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Research Article

On the Generalized Favard-Kantorovich and Favard-Durrmeyer Operators in Exponential Function Spaces

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We consider the Kantorovich- and the Durrmeyer-type modifications of the generalized Favard operators and we prove an inverse approximation theorem for functions f such that $w_\sigma f \in L_p(R)$, where $1 \le p \le \infty$ and $w_\sigma(x) = \exp(-\sigma x^2)$, $\sigma > 0$.

1. Preliminaries

Let

$$L_{p,\sigma}(R) = \{f : \|w_\sigma f\|_p < \infty\} \quad \text{for } 1 \le p \le \infty \tag{1.1}$$

be the weighted function space, where $w_\sigma(x) = \exp(-\sigma x^2)$, $\sigma > 0$,

$$\begin{aligned} \|g\|_p &= \left(\int_{-\infty}^{\infty} |g(x)|^p dx\right)^{1/p} \quad \text{if } 1 \le p < \infty, \\ \|g\|_\infty &= \operatorname*{essup}_{x\in R} |g(x)|. \end{aligned} \tag{1.2}$$

We define the generalized Favard operators F_n for functions $f : R \to R$ by

$$F_n f(x) = \sum_{k=-\infty}^{\infty} f(k/n) p_{n,k}(x;\gamma) \quad (x \in R,\ n \in N), \tag{1.3}$$

where $N = \{1, 2, \ldots\}$,

$$p_{n,k}(x;\gamma) = \frac{1}{n\gamma_n\sqrt{2\pi}} \exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{k}{n}-x\right)^2\right) \tag{1.4}$$

and $\gamma = (\gamma_n)_{n=1}^{\infty}$ is a positive sequence convergent to zero (see [1]). In the case where $\gamma_n^2 = \vartheta/(2n)$ with a positive constant ϑ, F_n become the known Favard operators introduced by Favard [2]. Some approximation properties of the classical Favard operators for continuous functions f on R are presented in [3, 4]. Some approximation properties of their generalization can be found, for example, in [1, 5]. Denote by F_n^* the Kantorovich-type modification of operators F_n, defined by

$$F_n^* f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt \quad (x \in R,\ n \in N), \tag{1.5}$$

and by $\widetilde{F}_n$ the Durrmeyer-type modification of operators F_n

$$\widetilde{F}_n f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)dt \qquad (x \in R,\ n \in N), \tag{1.6}$$

where $f \in L_{p,\sigma}(R)$. Some estimates concerning the rates of pointwise convergence of the operators $F_n^* f$ and $\widetilde{F}_n f$ can be found in [6, 7].

Recently, several autors investigated the conditions under which global smoothness of a function f, as measured by its modulus of continuity $\omega(f;\circ)$, is retained by the elements of approximating sequences $(L_n f)$ (see, e.g., [8, 9]). For example, Kratz and Stadtmüller considered in [10] a wide class of discrete operators L_n and derived estimates of the form

$$\omega(L_n f;t) \le K\omega(f;t) \quad (t > 0), \tag{1.7}$$

with a positive constant K independent of f, n, and t. For bounded functions $f \in C(R)$ and operators F_n satisfying

$$\gamma_1^2 \ge \frac{1}{2}\pi^{-2}\log 2, \qquad n^2\gamma_n^2 \ge \frac{1}{2}\pi^{-2}\log n \quad \text{if } n \ge 2, \tag{1.8}$$

they obtained the inequality

$$\omega(F_n f;t) \le 140\omega(f;t) + 16\pi \cdot t\|f\| \quad (t > 0), \tag{1.9}$$

where $\|f\| = \sup\{|f(x)| : x \in R\}$.

Forbounded functions $f \in C_m(R) = \{f : \|w_m f\|_\infty < \infty\}$, $w_m(x) = (1 + x^{2m})^{-1}$, $m \in N$ and for operators F_n satisfying $n\gamma_n^2 \ge c > 0$ for all $n \in N$, Pych-Taberska [5] obtained the inequality

$$\omega_2(F_n f;t)_m \le K\{(1+t_0^2)\omega_2(f;t)_m + t^2\|f\|_m\} \quad (0 < t \le t_0) \tag{1.10}$$

for all $n \in N$, $n \ge n_c$ where $n_c \in N$ and K is a constant.

In this paper, we obtained an analogous inequality for the rth weighted modulus of smoothness of the function $f \in L_{p,\sigma}(R), \sigma > 0, 1 \le p \le \infty$,

$$\omega_r(f;t)_{\sigma,p} = \sup_{0<h\le t} \|w_\sigma \Delta_h^r f\|_p \quad (r \in N), \tag{1.11}$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f(x + h(r/2 - i)). \tag{1.12}$$

Namely, suppose that (γ_n) is a positive null sequence satisfying $n\gamma_n^{r/2+1} \ge c \max_{n\in N}\{\gamma_n^{r/2-1}\} > 0$ for all $n \in N$ and $\sigma_1 > \sigma > 0$. Then there exist positive constants, K, K_1, such that for all $n \ge K_1$ and for arbitrary positive number t_0

$$\omega_r(L_n f,t)_{\sigma_1,p} \le K\left\{(1+t_0^2)\omega_r(f,t)_{\sigma,p} + t^r \|w_\sigma f\|_p\right\} \quad (0 < t \le t_0), \tag{1.13}$$

where L_n denotes the Favard-Kantorovich operator or the Favard-Durrmeyer operator.

