Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 72173, 9 pages
doi:10.1155/2007/72173

Research Article

Improvement of Aczél's Inequality and Popoviciu's Inequality

Shanhe Wu

Received 30 December 2006; Accepted 24 April 2007

Recommended by Laszlo I. Losonczi

We generalize and sharpen Aczél's inequality and Popoviciu's inequality by means of two classical inequalities, a unified improvement of Aczél's inequality and Popoviciu's inequality is given. As application, an integral inequality of Aczél-Popoviciu type is established.

1. Introduction

In 1956, Aczél [1] proved the following result:

$$\left(a_1^2-\sum_{i=2}^n a_i^2\right)\left(b_1^2-\sum_{i=2}^n b_i^2\right)\le\left(a_1b_1-\sum_{i=2}^n a_ib_i\right)^2, \tag{1.1}$$

where a_i, b_i $(i=1,2,\dots,n)$ are positive numbers such that $a_1^2-\sum_{i=2}^n a_i^2>0$ or $b_1^2-\sum_{i=2}^n b_i^2>0$. This inequality is called Aczél's inequality.

It is well known that Aczél's inequality has important applications in the theory of functional equations in non-Euclidean geometry. In recent years, this inequality has attracted the interest of many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations, improvements, and applications (see [2–11] and references therein). We state here a brief history on improvement of Aczél's inequality.

Popoviciu [12] first presented an exponential extension of Aczél's inequality, as follows.

Theorem 1.1. *Let $p>0$, $q>0$, $1/p+1/q=1$, and let a_i, b_i $(i=1,2,\ldots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\le a_1b_1-\sum_{i=2}^n a_ib_i. \tag{1.2}$$

Wu and Debnath [13] generalized inequality (1.2) in the following form.

Theorem 1.2. *Let $p>0$, $q>0$, and let a_i, b_i $(i=1,2,\ldots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\le n^{1-\min\{p^{-1}+q^{-1},1\}}a_1b_1-\sum_{i=2}^n a_ib_i. \tag{1.3}$$

In a recent paper [14], Wu established a sharp and generalized version of Popoviciu's inequality as follows.

Theorem 1.3. *Let $p>0$, $q>0$, $1/p+1/q\ge 1$, and let a_i, b_i $(i=1,2,\ldots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\le a_1b_1-\left(\sum_{i=2}^n a_ib_i\right)-\frac{a_1b_1}{\max\{p,q,1\}}\left(\sum_{i=2}^n\left(\frac{a_i^p}{a_1^p}-\frac{b_i^q}{b_1^q}\right)\right)^2. \tag{1.4}$$

In this paper, we show a new sharp and generalized version of Popoviciu's inequality, which is a unified improvement of Aczél's inequality and Popoviciu's inequality. In Section 4, the obtained result will be used to establish an integral inequality of Aczél-Popoviciu type.

2. Lemmas

In order to prove the theorem in Section 3, we first introduce the following lemmas.

Lemma 2.1 (generalized Hölder inequality [15, page 20]). *Let $a_{ij}>0$, $\lambda_j\ge 0$ $(i=1,2,\ldots,n,\ j=1,2,\ldots,m)$, and let $\lambda_1+\lambda_2+\cdots+\lambda_m=1$. Then*

$$\prod_{j=1}^m\left(\sum_{i=1}^n a_{ij}\right)^{\lambda_j}\ge\sum_{i=1}^n\prod_{j=1}^m a_{ij}^{\lambda_j} \tag{2.1}$$

with equality holding if and only if $a_{11}/a_{1j}=a_{21}/a_{2j}=\cdots=a_{n1}/a_{nj}$ $(j=2,3,\ldots,m)$ for $\lambda_1\lambda_2\cdots\lambda_n\ne 0$.

Lemma 2.2 (mean value inequality [16, page 17]). *Let $x_i>0$, $\lambda_i>0$ $(i=1,2,\ldots,n)$ and let $\lambda_1+\lambda_2+\cdots+\lambda_n=1$. Then*

$$\sum_{i=1}^n\lambda_ix_i\ge\prod_{i=1}^n x_i^{\lambda_i} \tag{2.2}$$

with equality holding if and only if $x_1=x_2=\cdots=x_n$.

