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Research Article

Bleimann, Butzer, and Hahn Operators Based on the q-Integers

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We give a new generalization of Bleimann, Butzer, and Hahn operators, which includes q-integers. We investigate uniform approximation of these new operators on some subspace of bounded and continuous functions. In Section 3, we show that the rates of convergence of the new operators in uniform norm are better than the classical ones. We also obtain a pointwise estimation in a general Lipschitz-type maximal function space. Finally, we de?fine a generalization of these new operators and study the uniform convergence of them.

1. Introduction

Recently, in 1997, Phillips [1] used the q-integers in approximation theory where it is considered q-based generalization of classical Bernstein polynomials. It was obtained by replacing the binomial expansion with the general one, the q-binomial expansion. Phillips has obtained the rate of convergence and Voronovskaja-type asymptotic formulae for these new Bernstein operators based on q-integers. Later, some results are established in due course by Phillips et al. (see [2, 3, 1]). In [4], Barbasu gave Stancu-type generalization of these operators and II‘inskii and Ostrovska [5] studied their different convergence properties. Also some results on the statistical and ordinary approximation of functions by Meyer-König and Zeller operators based on q-integers can be found in [6, 7], respectively.

In [8], Bleimann, Butzer, and Hahn introduced the following operators:

$$B_n(f)(x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k, \quad x>0,\ n\in\mathbb{N}. \tag{1.1}$$

There are several studies related to approximation properties of Bleimann, Butzer, and Hahn operators (or, briefly, BBH). There are many approximating operators that their Korovkin-type approximation properties and rates of convergence are investigated. The results involving Korovkin-type approximation properties can be found in [9] with details. In [10], Gadjiev and Çakar gave a Korovkin-type theorem using the test function $(t/(1+t))^{\nu}$ for $\nu = 0,1,2$. Some generalization of the operators (1.1) were given in [11–13].

In this paper, we derive a q-integers-type modification of BBH operators that we call q-BBH operators and investigate their Korovkin-type approximation properties by using the test function $(t/(1+t))^{\nu}$ for $\nu = 0,1,2$. Also, we define a space of generalized Lipschitz-type maximal function and give a pointwise estimation. Then, a Stancu-type formula of the remainder of q-BBH is given. We will also give a generalization of these new operators and study the approximation properties of this generalization. We emphasis that while Bernstein and Meyer-König and Zeller operators based on q-integers depend on a function defined on a bounded interval, these new operators are defined on unbounded intervals. Also, these new operators are more flexible than classical BBH operators. That is, depending on the selection of q, rate of convergence of the q-BBH operators is better than the classical one.

2. Construction of the operators

We first start by recalling some definitions about q-integers denoted by $[\cdot]$.

For any fixed real number $q > 0$ and nonnegative integer r, the q-integer of the number r is defined by

$$[r] = \begin{cases} \dfrac{1-q^r}{1-q}, & q \neq 1, \\ r, & q = 1. \end{cases} \tag{2.1}$$

Also we have $[0] = 0$.

The q-factorial is defined in the following:

$$[r]! = \begin{cases} [r][r-1]\cdots[1], & r = 1,2,\ldots, \\ 1, & r = 0, \end{cases} \tag{2.2}$$

and q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!} \tag{2.3}$$

for integers $n \geq r \geq 0$.

Also, let us recall the following Euler identity (see [14, page 293]):

$$\prod_{k=0}^{n-1}(1+q^k x) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{2.4}$$

It is clear that when $q = 1$, these q-binomial coefficients reduce to ordinary binomial coefficients.

According to these explanations, similarly in [6], we define a new Bleimann-, Butzer-, and Hahn-type operators based on q-integers as follows:

$$L_n(f;x) = \frac{1}{\ell_n(x)} \sum_{k=0}^{n} f\left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{2.5}$$

where

$$\ell_n(x) = \prod_{s=0}^{n-1} (1+q^s x) \tag{2.6}$$

and f is defined on semiaxis $[0,\infty)$.