Throughout the paper, the symbols $K(\sigma,\sigma_1,\dots)$, $K_j(\sigma,\sigma_1,\dots)$ $(j = 1,2,\dots)$ will mean some positive constants, not necessarily the same at each occurrence, depending only on the parameters indicated in parentheses.

2. Preliminary results

Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence and let $n\gamma_n^2 \ge c$ for all $n \in N$, with a positive absolute constant c. As is known [5], for $\nu \in N_0 = \{0\} \cup N$, $n \in N$, $x \in R$,

$$\sum_{k=-\infty}^{\infty} \left|\frac{k}{n} - x\right|^{\nu} p_{n,k}(x;\gamma) \le 15 A_c \left(\frac{2}{e}\right)^{\nu/2} \sqrt{(2\nu)!}\,\gamma_n^{\nu}, \tag{2.1}$$

where $A_c = \max\{1, (2c\pi^2)^{-1}\}$. A simple calculation and the known Schwarz inequality lead to

$$\int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right|^{\nu} p_{n,k}(t;\gamma)\,dt \le \sqrt{(2\nu)!!}\,\frac{\gamma_n^{\nu}}{n} \quad (k \in Z = \{0, \pm 1, \pm 2, \dots\}). \tag{2.2}$$

Let us choose $n \in N$, $j \in N_0$ and let us write

$$G_{n,j}^* f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\frac{k}{n} - x\right)^j \int_{k/n}^{(k+1)/n} f(t)\,dt, \tag{2.3}$$

$$\widetilde{G}_{n,j} f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\frac{k}{n} - x\right)^j \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)\,dt, \tag{2.4}$$

where $f \in L_{p,\sigma}(R)$, $1 \le p \le \infty$, $\sigma > 0$. Obviously, $G_{n,0}^* f(x) = F_n^* f(x)$ and $\widetilde{G}_{n,0} f(x) = \widetilde{F}_n f(x)$.

Lemma 2.1. *Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence convergent to 0 and let $n\gamma_n^2 \geq c$ for all $n \in N$, with a positive absolute constant c. Then for $j \in N_0$, $f \in L_{p,\sigma}(R)$, $\sigma > 0$, $1 \leq p \leq \infty$, and $\sigma_1 > \sigma$,*

$$\|w_{\sigma_1} G^*_{n,j} f\|_p \leq 15A_c \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{(2j)!2^{j/2}\gamma_n^j}\|w_\sigma f\|_p \tag{2.5}$$

for all $n \in N$ such that $\gamma_n^2 \leq (\sigma_1 - \sigma)/(4\sigma(\sigma+\sigma_1))$,

$$\|w_{\sigma_1} \widetilde{G}_{n,j} f\|_p \leq 30A_c\sqrt{(2j)!2^{j/2}\gamma_n^j}\|w_\sigma f\|_p \tag{2.6}$$

for all $n \in N$ such that $\gamma_n \leq \max\{(\sigma_1-\sigma)/(2\sqrt{\sigma}(\sigma+\sigma_1));(\sqrt{\sigma_1-\sigma})/(\sqrt{2}(\sigma+\sigma_1))\}$.

Proof. In view of definition (2.3),

$$\begin{aligned}\exp(-\sigma_1 x^2)|G^*_{n,j}f(x)| \leq n\sum_{k=-\infty}^{\infty}\exp(-\sigma_1x^2)p_{n,k}(x;\gamma)\left|\frac{k}{n}-x\right|^j \\ \times \exp\left(\sigma(|k|+1)^2/n^2\right)\int_{k/n}^{(k+1)/n}\exp(-\sigma t^2)|f(t)|dt.\end{aligned} \tag{2.7}$$