Lemma 2.3. *Let* $p_1 \geq p_2 \geq \cdots \geq p_m > 0$, $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1$, $0 < x_j < 1$ $(j = 1,2,\ldots,m)$, *and let* $x_{m+1} = x_1$, $p_{m+1} = p_1$. *Then*

$$\prod_{j=1}^{m} x_j + \prod_{j=1}^{m} \left(1 - x_j^{p_j}\right)^{1/p_j} \leq 1 - \frac{1}{2p_1} \sum_{j=1}^{m} \left(x_j^{p_j} - x_{j+1}^{p_{j+1}}\right)^2 \tag{2.3}$$

with equality holding if and only if $x_1^{p_1} = x_2^{p_2} = \cdots = x_m^{p_m}$.

Proof. From hypotheses in Lemma 2.3, it is easy to verify that

$$\begin{gathered}
\frac{1}{p_m} \geq \frac{1}{p_{m-1}} \geq \cdots \geq \frac{1}{p_2} \geq \frac{1}{p_1} > 0, \\
\frac{1}{2p_2} - \frac{1}{2p_1} \geq 0, \frac{1}{2p_3} - \frac{1}{2p_2} \geq 0, \ldots, \frac{1}{2p_m} - \frac{1}{2p_{m-1}} \geq 0, \frac{1}{2p_m} - \frac{1}{2p_1} \geq 0,
\end{gathered}$$

$$\begin{aligned}
&\frac{1}{2p_1} + \frac{1}{2p_1} + \left(\frac{1}{2p_2} - \frac{1}{2p_1}\right) + \frac{1}{2p_2} + \frac{1}{2p_2} + \left(\frac{1}{2p_3} - \frac{1}{2p_2}\right) + \cdots + \frac{1}{2p_{m-2}} + \frac{1}{2p_{m-2}} \\
&\quad + \left(\frac{1}{2p_{m-1}} - \frac{1}{2p_{m-2}}\right) + \frac{1}{2p_{m-1}} + \frac{1}{2p_{m-1}} + \left(\frac{1}{2p_m} - \frac{1}{2p_{m-1}}\right) + \frac{1}{2p_1} + \frac{1}{2p_1} + \left(\frac{1}{2p_m} - \frac{1}{2p_1}\right) \\
&= \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.
\end{aligned} \tag{2.4}$$

