$ satisfying (2.12). The additive mapping $U : X \to Y$ is given by

$$U(x) = \lim_{b\to\infty} q^b f\left(\frac{x}{q^b}\right) \tag{2.15}$$

for all $x \in X$.

It follows from (2.3) that

$$\begin{aligned} &\left| \left\| q^b f\left(\frac{x_1}{q^b}\right),\dots,q^b f\left(\frac{x_N}{q^b}\right)\right\|_Y - \|x_1,\dots,x_N\|_X \right| \\ &\quad = q^{bN} \left| \left\| f\left(\frac{x_1}{q^b}\right),\dots,f\left(\frac{x_N}{q^b}\right)\right\|_Y - \left\|\frac{x_1}{q^b},\dots,\frac{x_N}{q^b}\right\|_X \right| \\ &\quad \le q^{bN} \varphi\left(\frac{x_1}{q^b},\dots,\frac{x_N}{q^b}, \underbrace{0,\dots,0}_{d-N\text{ times}}\right), \end{aligned} \tag{2.16}$$

which tends to zero as $b \to \infty$ for all $x_1,\dots,x_N \in X$ by (2.11). By (2.15),

$$\|U(x_1),\dots,U(x_N)\|_Y = \lim_{b\longrightarrow\infty} \left\| q^b f\left(\frac{x_1}{q^b}\right),\dots,q^b f\left(\frac{x_N}{q^b}\right)\right\|_Y = \|x_1,\dots,x_N\|_X \tag{2.17}$$

for all $x_1,\dots,x_N \in X$. Since $U : X \to Y$ is additive,

$$\begin{aligned} &\|U(x_1) - U(y_1),\dots,U(x_N) - U(y_N)\|_Y \\ &\quad = \|U(x_1 - y_1),\dots,U(x_N - y_N)\|_Y = \|x_1 - y_1,\dots,x_N - y_N\|_X \end{aligned} \tag{2.18}$$

for all $x_1, y_1,\dots,x_N, y_N \in X$. So the additive mapping $U : X \to Y$ is an N-isometry, as desired. □

Corollary 2.5. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in (N,\infty)$ satisfying (2.9). Then there exists a unique additive N-isometry $U : X \to Y$ such that*

$$\|f(x) - U(x)\| \le \frac{(1+(d-1)r^p)\theta}{{}_{d-2}C_{l-1}(1 - q^{1-p})}\|x\|^p \tag{2.19}$$

for all $x \in X$.

Proof. Define $\varphi(x_1,\ldots,x_d) = \theta\sum_{j=1}^{d}\|x_j\|^p$, and apply Theorem 2.4. □

Similarly, we can prove the corresponding results for the case $N > d$.

Now, assume that m, n, k are integers with $1 < m < k < mn$, and that s, q are integers with $1 \le s \le [n/2]$ and $1 < 2q \le m$, where $[\cdot]$ denotes the Gauss symbol. Assume that $1 \le N \le mn$.

Theorem 2.6. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^{mn} \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x_1,\ldots,x_{mn}) := \sum_{j=0}^{\infty}\frac{1}{2^j}\varphi(2^j x_1,\ldots,2^j x_{mn}) < \infty, \tag{2.20}$$

$$\begin{aligned}\Bigg\| mn\,{}_{mn-2}C_{k-2}f\left(\frac{x_1+\cdots+x_{mn}}{mn}\right) + m\,{}_{mn-2}C_{k-1}\sum_{i=1}^{n}f\left(\frac{x_{mi-m+1}+\cdots+x_{mi}}{m}\right)\\ -k\sum_{1\le i_1<\cdots<i_k\le mn}f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\Bigg\| \le \varphi(x_1,\ldots,x_{mn}),\end{aligned} \tag{2.21}$$

$$\big|\,\|f(x_1),\ldots,f(x_N)\|_Y - \|x_1,\ldots,x_N\|_X\,\big| \le \varphi\Big(x_1,\ldots,x_N,\underbrace{0,\ldots,0}_{mn\text{-}N\ times}\Big) \tag{2.22}$$

for all $x_1,\ldots,x_{mn} \in X$. Then there exists a unique additive N-isometry $U : X \to Y$ such that