Using the inequality

$$(u+v)^2 \leq \frac{\sigma+\sigma_1}{2\sigma}u^2 + \frac{\sigma+\sigma_1}{\sigma_1-\sigma}v^2 \quad (u \in R,\ v \in R), \tag{2.8}$$

we can easily observe, that

$$\begin{aligned}p_{n,k}(x;\gamma)\exp(-\sigma_1x^2)\exp\left(\sigma\left(\frac{k+1}{n}\right)^2\right) &\leq \sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)p_{n,k}(x;\sqrt{2}\gamma), \\ p_{n,k}(x;\gamma)\exp(-\sigma_1x^2)\exp\left(\sigma\left(\frac{k}{n}\right)^2\right) &\leq \sqrt{2}p_{n,k}(x;\sqrt{2}\gamma),\end{aligned} \tag{2.9}$$

for $n \in N$ such that $\gamma_n^2 \leq (\sigma_1-\sigma)/(4\sigma(\sigma+\sigma_1))$ (see [9]), where the symbol $\sqrt{2}\gamma$ means the sequence $(\sqrt{2}\gamma_n)_{n=1}^{\infty}$. Therefore,

$$\begin{aligned}\exp(-\sigma_1x^2)|G^*_{n,j}f(x)| \leq \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{2}n\sum_{k=-\infty}^{\infty}p_{n,k}(x;\sqrt{2}\gamma) \\ \times\left|\frac{k}{n}-x\right|^j\int_{k/n}^{(k+1)/n}\exp(-\sigma t^2)|f(t)|dt.\end{aligned} \tag{2.10}$$

From (2.2), we have

$$\|w_{\sigma_1}G^*_{n,j}f\|_1 \leq \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{(2j)!!(\sqrt{2})^{j+1}\gamma_n^j}\|w_\sigma f\|_1. \tag{2.11}$$

Instead, for $p=\infty$, from (2.1) it follows that

$$\begin{aligned} ||w_{\sigma_1} G^*_{n,j} f||_\infty &\le \sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)||w_\sigma f||_\infty \operatorname*{essup}_{x\in R}\left(\sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma)\left|\frac{k}{n}-x\right|^j\right) \\ &\le 15\sqrt{2}A_c \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{2^j(2j)!}\gamma_n^j\|w_\sigma f\|_\infty. \end{aligned} \tag{2.12}$$

Finally, by Riesz-Thorin theorem, we have (2.5).

In view of definition (2.4) and the inequality

$$p_{n,k}(x;\gamma)p_{n,k}(t;\gamma)\exp(-\sigma_1 x^2)\exp(\sigma t^2) \le 2p_{n,k}(x;\sqrt{2}\gamma)p_{n,k}(t;\sqrt{2}\gamma), \tag{2.13}$$

for $n\in N$ such that $\gamma_n \le \max\{(\sigma_1-\sigma)/(2\sqrt{\sigma}(\sigma+\sigma_1)); \sqrt{\sigma_1-\sigma}/(\sqrt{2}(\sigma+\sigma_1))\}$ (see [6]), we have

$$\begin{aligned} &\exp(-\sigma_1 x^2)\,|\widetilde{G}_{n,j} f(x)| \\ &\quad\le 2n\sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma)\left|\frac{k}{n}-x\right|^j \int_{-\infty}^{\infty}\exp(-\sigma t^2)p_{n,k}(\sqrt{2}t;\gamma)\,|f(t)|\,dt. \end{aligned} \tag{2.14}$$

Applying (2.1) and (2.2), we get

$$\begin{aligned} \|w_{\sigma_1}\widetilde{G}_{n,j} f\|_1 &\le 30A_c\sqrt{(2j)!!}\gamma_n^j 2^{j/2}\|w_\sigma f\|_1, \\ \|w_{\sigma_1}\widetilde{G}_{n,j} f\|_\infty &\le 30A_c\sqrt{(2j)!}\gamma_n^j 2^{j/2}\|w_\sigma f\|_\infty. \end{aligned} \tag{2.15}$$

Finally, by Riesz-Thorin theorem, we have (2.6).

Further, for $\delta>0$, $x\in R$, and $r\in N$ we define Stieklov function of f

$$f_{(\delta,2r)}(x) = \frac{1}{\delta^{2r}}\frac{2}{\binom{2r}{r}}\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}\sum_{i=1}^{r}\binom{2r}{r-i}(-1)^{i-1}f(x+i(t_1+\cdots+t_{2r}))\,dt_1\cdots dt_{2r}. \tag{2.16}$$

□

Lemma 2.2. *For all $r=1,2,\ldots$, $0<\delta\le 1$, $\sigma_1>\sigma>0$, $1\le p\le\infty$, and $x\in R$,*

$$||w_{\sigma_1} f^{(r)}_{(\delta,2r)}||_p \le K(r,\sigma,\sigma_1)\frac{1}{\delta^r}\omega_r(f;\delta)_{\sigma,p}, \tag{2.17}$$

$$||w_{\sigma_1}(f_{(\delta,2r)}-f)||_p \le K(r,\sigma,\sigma_1)\omega_r(f;\delta)_{\sigma,p}. \tag{2.18}$$

Proof. It is easy to see by induction that

$$\begin{aligned} f^{(r)}_{(\delta,2r)}(x) &= \frac{2}{\binom{2r}{r}}\sum_{i=1}^{r}(-1)^{i-1}\binom{2r}{r-i}\frac{1}{(i\delta)^{2r}} \\ &\quad\times\int_{-i\delta/2}^{i\delta/2}\cdots\int_{-i\delta/2}^{i\delta/2}\Delta^r_{i\delta}f(x+u_1+\cdots+u_r)\,du_1\cdots du_r. \end{aligned} \tag{2.19}$$