Note that taking $f([k]/[n-k+1])$ instead of $f([k]/[n-k+1]q^k)$ in (2.5), we obtain usual generalization of Bleimann, Butzer, and Hahn operators based on q-integers. But in this case, it is impossible to obtain explicit expressions for the monomials t^ν and $(t/(1+t))^\nu$ for $\nu = 1,2$. If we define the Bleimann-, Butzer-, and Hahn-type operators as in (2.5), then we can obtain explicit formulas for the monomials $(t/(1+t))^\nu$ for $\nu = 0,1,2$.

By a simple calculation, we have

$$q^k[n-k+1] = [n+1]-[k], \qquad q[k-1] = [k]-1. \tag{2.7}$$

From (2.4), (2.5), and (2.7), we have

$$L_n(1;x) = 1, \tag{2.8}$$

$$\begin{aligned} L_n\left(\frac{t}{1+t};x\right) &= \frac{1}{\ell_n(x)} \sum_{k=1}^{n} \frac{[k]}{[n+1]} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k \\ &= \frac{1}{\ell_n(x)} \sum_{k=1}^{n} \frac{[n]}{[n+1]} q^{k(k-1)/2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^k \\ &= \frac{[n]}{[n+1]} x \frac{1}{\ell_n(x)} \sum_{k=0}^{n-1} q^{k(k-1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} (qx)^k \\ &= \frac{x}{x+1} \frac{[n]}{[n+1]}. \end{aligned} \tag{2.9}$$

We can also write

$$\begin{aligned} L_n\left(\frac{t^2}{(1+t)^2};x\right) &= \frac{1}{\ell_n(x)} \sum_{k=1}^{n} \frac{[k]^2}{[n+1]^2} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k \\ &= \frac{1}{\ell_n(x)} \sum_{k=2}^{n} \frac{q[k][k-1]}{[n+1]^2} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k \\ &\quad + \frac{1}{\ell_n(x)} \sum_{k=1}^{n} \frac{[k]}{[n+1]^2} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\ell_n(x)} \sum_{k=0}^{n-2} \frac{[n][n-1]}{[n+1]^2} q^{k(k-1)/2} \begin{bmatrix} n-2 \\ k \end{bmatrix} (q^2x)^k q^2 x^2 \\
&\quad + \frac{1}{\ell_n(x)} \sum_{k=0}^{n-1} \frac{[n]}{[n+1]^2} q^{k(k-1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} (qx)^k x \\
&= \frac{[n][n-1]}{[n+1]^2} q^2 \frac{x^2}{(1+x)(1+qx)} + \frac{[n]}{[n+1]^2} \frac{x}{x+1}.
\end{aligned}
\tag{2.10}
$$

Remark 2.1. Note that if we choose $q = 1$, then L_n operators turn out into classical Bleimann, Butzer, and Hahn operators given by (1.1). Also similarly as in [1, 6], to ensure that the convergence properties of L_n, we will assume $q = q_n$ as a sequence such that $q_n \to 1$ as $n \to \infty$ for $0 < q_n < 1$.

3. Properties of the operators

In this section, we will give the theorems on uniform convergence and rate of convergence of the operators (2.5). As in [10], for this purpose we give a space of function ω of the type of modulus of continuity which satisfies the following conditions:

(a) ω is a nonnegative increasing function on $[0, \infty)$,

(b) $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$,

(c) $\lim_{\delta \to 0} \omega(\delta) = 0$,

and H_ω is the subspace of real-valued function and satisfies the following condition.

For any $x, y \in [0, \infty)$,

$$
|f(x) - f(y)| \le \omega\left(\left| \frac{x}{1+x} - \frac{y}{1+y} \right| \right). \tag{3.1}
$$

Also $H_\omega \subset C_B[0, \infty)$, where $C_B[0, \infty)$ is the space of functions f which is continuous and bounded on $[0, \infty)$ endowed with norm $\|f\|_{C_B} = \sup_{x \ge 0} |f(x)|$.