Hence, by using Lemma 2.1 we obtain

$$\begin{aligned}
&\left[x_1^{p_1} + \left(1 - x_2^{p_2}\right)\right]^{1/2p_1} \left[x_2^{p_2} + \left(1 - x_1^{p_1}\right)\right]^{1/2p_1} \left[x_2^{p_2} + \left(1 - x_2^{p_2}\right)\right]^{1/2p_2 - 1/2p_1} \\
&\quad \times \left[x_2^{p_2} + \left(1 - x_3^{p_3}\right)\right]^{1/2p_2} \left[x_3^{p_3} + \left(1 - x_2^{p_2}\right)\right]^{1/2p_2} \left[x_3^{p_3} + \left(1 - x_3^{p_3}\right)\right]^{1/2p_3 - 1/2p_2} \\
&\qquad \vdots \\
&\quad \times \left[x_{m-2}^{p_{m-2}} + \left(1 - x_{m-1}^{p_{m-1}}\right)\right]^{1/2p_{m-2}} \\
&\quad \times \left[x_{m-1}^{p_{m-1}} + \left(1 - x_{m-2}^{p_{m-2}}\right)\right]^{1/2p_{m-2}} \left[x_{m-1}^{p_{m-1}} + \left(1 - x_{m-1}^{p_{m-1}}\right)\right]^{1/2p_{m-1} - 1/2p_{m-2}} \\
&\quad \times \left[x_{m-1}^{p_{m-1}} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_{m-1}} \left[x_m^{p_m} + \left(1 - x_{m-1}^{p_{m-1}}\right)\right]^{1/2p_{m-1}} \left[x_m^{p_m} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_m - 1/2p_{m-1}} \\
&\quad \times \left[x_m^{p_m} + \left(1 - x_1^{p_1}\right)\right]^{1/2p_1} \left[x_1^{p_1} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_1} \left[x_m^{p_m} + \left(1 - x_m^{p_m}\right)\right]^{1/2p_m - 1/2p_1} \\
&\geq x_1^{p_1/2p_1} x_2^{p_2/2p_1} x_2^{p_2/2p_2 - p_2/2p_1} x_2^{p_2/2p_2} \cdots x_{m-1}^{p_{m-1}/2p_{m-2}} x_{m-1}^{p_{m-1}/2p_{m-1} - p_{m-1}/2p_{m-2}} x_{m-1}^{p_{m-1}/2p_{m-1}} \\
&\quad \times x_m^{p_m/2p_{m-1}} x_m^{p_m/2p_m - p_m/2p_{m-1}} x_m^{p_m/2p_1} x_m^{p_m/2p_m - p_m/2p_1} x_1^{p_1/2p_1} \\
&\quad + \left(1 - x_1^{p_1}\right)^{1/2p_1} \left(1 - x_2^{p_2}\right)^{1/2p_1} \left(1 - x_2^{p_2}\right)^{1/2p_2 - 1/2p_1} \left(1 - x_2^{p_2}\right)^{1/2p_2} \\
&\quad \cdots \left(1 - x_{m-1}^{p_{m-1}}\right)^{1/2p_{m-2}} \left(1 - x_{m-1}^{p_{m-1}}\right)^{1/2p_{m-1} - 1/2p_{m-2}} \left(1 - x_{m-1}^{p_{m-1}}\right)^{1/2p_{m-1}} \\
&\quad \times \left(1 - x_m^{p_m}\right)^{1/2p_{m-1}} \left(1 - x_m^{p_m}\right)^{1/2p_m - 1/2p_{m-1}} \left(1 - x_m^{p_m}\right)^{1/2p_1} \left(1 - x_m^{p_m}\right)^{1/2p_m - 1/2p_1} \\
&\quad \times \left(1 - x_1^{p_1}\right)^{1/2p_1},
\end{aligned} \tag{2.5}$$

which is equivalent to

$$\begin{aligned}&\left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]^{1/2p_1}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]^{1/2p_2}\\&\qquad\cdots\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right]^{1/2p_{m-1}}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]^{1/2p_1}\\&\qquad\geq x_1x_2\cdots x_m+\left(1-x_1^{p_1}\right)^{1/p_1}\left(1-x_2^{p_2}\right)^{1/p_2}\cdots\left(1-x_m^{p_m}\right)^{1/p_m}.\end{aligned}\tag{2.6}$$

On the other hand, it follows from Lemma 2.2 that

$$\begin{aligned}&\frac{1}{2p_1}\left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]+\frac{1}{2p_2}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]+\cdots+\frac{1}{2p_{m-1}}\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right]\\&\qquad+\frac{1}{2p_1}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]+\left(\frac{1}{2p_2}+\frac{1}{2p_3}+\cdots+\frac{1}{2p_{m-1}}+\frac{1}{p_m}\right)\cdot 1\\&\qquad\geq\left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]^{1/2p_1}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]^{1/2p_2}\\&\qquad\quad\cdots\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right]^{1/2p_{m-1}}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]^{1/2p_1},\end{aligned}\tag{2.7}$$

this yields

$$\begin{aligned}&\left[1-\left(x_1^{p_1}-x_2^{p_2}\right)^2\right]^{1/2p_1}\left[1-\left(x_2^{p_2}-x_3^{p_3}\right)^2\right]^{1/2p_2}\cdots\left[1-\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2\right]^{1/2p_{m-1}}\left[1-\left(x_m^{p_m}-x_1^{p_1}\right)^2\right]^{1/2p_1}\\&\qquad\leq\left(\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}\right)-\frac{1}{2p_1}\left(x_1^{p_1}-x_2^{p_2}\right)^2-\frac{1}{2p_2}\left(x_2^{p_2}-x_3^{p_3}\right)^2\\&\qquad\quad-\cdots-\frac{1}{2p_{m-1}}\left(x_{m-1}^{p_{m-1}}-x_m^{p_m}\right)^2-\frac{1}{2p_1}\left(x_m^{p_m}-x_1^{p_1}\right)^2\\&\qquad\leq 1-\frac{1}{2p_1}\left[\left(x_1^{p_1}-x_2^{p_2}\right)^2+\left(x_2^{p_2}-x_3^{p_3}\right)^2+\cdots+\left(x_{m-1}^{p_{m-1}}+x_m^{p_m}\right)^2+\left(x_m^{p_m}-x_1^{p_1}\right)^2\right].\end{aligned}\tag{2.8}$$