Let $\sigma_2 = (2\sigma_1 + \sigma)/3$. In view of the inequality

$$\exp\left(- \sigma_1 x^2 + \sigma_2 (x+u)^2 \right) \le \exp\left(\frac{\sigma_2\sigma_1}{\sigma_1 - \sigma_2} u^2 \right), \tag{2.20}$$

where $0 < \delta \le 1$ and $u = u_1 + \cdots + u_r$, $(u \le r^2/2)$, we have

$$\|w_{\sigma_1} f^{(r)}_{(\delta,2r)}\|_p \le \frac{2}{\binom{2r}{r}} \exp\left(\frac{\sigma_1\sigma_2}{\sigma_1 - \sigma_2} \frac{r^4}{4} \right) \sum_{i=1}^{r} \binom{2r}{r-i} \frac{1}{(i\delta)^r} \|w_{\sigma_2} \Delta^r_{i\delta} f\|_p. \tag{2.21}$$

Applying the Minkowski inequality and the fact that for $0 \le l_i \le i-1$ $(0 \le i \le r)$, $0 < h \le 1$,

$$\exp\left(- \sigma_2 x^2 + \sigma \left(x + h\left(l_1 + \cdots + l_r - \frac{r(i-1)}{2} \right) \right)^2 \right) \le \exp\left(\frac{\sigma\sigma_2}{\sigma_2 - \sigma} \left(\frac{r(i-1)}{2} \right)^2 \right), \tag{2.22}$$

we obtain

$$\begin{aligned} &\|w_{\sigma_2} \Delta^r_{i\delta} f\|_p \\ &\quad = \sup_{|h| \le \delta} \left\{ \int_{-\infty}^{\infty} \left| \exp\left(- \sigma_2 x^2 \right) \sum_{l_1=0}^{i-1} \cdots \sum_{l_r=0}^{i-1} \Delta^r_h f\left(x + h\left(l_1 + \cdots + l_r - \frac{r(i-1)}{2} \right) \right) \right|^p dx \right\}^{1/p} \\ &\quad \le \exp\left(\frac{\sigma\sigma_2}{\sigma_2 - \sigma} \frac{r^2(i-1)^2}{4} \right) i^r \omega_r(f;\delta)_{\sigma,p}. \end{aligned} \tag{2.23}$$

So (2.17) is evident. It is easy to see that

$$f_{(\delta,2r)}(x) - f(x) = \frac{(-1)^{r-1}}{\delta^{2r}} \frac{1}{\binom{2r}{r}} \int_{-\delta/2}^{\delta/2} \cdots \int_{-\delta/2}^{\delta/2} \Delta^{2r}_{t_1 + \cdots + t_{2r}} f(x)\, dt_1 \cdots dt_{2r}. \tag{2.24}$$

By Minkowski inequality, for $1 \le p \le \infty$, we have (2.18). □

Lemma 2.3. *Suppose that $\gamma = (\gamma_n)_{n=1}^{\infty}$ is a positive sequence convergent to 0 and that $n\gamma_n^{r/2+1} \ge cK(r)$, where $r \in N$, $r \ge 2$, $K(r) = \max_{n \in N}\{\gamma_n^{r/2-1}\}$, c is a positive absolute constant and let $a_r = 1$ for even r and $a_r = 2$ for odd r. Then for $f \in L_{p,\sigma}(R)$, $\sigma > 0$, $1 \le p \le \infty$ and $\sigma_1 > \sigma$, we have*

$$\|w_{\sigma_1}((F_n^* f)^{(r)} - (n/a_r)^r F_n^* \Delta^r_{a_r/n} f)\|_p \le K(\sigma,\sigma_1,c,r) \|w_\sigma f\|_p \tag{2.25}$$

for all $n \in N$ such that $\gamma_n^2 \le (\sigma_1 - \sigma)/(4\sigma(\sigma + \sigma_1))$ and $n\gamma_n > 4a_r^2 r^2$, and

$$\|w_{\sigma_1}((\widetilde{F}_n f)^{(r)} - (n/a_r)^r \widetilde{F}_n \Delta^r_{a_r/n} f)\|_p \le K(\sigma,\sigma_1,c,r) \|w_\sigma f\|_p \tag{2.26}$$

for all $n \in N$ such that $\gamma_n \le \max\{(\sigma_1 - \sigma)/(2\sqrt{\sigma}(\sigma + \sigma_1)); \sqrt{\sigma_1 - \sigma}/(\sqrt{2}(\sigma + \sigma_1))\}$ and $n\gamma_n > r^2/4$.