It is easy to show that from condition (b), the function ω satisfies the inequality

$$
\omega(n\delta) \le n\omega(\delta), \quad n \in \mathbb{N}, \tag{3.2}
$$

and from condition (a) for $\lambda > 0$, we have

$$
\omega(\lambda\delta) \le \omega((1 + [|\lambda|])\delta) \le (1 + \lambda)\omega(\delta), \tag{3.3}
$$

where $[|\lambda|]$ is the greatest integer of λ.

Remark 3.1. The operator L_n maps H_ω into $C_B[0, \infty)$ and it is continuous with respect to supnorm.

The properties of linear positive operators acting from H_ω to $C_B[0, \infty)$ and Korovkin-type theorems for them have been studied by Gadjiev and Çakar who have established the following theorem (see [10]).

Theorem 3.2. *If A_n is the sequence of positive linear operators acting from H_ω to $C_B[0,\infty)$ and satisfying the following condition for $v = 0,1,2$:*

$$\left\|\left(A_n\left(\frac{t}{1+t}\right)^v\right)(x) - \left(\frac{x}{1+x}\right)^v\right\|_{C_B} \longrightarrow 0, \quad \text{for } n \longrightarrow \infty, \tag{3.4}$$

then, for any function f in H_ω, one has

$$||A_n f - f||_{C_B} \longrightarrow 0, \quad \text{for } n \longrightarrow \infty. \tag{3.5}$$

Theorem 3.3. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. If L_n is defined by (2.5), then for any $f \in H_\omega$,*

$$\lim_{n\to\infty} ||L_n f - f||_{C_B} = 0. \tag{3.6}$$

Proof. Using Theorem 3.2, we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|L_n\left(\left(\frac{t}{1+t}\right)^v; x\right) - \left(\frac{x}{1+x}\right)^v\right\|_{C_B} = 0, \quad v = 0,1,2. \tag{3.7}$$

From (2.8), the first condition of (3.7) is fulfilled for $v = 0$. Now it is easy to see that from (2.9),

$$\left\|L_n\left(\left(\frac{t}{1+t}\right); x\right) - \frac{x}{1+x}\right\|_{C_B} \le \left|\frac{[n]}{[n+1]} - 1\right| \le \left|\frac{1}{q_n} - \frac{1}{q_n[n+1]} - 1\right|, \tag{3.8}$$

and since $[n+1] \to \infty$, $q_n \to 1$ as $n \to \infty$, condition (3.7) holds for $v = 1$.

To verify this condition for $v = 2$, consider (2.10). We see that

$$\begin{aligned} &\left\|L_n\left(\left(\frac{t}{1+t}\right)^2; x\right) - \left(\frac{x}{1+x}\right)^2\right\|_{C_B} \\ &\quad = \sup_{x\ge 0}\left(\frac{x^2}{(1+x)^2}\left(\frac{[n][n-1]}{[n+1]^2} q_n^2 \frac{1+x}{1+q_n x} - 1\right) + \frac{[n]}{[n+1]^2}\frac{x}{1+x}\right). \end{aligned} \tag{3.9}$$

A small calculation shows that

$$\frac{[n][n-1]}{[n+1]^2} = \frac{1}{q_n^3}\left(1 - \frac{2+q_n}{[n+1]} + \frac{1+q_n}{[n+1]^2}\right). \tag{3.10}$$

Thus we have

$$\left\|L_n\left(\left(\frac{t}{1+t}\right)^2; x\right) - \left(\frac{x}{1+x}\right)^2\right\|_{C_B} \le \frac{1}{q_n^2}\left(1 - q_n^2 - \frac{2}{[n+1]} + \frac{1}{[n+1]^2}\right). \tag{3.11}$$

This means that condition (3.7) holds also for $v = 2$ and the proof is completed by the Theorem 3.2. □