$$\begin{aligned}&\|f(x) - U(x)\|\\ &\le \frac{1}{2ms\,{}_{mn-2}C_{k-1}}\widetilde{\varphi}\Bigg(\underbrace{0,\ldots,0}_{m-2q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{0,\ldots,0}_{q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{0,\ldots,0}_{m-q\ times},\ldots,\\ &\qquad\underbrace{0,\ldots,0}_{m-2q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{0,\ldots,0}_{q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{0,\ldots,0}_{m-q\ times},\underbrace{0,\ldots,0}_{mn-2ms\ times}\Bigg)\\ &+\frac{1}{2ms\,{}_{mn-2}C_{k-1}}\widetilde{\varphi}\Bigg(\underbrace{0,\ldots,0}_{m-2q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{0,\ldots,0}_{q\ times},\underbrace{0,\ldots,0}_{m-q\ times},\ldots,\\ &\qquad\underbrace{0,\ldots,0}_{m-2q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q\ times},\underbrace{0,\ldots,0}_{q\ times},\underbrace{0,\ldots,0}_{m-q\ times},\underbrace{0,\ldots,0}_{mn-2ms\ times}\Bigg)\end{aligned} \tag{2.23}$$

for all $x \in X$.

Proof. From [17, Theorem 3.1], it follows from (2.20) and (2.21) that there exists a unique additive mapping $U : X \to Y$ satisfying (2.23). The additive mapping $U : X \to Y$ is given by

$$U(x) = \lim_{d\to\infty} \frac{1}{2^d} f(2^d x) \tag{2.24}$$

for all $x \in X$.

It follows from (2.22) that

$$\begin{aligned} &\left| \left\| \frac{1}{2^d} f(2^d x_1),\dots,\frac{1}{2^d} f(2^d x_N) \right\|_Y - \|x_1,\dots,x_N\|_X \right| \\ &\quad = \frac{1}{2^{dN}} \left| \|f(2^d x_1),\dots,f(2^d x_N)\|_Y - \|2^d x_1,\dots,2^d x_N\|_X \right| \\ &\quad \le \frac{1}{2^{dN}} \varphi\Bigg(2^d x_1,\dots,2^d x_N, \underbrace{0,\dots,0}_{mn-N\text{ times}}\Bigg) \\ &\quad \le \frac{1}{2^{d}} \varphi\Bigg(2^d x_1,\dots,2^d x_N, \underbrace{0,\dots,0}_{mn-N\text{ times}}\Bigg), \end{aligned} \tag{2.25}$$

which tends to zero for all $x_1,\dots,x_N \in X$ by (2.20). By (2.24),

$$\|U(x_1),\dots,U(x_N)\|_Y = \lim_{d\to\infty} \left\| \frac{1}{2^d} f(2^d x_1),\dots,\frac{1}{2^d} f(2^d x_N) \right\|_Y = \|x_1,\dots,x_N\|_X \tag{2.26}$$

for all $x_1,\dots,x_N \in X$. Since $U : X \to Y$ is additive,

$$\begin{aligned} &\|U(x_1) - U(y_1),\dots,U(x_N) - U(y_N)\|_Y \\ &\quad = \|U(x_1 - y_1),\dots,U(x_N - y_N)\|_Y = \|x_1 - y_1,\dots,x_N - y_N\|_X \end{aligned} \tag{2.27}$$

for all $x_1, y_1,\dots,x_N, y_N \in X$. So the additive mapping $U : X \to Y$ is an N-isometry, as desired. □

Corollary 2.7. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0,1)$ such that*

$$\begin{aligned} &\Bigg\| mn\, {}_{mn-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_{mn}}{mn}\right) + m\, {}_{mn-2}C_{k-1} \sum_{i=1}^{n} f\left(\frac{x_{mi-m+1} + \cdots + x_{mi}}{m}\right) \\ &\qquad - k \sum_{1\le i_1<\cdots<i_k\le mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \Bigg\| \le \theta \sum_{j=1}^{mn} \|x_j\|^p, \end{aligned} \tag{2.28}$$

$$\left| \|f(x_1),\dots,f(x_N)\|_Y - \|x_1,\dots,x_N\|_X \right| \le \theta \sum_{j=1}^{N} \|x_j\|^p$$

for all $x_1,\dots,x_{mn} \in X$. Then there exists a unique additive N-isometry $U : X \to Y$ such that

$$\|f(x) - U(x)\| \le \frac{4m^{p-1}q^{1-p}\theta}{(2-2^p)\,{}_{mn-2}C_{k-1}}\|x\|^p \tag{2.29}$$

for all $x \in X$.