Combining inequalities (2.6) and (2.8) leads to inequality (2.3). In addition, from Lemmas 2.1 and 2.2, we can easily deduce that the equality holds in both (2.6) and (2.8) if and only if $x_1^{p_1}=x_2^{p_2}=\cdots=x_m^{p_m}$, and thus we obtain the condition of equality in (2.3). The proof of Lemma 2.3 is complete. □

3. Improvement of Aczél's inequality and Popoviciu's inequality

Theorem 3.1. *Let $p_1\geq p_2\geq\cdots\geq p_m>0$, $1/p_1+1/p_2+\cdots+1/p_m=1$, $a_{ij}>0$, $a_{1j}^{p_j}-\sum_{i=2}^{n}a_{ij}^{p_j}>0$ $(i=1,2,\ldots,n,\ j=1,2,\ldots,m)$, and let $p_{m+1}=p_1$, $a_{im+1}=a_{i1}$ $(i=1,2,\ldots,n)$. Then one has the following inequality:*

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j}-\sum_{i=2}^{n}a_{ij}^{p_j}\right)^{1/p_j}\leq\prod_{j=1}^{m}a_{1j}-\sum_{i=2}^{n}\prod_{j=1}^{m}a_{ij}-\frac{a_{11}a_{12}\cdots a_{1m}}{2p_1}\sum_{j=1}^{m}\left(\sum_{i=2}^{n}\left(\frac{a_{ij}^{p_j}}{a_{1j}^{p_j}}-\frac{a_{ij+1}^{p_{j+1}}}{a_{1j+1}^{p_{j+1}}}\right)\right)^2.\tag{3.1}$$

Equality holds in (3.1) if and only if $a_{11}^{p_1}/a_{1j}^{p_j}=a_{21}^{p_1}/a_{2j}^{p_j}=\cdots=a_{n1}^{p_1}/a_{nj}^{p_j}$ $(j=2,3,\ldots,m)$.

Proof. Since by hypotheses in Theorem 3.1 we have

$$0 < \frac{\left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j}}{\left(a_{1j}^{p_j}\right)^{1/p_j}} < 1 \quad (j = 1,2,\dots,m), \tag{3.2}$$

it follows from Lemma 2.3, with a substitution $x_j = (a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j})^{1/p_j}/(a_{1j}^{p_j})^{1/p_j}$ $(j = 1,2,\dots,m)$ in (2.3), that

$$\begin{aligned}
&\prod_{j=1}^{m}\left(\frac{a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}}{a_{1j}^{p_j}}\right)^{1/p_j} + \prod_{j=1}^{m}\left(\frac{\sum_{i=2}^{n} a_{ij}^{p_j}}{a_{1j}^{p_j}}\right)^{1/p_j} \\
&\quad \le 1 - \frac{1}{2p_1}\sum_{j=1}^{m}\left(\frac{a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}}{a_{1j}^{p_j}} - \frac{a_{1j+1}^{p_{j+1}} - \sum_{i=2}^{n} a_{ij+1}^{p_{j+1}}}{a_{1j+1}^{p_{j+1}}}\right)^{2},
\end{aligned} \tag{3.3}$$

which is equivalent to

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j} \le \prod_{j=1}^{m} a_{1j} - \prod_{j=1}^{m}\left(\sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j} - \frac{a_{11}a_{12}\cdots a_{1m}}{2p_1}\sum_{j=1}^{m}\left(\sum_{i=2}^{n}\left(\frac{a_{ij}^{p_j}}{a_{1j}^{p_j}} - \frac{a_{ij+1}^{p_{j+1}}}{a_{1j+1}^{p_{j+1}}}\right)\right)^{2}, \tag{3.4}$$