THEOREM 3.4. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. If L_n is defined by (2.5), then for each $x \geq 0$ and for any $f \in H_\omega$, the following inequality:*

$$|L_n(f;x) - f(x)| \leq 2\omega\left(\sqrt{\mu_n(x)}\right) \tag{3.12}$$

holds, where

$$\mu_n(x) = \left(\frac{x}{1+x}\right)^2 \left(1 - 2\frac{[n]}{[n+1]} + \frac{[n][n-1]}{[n+1]^2} q_n^2 \frac{(1+x)}{(1+q_n x)}\right) + \frac{[n]}{[n+1]^2} \frac{x}{1+x}. \tag{3.13}$$

Proof. Since $L_n(1;x) = 1$, we can write

$$|L_n(f;x) - f(x)| \leq L_n(|f(t) - f(x)|;x). \tag{3.14}$$

On the other hand, from (3.1) and (3.3),

$$|f(t) - f(x)| \leq \omega\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|\right) \leq \left(1 + \frac{|t/(1+t) - x/(1+x)|}{\delta}\right)\omega(\delta), \tag{3.15}$$

where we choose $\lambda = \delta^{-1}|t/(1+t) - x/(1+x)|$. This inequality and (3.14) imply that

$$|L_n(f;x) - f(x)| \leq \omega(\delta)\left(1 + \frac{1}{\delta} L_n\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|;x\right)\right). \tag{3.16}$$

According to the Cauchy-Schwarz inequality, we have

$$|L_n(f;x) - f(x)| \leq \omega(\delta)\left(1 + \frac{1}{\delta} L_n\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^2;x\right)^{1/2}\right). \tag{3.17}$$

By choosing $\delta = \mu_n(x) = L_n(|t/(1+t) - x/(1+x)|^2;x)$, we obtain desired result. $\square$

Remark 3.5. Using (3.13) and taking into consideration $[n-1]q_n + 1 = [n]$ and $[n+1] - [n] = q_n < 1$, then we have that

$$\sup_{x\geq 0} \mu_n(x) \leq 1 - 2\frac{[n]}{[n+1]} + \frac{[n]}{[n+1]^2}([n-1]q_n + 1) = \left(\frac{[n+1] - [n]}{[n+1]}\right)^2 \leq \frac{1}{[n+1]^2} \tag{3.18}$$

holds for n large enough. Thus, if the assumptions of Theorem 3.4 hold, then, depending on the selection of q_n, the rate of convergence of the operators (2.5) to f is $1/[n+1]^2$ that is better than $1/(n+1)^2$, which is the rate of convergence of the BBH operators. Indeed, if we take $q_n = 1 - 1/(n+2)$, since $\lim_{n\to\infty} q_n^n = e^{-1}$, the rate of convergence of q-BBH operators to f is exactly of order $(1 - q_n)^2 = 1/(n+2)^2$ that is better than $1/(n+1)^2$.

Now we will give an estimate concerning the rate of convergence as given in [13, 15, 16]. We define the space of general Lipschitz-type maximal functions on $E \subset [0,\infty)$ by $W_{\alpha,E}^{\sim}$ as

$$W_{\alpha,E}^{\sim} = \left\{ f : \sup(1+x)^{\alpha} f_{\alpha}(x,y) \leq M \frac{1}{(1+y)^{\alpha}},\ x \geq 0,\ y \in E \right\}, \tag{3.19}$$

where f is bounded and continuous on $[0,\infty)$, M is a positive constant, $0 < \alpha \leq 1$, and f_{α} is the following function:

$$f_{\alpha}(x,t) = \frac{|f(t) - f(x)|}{|x-t|^{\alpha}}. \tag{3.20}$$

Also, let $d(x,E)$ be the distance between x and E, that is,

$$d(x,E) = \inf\{|x-y|; y \in E\}. \tag{3.21}$$

Theorem 3.6. *If L_n is defined by (2.5), then for all $f \in W_{\alpha,E}^{\sim}$ we have*

$$|L_n(f;x) - f(x)| \leq M\left(\mu_n^{\alpha/2}(x) + 2\left(d(x,E)\right)^{\alpha}\right), \tag{3.22}$$

where $\mu_n(x)$ defined in (3.13).