Proof. Define $\varphi(x_1,\dots,x_{mn}) = \theta\sum_{j=1}^{mn}\|x_j\|^p$, and apply Theorem 2.6. □

Theorem 2.8. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^{mn} \to [0,\infty)$ satisfying (2.21) and (2.22) such that*

$$\sum_{j=1}^{\infty} 2^{jN}\varphi\left(\frac{x_1}{2^j},\dots,\frac{x_{mn}}{2^j}\right) < \infty \tag{2.30}$$

for all $x_1,\dots,x_{mn} \in X$. Then there exists a unique additive N-isometry $U : X \to Y$ such that

$$\begin{aligned}\|f(x) - U(x)\| \\ \le \frac{1}{2ms\,{}_{mn-2}C_{k-1}}\widetilde{\varphi}\Bigg(&\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\dots, \\ &\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\underbrace{0,\dots,0}_{mn-2ms\text{ times}}\Bigg) \\ + \frac{1}{2ms\,{}_{mn-2}C_{k-1}}\widetilde{\varphi}\Bigg(&\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\dots, \\ &\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\underbrace{0,\dots,0}_{mn-2ms\text{ times}}\Bigg)\end{aligned} \tag{2.31}$$

for all $x \in X$, where

$$\widetilde{\varphi}(x_1,\dots,x_{mn}) := \sum_{j=1}^{\infty} 2^j\varphi\left(\frac{x_1}{2^j},\dots,\frac{x_{mn}}{2^j}\right) \tag{2.32}$$

for all $x_1,\dots,x_{mn} \in X$.

Proof. Note that

$$2^j\varphi\left(\frac{x_1}{2^j},\dots,\frac{x_{mn}}{2^j}\right)\le 2^{jN}\varphi\left(\frac{x_1}{2^j},\dots,\frac{x_{mn}}{2^j}\right) \tag{2.33}$$

for all $x_1,\dots,x_N\in X$ and all positive integers j. From [17, Theorem 3.3], it follows from (2.21), (2.30), and (2.33) that there exists a unique additive mapping $U:X\to Y$ satisfying (2.31). The additive mapping $U:X\to Y$ is given by

$$U(x)=\lim_{d\to\infty}2^d f\left(\frac{x}{2^d}\right) \tag{2.34}$$

for all $x\in X$.

It follows from (2.22) that

$$\begin{aligned}&\left|\left\|2^l f\left(\frac{x_1}{2^l}\right),\dots,2^l f\left(\frac{x_N}{2^l}\right)\right\|_Y-\|x_1,\dots,x_N\|_X\right|\\&=2^{lN}\left|\left\|f\left(\frac{x_1}{2^l}\right),\dots,f\left(\frac{x_N}{2^l}\right)\right\|_Y-\left\|\frac{x_1}{2^l},\dots,\frac{x_N}{2^l}\right\|_X\right|\\&\le 2^{lN}\varphi\left(\frac{x_1}{2^l},\dots,\frac{x_N}{2^l},\underbrace{0,\dots,0}_{mn-N\text{ times}}\right),\end{aligned} \tag{2.35}$$

which tends to zero $l\to\infty$ for all $x_1,\dots,x_N\in X$ by (2.30). By (2.34),

$$\|U(x_1),\dots,U(x_N)\|_Y=\lim_{l\to\infty}\left\|2^l f\left(\frac{x_1}{2^l}\right),\dots,2^l f\left(\frac{x_N}{2^l}\right)\right\|_Y=\|x_1,\dots,x_N\|_X \tag{2.36}$$

for all $x_1,\dots,x_N\in X$. Since $U:X\to Y$ is additive,

$$\begin{aligned}&\|U(x_1)-U(y_1),\dots,U(x_N)-U(y_N)\|_Y\\&\quad=\|U(x_1-y_1),\dots,U(x_N-y_N)\|_Y=\|x_1-y_1,\dots,x_N-y_N\|_X\end{aligned} \tag{2.37}$$

for all $x_1,y_1,\dots,x_N,y_N\in X$. So the additive mapping $U:X\to Y$ is an N-isometry, as desired. □

Corollary 2.9. *Let $f:X\to Y$ be a mapping with $f(0)=0$ for which there exist constants $\theta\ge 0$ and $p\in(N,\infty)$ satisfying (2.28). Then there exists a unique additive N-isometry $U:X\to Y$ such that*

$$\|f(x)-U(x)\|\le\frac{4m^{p-1}q^{1-p}\theta}{(2^p-2)\,{}_{mn-2}C_{k-1}}\|x\|^p p \tag{2.38}$$

for all $x\in X$.

Proof. Define $\varphi(x_1,\dots,x_{mn})=\theta\sum_{j=1}^{mn}\|x_j\|^p$, and apply Theorem 2.8. □

Similarly, we can prove the corresponding results for the case $N>mn$.

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Choonkil Park: Department of Mathematics, Hanyang University, Seoul 133-791, South Korea
Email address: baak@hanyang.ac.kr

Themistocles M. Rassias: Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece
Email address: trassias@math.ntua.gr