where equality holds if and only if $(\sum_{i=2}^{n} a_{ij}^{p_j})/a_{1j}^{p_j} = (\sum_{i=2}^{n} a_{ij+1}^{p_{j+1}})/a_{1j+1}^{p_{j+1}}$ $(j = 1,2,\dots,m)$, that is, if and only if $a_{11}^{p_1}/a_{1j}^{p_j} = (\sum_{i=2}^{n} a_{i1}^{p_1})/(\sum_{i=2}^{n} a_{ij}^{p_j})$ $(j = 2,3,\dots,m)$.

On the other hand, using Lemma 2.1 gives

$$\prod_{j=1}^{m}\left(\sum_{i=2}^{n} a_{ij}^{p_j}\right)^{1/p_j} \ge \sum_{i=2}^{n}\prod_{j=1}^{m} a_{ij}, \tag{3.5}$$

where equality holds if and only if $a_{21}^{p_1}/a_{2j}^{p_j} = a_{31}^{p_1}/a_{3j}^{p_j} = \cdots = a_{n1}^{p_1}/a_{nj}^{p_j}$ $(j = 2,3,\dots,m)$.

Combining inequalities (3.4) and (3.5) leads to the desired inequality (3.1). By means of the conditions of equality in (3.4) and (3.5), it is easy to conclude that there is equality in (3.1) if and only if $a_{11}^{p_1}/a_{1j}^{p_j} = a_{21}^{p_1}/a_{2j}^{p_j} = \cdots = a_{n1}^{p_1}/a_{nj}^{p_j}$ $(j = 2,3,\dots,m)$. This completes the proof of Theorem 3.1. □

As a consequence of Theorem 3.1, putting $m = 2$, $p_1 = p$, $p_2 = q$, $a_{i1} = a_i$, $a_{i2} = b_i$ $(i = 1,2,\dots,n)$ in (3.1), we get the following.

Corollary 3.2. *Let $p \ge q > 0$, $1/p + 1/q = 1$, and let a_i, b_i $(i = 1,2,\dots,n)$ be positive numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p}\left(b_1^q - \sum_{i=2}^{n} b_i^q\right)^{1/q} \le a_1 b_1 - \left(\sum_{i=2}^{n} a_i b_i\right) - \frac{a_1 b_1}{p}\left(\sum_{i=2}^{n}\left(\frac{a_i^p}{a_1^p} - \frac{b_i^q}{b_1^q}\right)\right)^{2} \tag{3.6}$$

with equality holding if and only if $a_1^p/b_1^q = a_2^p/b_2^q = \cdots = a_n^p/b_n^q$.

A simple application of Corollary 3.2 yields the following sharp version of Popoviciu's inequality.

Corollary 3.3. *Let $p>0$, $q>0$, $1/p+1/q=1$, and let a_i, b_i $(i=1,2,\ldots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\leq a_1b_1-\left(\sum_{i=2}^n a_ib_i\right)-\frac{a_1b_1}{\max\{p,q\}}\left(\sum_{i=2}^n\left(\frac{a_i^p}{a_1^p}-\frac{b_i^q}{b_1^q}\right)\right)^2, \tag{3.7}$$

with equality holding if and only if $a_1^p/b_1^q=a_2^p/b_2^q=\cdots=a_n^p/b_n^q$.

Obviously, inequalities (3.1), (3.6), and (3.7) are the improvement of Aczél's inequality and Popoviciu's inequality.

4. Integral version of Aczél-Popoviciu-type inequality

As application of Theorem 3.1, we establish here an interesting integral inequality of Aczél-Popoviciu type.