Proof. We consider an even r. Let $r=2r_1$, $r_1\in N$, $x\in R$. Then

$$\begin{aligned}
n^r F_n^*\left(\Delta_{1/n}^r f(x)\right) &= n^{2r_1+1}\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\sum_{i=0}^{2r_1}\binom{2r_1}{i}(-1)^i\int_{(k+r_1-i)/n}^{(k+r_1-i+1)/n} f(t)dt \\
&= n^{2r_1+1}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i \\
&\quad\times\sum_{k=-\infty}^{\infty}\left(p_{n,k-(r_1-i)}(x;\gamma)+p_{n,k+(r_1-i)}(x;\gamma)\right)\int_{k/n}^{(k+1)/n} f(t)dt \\
&\quad+n^{2r_1+1}\sum_{k=-\infty}^{\infty}\binom{2r_1}{r_1}(-1)^{r_1}p_{n,k}(x;\gamma)\int_{k/n}^{(k+1)/n} f(t)dt.
\end{aligned} \tag{2.27}$$

It is easy to see that

$$\begin{aligned}
&p_{n,k-(r_1-i)}(x;\gamma)+p_{n,k+(r_1-i)}(x;\gamma) \\
&\quad= p_{n,k}(x;\gamma)\left\{\exp\left(\frac{r_1-i}{n\gamma_n^2}\left(\frac{k}{n}-x\right)-\frac{(r_1-i)^2}{2n^2\gamma_n^2}\right)+\exp\left(-\frac{r_1-i}{n\gamma_n^2}\left(\frac{k}{n}-x\right)-\frac{(r_1-i)^2}{2n^2\gamma_n^2}\right)\right\} \\
&\quad= p_{n,k}(x;\gamma)\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\sum_{j=0}^{[l/2]}\binom{l}{2j}2^{2j+1-l}\left(\frac{k}{n}-x\right)^{2j}n^{2j-2l}\gamma_n^{-2l}(r_1-i)^{2l-2j}+2p_{n,k}(x;\gamma).
\end{aligned} \tag{2.28}$$

Consequently, using definition (2.3), we get

$$\begin{aligned}
n^r F_n^*\left(\Delta_{1/n}^r f(x)\right) &= \sum_{l=1}^{2r_1}\sum_{j=0}^{[l/2]} n^{2(r_1+j-l)}\gamma_n^{-2l}\frac{(-1)^l 2^{2j+1-l}}{(2j)!(l-2j)!} \\
&\quad\times\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2l-2j}G_{n,2j}^* f(x) \\
&\quad+\sum_{l=2r_1+1}^{\infty}\sum_{j=0}^{[l/2]} n^{2(r_1+j-l)}\gamma_n^{-2l}\frac{(-1)^l 2^{2j+1-l}}{(2j)!(l-2j)!} \\
&\quad\times\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2l-2j}G_{n,2j}^* f(x) \\
&= S_{n,1}f(x)+S_{n,2}f(x).
\end{aligned} \tag{2.29}$$

In view of (2.5) and using Stirling formula, we obtain

$$\begin{aligned}
||w_{\sigma_1}S_{n,2}f||_p &\le K_1(\sigma,\sigma_1,c)||w_\sigma f||_p 4^{r_1}n^{2r_1}\sum_{l=2r_1+1}^{\infty}\frac{r_1^{2l}}{n^{2l}\gamma_n^{2l}2^l}\sum_{j=0}^{[l/2]}\frac{\sqrt{(4j)!}2^j}{(2j)!(l-2j)!}n^{2j}\gamma_n^{2j}4^j r_1^{-2j} \\
&\le K_2(\sigma,\sigma_1,c,r)||w_\sigma f||_p n^{2r_1}\sum_{l=2r_1+1}^{\infty}\frac{(r_1^2/2)^l}{(n^2\gamma_n^2)^l}\sum_{j=0}^{[l/2]}(n^2\gamma_n^2)^j 64^j \\
&\le K_3(\sigma,\sigma_1,c,r)||w_\sigma f||_p\left\{\frac{(16r_1^2)^{2r_1+1}}{n^2\gamma_n^{2r_1+2}}+n^{2r_1}\sum_{l=2r_1+2}^{\infty}\left(\frac{16r_1^2}{n\gamma_n}\right)^l\right\}.
\end{aligned} \tag{2.30}$$

Assuming $(16r_1^2)/(n\gamma_n) < 1$ and using the condition $n\gamma_n^{r_1+1} \ge cK(r)$, we get

$$\|w_{\sigma_1}S_{n,2}f\|_p \le K_4(\sigma,\sigma_1,c,r)\|w_\sigma f\|_p. \tag{2.31}$$

Now observe that

$$\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2s} = \begin{cases} 0 & \text{if } 0 < s < r_1, \\ (2r_1)!/2 & \text{if } s = r_1. \end{cases} \tag{2.32}$$