Proof. Let $\overline{E}$ denote the closure of the set E. Then there exists an $x_0 \in \overline{E}$ such that $|x - x_0| = d(x,E)$, where $x \in [0,\infty)$. Thus, we can write

$$|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|. \tag{3.23}$$

Since L_n is a positive and linear operator and $f \in W_{\alpha,E}^{\sim}$ by using above inequality, then we have

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq L_n(|f - f(x_0)|;x) + |f(x_0) - f(x)| \\ &\leq M L_n\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha};x\right) + M\frac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}. \end{aligned} \tag{3.24}$$

If we use the classical inequality $(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha}$ for $a \geq 0, b \geq 0$, one can write

$$\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha} \leq \left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\alpha} + \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha} \tag{3.25}$$

for $0 < \alpha \leq 1$ and $t \in [0,\infty)$. Consequently, we obtain

$$L_n\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha};x\right) \leq L_n\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\alpha};x\right) + \frac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}. \tag{3.26}$$

Since $L_n(1;x)=1$, applying Hölder inequality with $p=2/\alpha$ and $q=2/(2-\alpha)$, we have

$$L_n\left(\left|\frac{t}{1+t}-\frac{x_0}{1+x_0}\right|^{\alpha};x\right)\le L_n\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2;x\right)^{\alpha/2}+\frac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}. \tag{3.27}$$

Thus, in view of (3.24), we get (3.22). □

As a particular case of Theorem 3.6, when $E=[0,\infty)$, the following is true.

Corollary 3.7. *If $f\in W^{\sim}_{\alpha,[0,\infty)}$, then one has*

$$|L_n(f;x)-f(x)|\le M\mu_n^{\alpha/2}(x), \tag{3.28}$$

where $\mu_n(x)$ is defined in (3.13).

In the following theorem, a Stancu-type formula for the remainder of q-BBH operators is obtained which reduce to the formula of remainder of classical BBH operators (see [17, page 151]). Similar formula is obtained for q-Szasz Mirakyan operators in [18].

Here, $[x_0,x_1,\dots,x_n;f]$ denotes the divided difference of the function f with respect to distinct points in the domain of f and can be expressed as the following formula:

$$[x_0,x_1,\dots,x_n;f]=\frac{[x_1,\dots,x_n;f]-[x_0,\dots,x_{n-1};f]}{x_n-x_0}. \tag{3.29}$$

Theorem 3.8. *If $x\in(0,\infty)\setminus\{[k]/[n-k+1]q^k\mid k=0,1,2,\dots,n\}$, then the following identity holds:*

$$\begin{aligned}L_n(f;x)-f\left(\frac{x}{q}\right)&=-\frac{x^{n+1}}{\ell_n(x)}\left[\frac{x}{q},\frac{[n]}{q^n};f\right]\\&\quad+\frac{x}{\ell_n(x)}\sum_{k=0}^{n-1}\left[\frac{x}{q},\frac{[k]}{[n-k+1]q^k},\frac{[k+1]}{[n-k]q^{k+1}};f\right]\frac{q^{k(k+1)/2-2}}{[n-k]}\begin{bmatrix}n+1\\k\end{bmatrix}x^k.\end{aligned} \tag{3.30}$$

Proof. By using (2.5), we have

$$\begin{aligned}L_n(f;x)-f\left(\frac{x}{q}\right)&=\frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left[f\left(\frac{[k]}{[n-k+1]q^k}\right)-f\left(\frac{x}{q}\right)\right]q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}x^k\\&=-\frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left(\frac{x}{q}-\frac{[k]}{[n-k+1]q^k}\right)\left[\frac{x}{q},\frac{[k]}{[n-k+1]q^k};f\right]q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}x^k.\end{aligned} \tag{3.31}$$