Theorem 4.1. *Let $p_1\geq p_2\geq\cdots\geq p_m>0$, $1/p_1+1/p_2+\cdots+1/p_m=1$, $B_j>0$ $(j=1,2,\ldots,m)$, let f_j be positive Riemann integrable functions on $[a,b]$ such that $B_j^{p_j}-\int_a^b f_j^{p_j}(x)dx>0$ for all $j=1,2,\ldots,m$, and let $B_{m+1}=B_1$, $p_{m+1}=p_1$, $f_{m+1}=f_1$. Then one has the following inequality:*

$$\begin{aligned}&\prod_{j=1}^m\left(B_j^{p_j}-\int_a^b f_j^{p_j}(x)dx\right)^{1/p_j}\\&\quad\leq\prod_{j=1}^m B_j-\int_a^b\left(\prod_{j=1}^m f_j(x)\right)dx-\frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^m\left(\int_a^b\left(\frac{f_j^{p_j}(x)}{B_j^{p_j}}-\frac{f_{j+1}^{p_{j+1}}(x)}{B_{j+1}^{p_{j+1}}}\right)dx\right)^2.\end{aligned} \tag{4.1}$$

Proof. For any positive integer n, we choose an equidistant partition of $[a,b]$ as

$$\begin{gathered}a<a+\frac{b-a}{n}<\cdots<a+\frac{b-a}{n}i<\cdots<a+\frac{b-a}{n}(n-1)<b,\\ \Delta x_i=\frac{b-a}{n},\quad i=1,2,\ldots,n.\end{gathered} \tag{4.2}$$

Since the hypothesis $B_j^{p_j}-\int_a^b f_j^{p_j}(x)dx>0$ $(j=1,2,\ldots,m)$ implies that

$$B_j^{p_j}-\lim_{n\to\infty}\sum_{i=1}^n f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}>0\quad(j=1,2,\ldots,m), \tag{4.3}$$

there exists a positive integer N such that

$$B_j^{p_j}-\sum_{i=1}^n f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}>0\quad\forall n>N,\ j=1,2,\ldots,m. \tag{4.4}$$

Applying Theorem 3.1, one obtains for any $n > N$ the following inequality:

$$\begin{aligned}
\prod_{j=1}^{m}\left[B_j^{p_j} - \sum_{i=1}^{n} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{1/p_j}
&\le \prod_{j=1}^{m} B_j - \sum_{i=1}^{n}\left(\prod_{j=1}^{m} f_j\left(a+\frac{i(b-a)}{n}\right)\right)\left(\frac{b-a}{n}\right)^{1/p_1+1/p_2+\cdots+1/p_m} \\
&\quad - \frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^{m}\left[\sum_{i=1}^{n}\left(\frac{1}{B_j^{p_j}} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right.\right. \\
&\quad \left.\left. - \frac{1}{B_{j+1}^{p_{j+1}}} f_{j+1}^{p_{j+1}}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right)\right]^2.
\end{aligned} \tag{4.5}$$

Note that $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1$, the above inequality can be transformed to

$$\begin{aligned}
\prod_{j=1}^{m}\left[B_j^{p_j} - \sum_{i=1}^{n} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\frac{b-a}{n}\right]^{1/p_j}
&\le \prod_{j=1}^{m} B_j - \sum_{i=1}^{n}\left(\prod_{j=1}^{m} f_j\left(a+\frac{i(b-a)}{n}\right)\right)\left(\frac{b-a}{n}\right) \\
&\quad - \frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^{m}\left[\sum_{i=1}^{n}\left(\frac{1}{B_j^{p_j}} f_j^{p_j}\left(a+\frac{i(b-a)}{n}\right)\right.\right. \\
&\quad \left.\left. - \frac{1}{B_{j+1}^{p_{j+1}}} f_{j+1}^{p_{j+1}}\left(a+\frac{i(b-a)}{n}\right)\right)\frac{b-a}{n}\right]^2,
\end{aligned} \tag{4.6}$$

where equality holds if and only if $f_j^{p_j}(a+i(b-a)/n)/B_j^{p_j} = f_{j+1}^{p_{j+1}}(a+i(b-a)/n)/B_{j+1}^{p_{j+1}}$ for all $i = 1,2,\ldots,n$ $(j = 1,2,\ldots,m)$.