The equality follows simply from properties of finite differences since the left-hand side of the equation is a half of the finite difference of the polynomial $(r_1-x)^{2s}$. Therefore,

$$\begin{aligned}
S_{n,1}f(x) &= \sum_{l=r_1}^{2r_1}\frac{(-1)^l 2^{2j+1-l}}{l!n^{2l-2j-2r_1}\gamma_n^{2l}}\binom{l}{2j}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2l-2j}G_{n,2j}^*f(x) \\
&= \sum_{l=0}^{r_1}\sum_{j=0}^{l-1}\frac{(-1)^{r_1+l}2^{2j+1-l-r_1}}{(r_1+l)!n^{2l-2j}\gamma_n^{2l+2r_1}}\binom{r_1+l}{2j}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2r_1+2l-2j}G_{n,2j}^*f(x) \\
&\quad+\sum_{l=0}^{r_1}\frac{(-1)^{2r_1-l}}{\gamma_n^{4r_1-2l}}\frac{(2r_1)!}{2^l l!(2r_1-2l)!}G_{n,2j}^*f(x).
\end{aligned} \tag{2.33}$$

It is easy to see, by the method of induction, that

$$p_{n,k}^{(\nu)}(x;\gamma) = p_{n,k}(x;\gamma)\sum_{i=0}^{[\nu/2]}\frac{\nu!(-1)^i}{(\nu-2i)!(2i)!!}\frac{1}{\gamma_n^{2\nu-2i}}\left(\frac{k}{n}-x\right)^{\nu-2i}, \quad \nu\in N. \tag{2.34}$$

Therefore,

$$\begin{aligned}
S_{n,1}f(x) &= \sum_{l=0}^{r_1}\sum_{j=0}^{l-1}\frac{(-1)^{r_1+l}2^{2j+1-l-r_1}}{(r_1+l)!n^{2l-2j}\gamma_n^{2l+2r_1}}\binom{r_1+l}{2j}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2r_1+2l-2j}G_{n,2j}^*f(x) \\
&\quad+(F_n^*f(x))^{(2r_1)}.
\end{aligned} \tag{2.35}$$

Consequently, from (2.29)

$$\begin{aligned} &\left|\left(F_n^* f\right)^{(2r_1)}(x) - n^{2r_1} F_n^* \Delta_{1/n}^{2r_1} f(x)\right| \\ &\quad \le K_5(r) \sum_{j=0}^{r_1-1} \sum_{l=j+1}^{r_1} \frac{n^{2j}}{\left(n\gamma_n\right)^{2l} \gamma_n^{2r_1}} \left| G_{n,2j}^* f(x)\right| + \left| S_{n,2} f(x)\right|. \end{aligned} \tag{2.36}$$

The condition $n\gamma_n^{r_1+1} \ge cK(r)$ and the boundedness of the sequence (γ_n) lead to

$$\left|\left(F_n^* f\right)^{(2r_1)}(x) - n^{2r_1} F_n^* \Delta_{1/n}^{2r_1} f(x)\right| \le K_6(r,c) \sum_{j=0}^{r_1-1} \gamma_n^{-2j} \left| G_{n,2j}^* f(x)\right| + \left| S_{n,2} f(x)\right|. \tag{2.37}$$

Collecting the results we get estimate (2.25) for even r, immediately.

Now, we will prove inequality (2.25) for odd r. Namely, let $r = 2r_2 + 1$, $r_2 \in N$, $x \in R$. Then

$$\begin{aligned} n^r F_n^*\left(\Delta_{2/n}^r f(x)\right) &= n^{2r_2+2} \sum_{i=0}^{r_2} \sum_{k=-\infty}^{\infty} \binom{2r_2+1}{i} (-1)^i \\ &\quad \times \left(p_{n,k-(2r_2+1-2i)}(x;\gamma) - p_{n,k+(2r_2+1-2i)}(x;\gamma)\right) \int_{k/n}^{(k+1)/n} f(t)\,dt. \end{aligned} \tag{2.38}$$

It is easy to see that

$$\begin{aligned} &p_{n,k-(2r_2+1-2i)}(x;\gamma) - p_{n,k+(2r_2+1-2i)}(x;\gamma) \\ &\quad = p_{n,k}(x;\gamma) \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l!} \sum_{j=0}^{[(l-1)/2]} \binom{l}{2j+1} 2^{2j+2-l} \left(\frac{k}{n} - x\right)^{2j+1} \frac{n^{2j+1-2l}}{\gamma_n^{2l} (2r_2+1-2i)^{2j-2l+1}}. \end{aligned} \tag{2.39}$$