Since

$$\frac{[k]}{[n-k+1]}\begin{bmatrix}n\\k\end{bmatrix}=\begin{bmatrix}n\\k-1\end{bmatrix}, \tag{3.32}$$

then we have

$$
\begin{aligned}
L_n(f;x) - f\left(\frac{x}{q}\right) = {} & -\frac{1}{\ell_n(x)} \sum_{k=0}^{n} \left[\frac{x}{q}, \frac{[k]}{[n-k+1]q^k}; f\right] q^{k(k-1)/2-1} \begin{bmatrix} n \\ k \end{bmatrix} x^{k+1} \\
& + \frac{1}{\ell_n(x)} \sum_{k=1}^{n} \left[\frac{x}{q}, \frac{[k]}{[n-k+1]q^k}; f\right] q^{k(k-1)/2-k} \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{k}.
\end{aligned} \tag{3.33}
$$

Rearranging the above equality, we can write

$$
\begin{aligned}
& L_n(f;x) - f\left(\frac{x}{q}\right) \\
& \quad = -\frac{x^{n+1}}{\ell_n(x)} \left[\frac{x}{q}, \frac{[n]}{q^n}; f\right] q^{n(n-1)/2-1} \\
& \qquad + \frac{1}{\ell_n(x)} \sum_{k=0}^{n-1} \left(\left[\frac{x}{q}, \frac{[k+1]}{[n-k]q^{k+1}}; f\right] - \left[\frac{x}{q}, \frac{[k]}{[n-k+1]q^k}; f\right] \right) q^{k(k-1)/2-1} \begin{bmatrix} n \\ k \end{bmatrix} x^{k+1}.
\end{aligned} \tag{3.34}
$$

Using the equality

$$
\frac{[k+1]}{[n-k]q^{k+1}} - \frac{[k]}{[n-k+1]q^k} = \frac{[n+1]}{[n-k][n-k+1]q^{k+1}}, \tag{3.35}
$$

we have the following formula for divided differences:

$$
\begin{aligned}
& \left[\frac{x}{q}, \frac{[k]}{[n-k+1]q^k}, \frac{[k+1]}{[n-k]q^{k+1}}; f\right] \frac{[n+1]}{[n-k][n-k+1]q^{k+1}} \\
& \quad = \left[\frac{x}{q}, \frac{[k+1]}{[n-k]q^{k+1}}; f\right] - \left[\frac{x}{q}, \frac{[k]}{[n-k+1]q^k}; f\right],
\end{aligned} \tag{3.36}
$$

and therefore, we obtain that the remainder formula for q-BBH can be written as (3.30). □

We know that a function is convex on an interval if and only if all second-order divided differences of f are nonnegative. From this property and Theorem 3.8, we have the following result.

COROLLARY 3.9. *If f is convex and nonincreasing, then*

$$
f\left(\frac{x}{q}\right) \le L_n(f;x) \quad (n = 0, 1, \dots). \tag{3.37}
$$

4. Some generalization of L_n

In this section, similarly as in [13], we will define some generalization of the operators L_n.

We consider a sequence of linear positive operators as follows:

$$L_n^{\gamma}(f;x)=\frac{1}{\ell_n(x)}\sum_{k=0}^{n} f\left(\frac{[k]+\gamma}{b_{n,k}}\right)q^{k(k-1)/2}\begin{bmatrix} n\\ k\end{bmatrix}x^k \quad (\gamma\in\mathbb{R}), \tag{4.1}$$

where $b_{n,k}$ satisfies the following condition:

$$[k]+b_{n,k}=c_n, \quad \frac{[n]}{c_n}\longrightarrow 1, \quad \text{for } n\longrightarrow\infty. \tag{4.2}$$

It is easy to check that if $b_{n,k}=[n-k+1]q^k+\beta$ for any n,k and $0<q<1$, then $c_n=[n+1]+\beta$ and these operators turn out into Stancu-type generalization of Bleimann, Butzer, and Hahn operators based on q-integers (see [19]). If we choose $\gamma=0$ and $q=1$, then the operators become the special case of Balázs-type generalization of the operators (1.1), which is given in [13].