In view of the hypotheses that f_j are positive Riemann integrable functions on $[a,b]$ and $p_j > 0$ $(j = 1,2,\ldots,m)$, we conclude that $\prod_{j=1}^{m} f_j$ and $f_j^{p_j}$ $(j = 1,2,\ldots,m)$ are also integrable on $[a,b]$. Passing the limit as $n \to \infty$ in both sides of inequality (4.6), we obtain the inequality (4.1). The proof of Theorem 4.1 is complete. □

Remark 4.2. Motivated by the proof of Theorem 4.1, we propose here a conjecture.

Conjecture 4.3. Suppose that $p_1 \ge p_2 \ge \cdots \ge p_m > 0$, $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1$, $B_j > 0$ $(j = 1,2,\ldots,m)$, suppose also that $f_j \in L^{p_j}[a,b]$, $B_j^{p_j} - \int_a^b |f_j(x)|^{p_j} dx > 0$ for all $j = 1,2,\ldots,m$, let $B_{m+1} = B_1$, $p_{m+1} = p_1$, $f_{m+1} = f_1$. Then the following inequality holds true:

$$\begin{aligned}
&\prod_{j=1}^{m}\left(B_j^{p_j} - \int_a^b |f_j(x)|^{p_j} dx\right)^{1/p_j} \\
&\quad\le \prod_{j=1}^{m} B_j - \int_a^b \left(\prod_{j=1}^{m} |f_j(x)|\right) dx - \frac{B_1B_2\cdots B_m}{2p_1}\sum_{j=1}^{m}\left(\int_a^b\left(\frac{|f_j(x)|^{p_j}}{B_j^{p_j}} - \frac{|f_{j+1}(x)|^{p_{j+1}}}{B_{j+1}^{p_{j+1}}}\right)dx\right)^2
\end{aligned} \tag{4.7}$$

with equality holding if and only if $|f_j(x)|^{p_j}/B_j^{p_j} = |f_{j+1}(x)|^{p_{j+1}}/B_{j+1}^{p_{j+1}}$ $(j = 1,2,\ldots,m)$ almost everywhere on $[a,b]$.

As a consequence of Theorem 4.1, putting $m = 2$, $p_1 = p$, $p_2 = q$, $B_1 = A$, $B_2 = B$, $f_1 = f$, $f_2 = g$ in (4.1), we obtain the following.

Corollary 4.4. *Let $p \ge q > 0$, $1/p + 1/q = 1$, $A > 0$, $B > 0$, and let f, g be positive Riemann integrable functions on $[a,b]$ such that $A^p - \int_a^b f^p(x)dx > 0$ and $B^q - \int_a^b g^q(x)dx > 0$. Then*

$$\begin{aligned}\left(A^p - \int_a^b f^p(x)dx\right)^{1/p}\left(B^q - \int_a^b g^q(x)dx\right)^{1/q} \\ \le AB - \int_a^b f(x)g(x)dx - \frac{AB}{p}\left(\int_a^b\left(\frac{f^p(x)}{A^p} - \frac{g^q(x)}{B^q}\right)dx\right)^2.\end{aligned} \tag{4.8}$$

Further, from Corollary 4.4 we have the following.

Corollary 4.5. *Let $p > 0$, $q > 0$, $1/p + 1/q = 1$, $A > 0$, $B > 0$, and let f, g be positive Riemann integrable functions on $[a,b]$ such that $A^p - \int_a^b f^p(x)dx > 0$ and $B^q - \int_a^b g^q(x)dx > 0$. Then*

$$\begin{aligned}\left(A^p - \int_a^b f^p(x)dx\right)^{1/p}\left(B^q - \int_a^b g^q(x)dx\right)^{1/q} \\ \le AB - \int_a^b f(x)g(x)dx - \frac{AB}{\max\{p,q\}}\left(\int_a^b\left(\frac{f^p(x)}{A^p} - \frac{g^q(x)}{B^q}\right)dx\right)^2.\end{aligned} \tag{4.9}$$

Acknowledgment

The author would like to express hearty thanks to the anonymous referees for valuable comments on this paper.

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Shanhe Wu: Department of Mathematics, Longyan College, Longyan, Fujian 364012, China
Email address: wushanhe@yahoo.com.cn