Consequently,

$$\begin{aligned} n^r F_n^*\left(\Delta_{2/n}^r f(x)\right) = &\sum_{l=1}^{2r_2+1} n^{2r_2+2} \sum_{j=0}^{[(l-1)/2]} n^{2j-2l} \gamma_n^{-2l} \frac{(-1)^{l+1} 2^{2j+2-l}}{(2j+1)!(l-2j-1)!} \\ &\times \sum_{i=0}^{r_2} \binom{2r_2+1}{i} (-1)^i (2r_2+1-2i)^{2l-2j-1} G_{n,2j+1}^* f(x) \\ &+ \sum_{l=2r_2+2}^{\infty} n^{2r_2+2} \sum_{j=0}^{[(l-1)/2]} n^{2j-2l} \gamma_n^{-2l} \frac{(-1)^{l+1} 2^{2j+2-l}}{(2j+1)!(l-2j-1)!} \\ &\times \sum_{i=0}^{r_2} \binom{2r_2+1}{i} (-1)^i (2r_2+1-2i)^{2l-2j-1} G_{n,2j+1}^* f(x) \\ = &\, S_{n,1}^* f(x) + S_{n,2}^* f(x). \end{aligned} \tag{2.40}$$

Some simple calculation, Stirling formula and (2.5) give

$$\left|\left|w_{\sigma_1} S_{n,2}^* f\right|\right|_p \le K_7(\sigma,\sigma_1,c,r) \left|\left|w_\sigma f\right|\right|_p \tag{2.41}$$

for $n \in N$ such that $(16r^2)/(n\gamma_n) < 1$. Next, in view of (2.25) and the equality

$$\sum_{i=0}^{r_2} \binom{2r_2+1}{i} (-1)^i (r_2 - i + 1/2)^{2s-1} = \begin{cases} 0 & \text{if } 0 < s < r_2+1, \\ (2r_2+1)!/2 & \text{if } s = r_2+1 \end{cases} \tag{2.42}$$

we obtain

$$\begin{aligned} S_{n,1}^* f(x) &= \sum_{l=0}^{r_2} \sum_{j=0}^{l-1} \frac{(-1)^{r_2+l} 2^{2j+1-l-r_2}}{(2j+1)!(l+r_2-2j)!} n^{2j-2l} \gamma_n^{-2l-2r_2-2} \\ &\quad \times \sum_{i=0}^{r_2} \binom{2r_2+1}{i} (-1)^i (2r_2+1-2i)^{2r_2+2l-2j+1} \\ &\quad \times G_{n,2j+1}^* f(x) + 2^{2r_2+1} (F_n^* f)^{(2r_2+1)}(x). \end{aligned} \tag{2.43}$$

Using (2.40) and the condition $n\gamma_n^{r_2+3/2} \ge cK(r)$, we have

$$\begin{aligned} &\left| (F_n^* f)^{(2r_2+1)}(x) - (n/2)^{2r_2+1} F_n^* \Delta_{2/n}^{2r_2+1} f(x) \right| \\ &\quad \le K_8(r,c) \sum_{j=0}^{r_2-1} \frac{1}{\gamma_n^{2j+1}} \left| G_{n,2j+1}^* f(x) \right| + \left| S_{n,2}^* f(x) \right|. \end{aligned} \tag{2.44}$$

Applying (2.5), we get (2.25) for odd r. Therefore, inequality (2.25) is proved.

Now we will prove (2.26). Let $r = 2r_1$, $r_1 \in N$. A simple calculation and the equality $p_{n,k}(t - (r_1 - i)/n; \gamma) = p_{n,k+r_1-i}(t;\gamma)$ give

$$\begin{aligned} n^r \widetilde{F}_n (\Delta_{1/n}^r f(x)) &= n^{2r_1+1} \sum_{i=0}^{r_1-1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{i} (-1)^i \left(p_{n,k-(r_1-i)}(x;\gamma) + p_{n,k+(r_1-i)}(x;\gamma) \right) \\ &\quad \times \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t) dt + n^{2r_1+1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{r_1} (-1)^i p_{n,k}(x;\gamma) \\ &\quad \times \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t) dt. \end{aligned} \tag{2.45}$$

The estimate (2.26) follows now the same way as (2.25). □

3. Main result

Theorem 3.1. *Suppose that $r \in N$, (γ_n) is a positive null sequence satisfying $n\gamma_n^{r/2+1} \ge cK(r)$ for all $n \in N$ with some $c > 0$ where $K(r) = \max_{n\in N}\{\gamma_n^{r/2-1}\}$. Then there exists a constant $K > 0$, such that for all $f \in L_{p,\sigma}(R)$, $\sigma_1 > \sigma > 0$, $1 \le p \le \infty$, and for an arbitrary positive number t_0,*

$$\omega_r (F_n^* f, t)_{\sigma_1,p} \le K(\sigma,\sigma_1,r,c) \{ (1+t_0^2) \omega_r(f,t)_{\sigma,p} + t^r \| w_\sigma f \|_p \} \quad (0 < t \le t_0) \tag{3.1}$$

for all $n \in N$ such that $\gamma_n^2 \le (\sigma_1 - \sigma)/(4\sigma(\sigma+\sigma_1))$ and $n\gamma_n > 16r^2$, and

$$\omega_r (\widetilde{F}_n f, t)_{\sigma_1,p} \le K(\sigma,\sigma_1,r,c) \{ (1+t_0^2) \omega_r(f,t)_{\sigma,p} + t^r \| w_\sigma f \|_p \} \quad (0 < t \le t_0) \tag{3.2}$$

for all $n \in N$ *such that* $\gamma_n \le \max\{(\sigma_1 - \sigma)/(2\sqrt{\sigma}(\sigma+\sigma_1)); \sqrt{\sigma_1-\sigma}/(\sqrt{2}(\sigma+\sigma_1))\}$ *and* $n\gamma_n > r^2/4$.