Theorem 4.1. *Let $q=q_n$ satisfies $0<q_n\le 1$ and let $q_n\to 1$ as $n\to\infty$. If $f\in W^{\sim}_{\alpha,[0,\infty)}$, then the following inequality:*

$$\begin{aligned}&\left\|L_n^{\gamma}(f;x)-f(x)\right\|_{C_B}\\ &\le 3M\max\left\{\left(\frac{[n]}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]}\right)^{\alpha},\left|1-\frac{[n+1]}{c_n+\gamma}\right|^{\alpha}\left(\frac{[n]}{[n+1]}\right)^{\alpha},1-2\frac{[n]}{[n+1]}+\frac{[n][n-1]}{[n+1]^2}q_n\right\}\end{aligned} \tag{4.3}$$

holds for a large n.

Proof. Using (2.5) and (4.1), we have

$$\begin{aligned}\left|L_n^{\gamma}(f;x)-f(x)\right| &\le \frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left|f\left(\frac{[k]+\gamma}{b_{n,k}}\right)-f\left(\frac{[k]}{\gamma+b_{n,k}}\right)\right|q_n^{k(k-1)/2}\begin{bmatrix} n\\ k\end{bmatrix}x^k\\ &\quad+\frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left|f\left(\frac{[k]}{\gamma+b_{n,k}}\right)-f\left(\frac{[k]}{[n-k+1]q_n^k}\right)\right|q_n^{k(k-1)/2}\begin{bmatrix} n\\ k\end{bmatrix}x^k\\ &\quad+\left|L_n(f;x)-f(x)\right|.\end{aligned} \tag{4.4}$$

Since $f\in W^{\sim}_{\alpha,[0,\infty)}$ and by using Corollary 3.7 , we can write

$$\begin{aligned}\left|L_n^{\gamma}(f;x)-f(x)\right| &\le \frac{M}{\ell_n(x)}\sum_{k=0}^{n}\left|\frac{[k]+\gamma}{[k]+\gamma+b_{n,k}}-\frac{[k]}{\gamma+[k]+b_{n,k}}\right|^{\alpha}q_n^{k(k-1)/2}\begin{bmatrix} n\\ k\end{bmatrix}x^k\\ &\quad+\frac{M}{\ell_n(x)}\sum_{k=0}^{n}\left|\frac{[k]}{[k]+\gamma+b_{n,k}}-\frac{[k]}{[n+1]}\right|^{\alpha}q_n^{k(k-1)/2}\begin{bmatrix} n\\ k\end{bmatrix}x^k+\mu_n^{\alpha/2}(x)\end{aligned}$$

$$\leq \left(\frac{[n]}{c_n+\gamma}\right)^\alpha \left(\frac{\gamma}{[n]}\right)^\alpha + \left|1-\frac{[n+1]}{c_n+\gamma}\right|^\alpha$$
$$\times \frac{1}{\ell_n(x)} \sum_{k=0}^{n} \left(\frac{[k]}{[n+1]}\right)^\alpha q_n^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k + \mu_n^{\alpha/2}(x). \tag{4.5}$$

Using the Hölder inequality for $p = 1/\alpha$, $q = 1/(1-\alpha)$, and (2.9), we obtain

$$\left|L_n^\gamma(f;x) - f(x)\right| \leq M\left(\frac{[n]}{c_n+\gamma}\right)^\alpha \left(\frac{\gamma}{[n]}\right)^\alpha + M\left|1-\frac{[n+1]}{c_n+\gamma}\right|^\alpha \left(\frac{x}{x+1}\frac{[n]}{[n+1]}\right)^\alpha + \mu_n^{\alpha/2}(x). \tag{4.6}$$

Thus, inequality (4.3) holds for $x \in [0,\infty)$. □

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