Proof. Let $\sigma_2 = (3\sigma_1+\sigma)/4$. In view of the inequality

$$\exp\left(-\sigma_1 x^2 + \sigma_2(x+u)^2\right) \le \exp\left(\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}u^2\right) \quad (u \in R) \tag{3.3}$$

and the generalized Minkowski inequality it is easy to see that for $0 < h \le 1$

$$\begin{aligned} \|w_{\sigma_1}\Delta_h^r f\|_p &= \Big\| w_{\sigma_1}\int_{-h/2}^{h/2}\cdots\int_{-h/2}^{h/2} f^{(r)}(\circ + s_1+\cdots+s_r)\exp\left(\sigma_2(\circ+s_1+\cdots+s_r)^2\right) \\ &\qquad \times \exp\left(-\sigma_2(\circ+s_1+\cdots+s_r)^2\right)ds_1\cdots ds_r\Big\|_p \\ &\le \exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)h^r\|w_{\sigma_2}f^{(r)}\|_p, \end{aligned} \tag{3.4}$$

$$\|w_{\sigma_1}\Delta_h^r f\|_p \le 2^r\exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)\|w_{\sigma_2}f\|_p. \tag{3.5}$$

Applying these inequalities, we get

$$\begin{aligned} \|w_{\sigma_1}\Delta_h^r f\|_p &\le \|w_{\sigma_1}\Delta_h^r(f - f_{(\delta,2r)})\|_p + \|w_{\sigma_1}\Delta_h^r f_{(\delta,2r)}\|_p \\ &\le 2^r\exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)\left(\|w_{\sigma_2}(f-f_{(\delta,2r)})\|_p + h^r\|w_{\sigma_2}f_{(\delta,2r)}^{(r)}\|_p\right), \end{aligned} \tag{3.6}$$

where $f_{(\delta,2r)}(x)$ ($\delta>0$, $x\in R$, $r\in N$) is defined by (2.16).

Hence, applying this inequality for $F_n^* f$ we have

$$\omega_r(F_n^* f,t)_{\sigma_1,p} \le 2^r\exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)\left\{\|w_{\sigma_2}F_n^*(f-f_{(\delta,2r)})\|_p + t^r\|w_{\sigma_2}(F_n^* f_{(\delta,2r)})^{(r)}\|_p\right\}. \tag{3.7}$$

Hence, $\|w_{\sigma_2}(F_n^* f_{(\delta,2r)})^{(r)}\|_p$ can be estimated by (2.5) for $j=0$, (2.25), and (3.4). Let $\sigma_3 = (2\sigma_1+\sigma)/3$, then

$$\begin{aligned} &\|w_{\sigma_2}(F_n^* f_{(\delta,2r)})^{(r)}\|_p \\ &\quad\le \|w_{\sigma_2}((F_n^* f_{(\delta,2r)})^{(r)} - (n/a_r)F_n^*\Delta_{a_r/n}^r f_{(\delta,2r)})\|_p + n^r\|w_{\sigma_2}F_n^*\Delta_{a_r/n}^r f_{(\delta,2r)}\|_p \\ &\quad\le K(\sigma_2,\sigma_3,r,c)\left(\|w_{\sigma_3}f_{(\delta,2r)}\|_p + \|w_{\sigma_3}f_{(\delta,2r)}^{(r)}\|_p\right). \end{aligned} \tag{3.8}$$

Using (2.5) for $j=0$ and (3.7) we have

$$\omega_r(F_n^* f,t)_{\sigma_1,p} \le K(\sigma,\sigma_1,r,c)\left\{\|w_{\sigma_3}(f-f_{(\delta,2r)})\|_p + t^r\|w_{\sigma_3}f_{(\delta,2r)}^{(r)}\|_p + t^r\|w_{\sigma_3}f_{(\delta,2r)}\|_p\right\}. \tag{3.9}$$

Consequently by (2.17), (2.18) and assuming now $0 < t \le t_0$, we have

$$\omega_r(F_n^* f, t)_{\sigma_1,p} \le K(\sigma,\sigma_1,r,c)\left((1+t_0^r)\omega_r(f;t)_{\sigma,p} + t^r \|w_\sigma f\|_p\right). \tag{3.10}$$

On the same way we can prove (3.2) for $\widetilde{F}_n f$, using (2.6) and (2.26). □

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