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Research Article

A Reverse Hardy-Hilbert-Type Inequality

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By estimating the weight coefficient, a reverse Hardy-Hilbert-type inequality is proved. As applications, some equivalent forms and a number of particular cases are obtained.

1. Introduction

Let $p > 1$, $1/p + 1/q = 1$, $a_n \geq 0$, $b_n \geq 0$, and $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} b_n^q < \infty$. Then the Hardy-Hilbert inequality is as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=0}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=0}^{\infty} b_n^q\right)^{1/q}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi p)$ is the best possible [1].

For (1.1), Yang et al. [2–6] gave some strengthened versions and extensions as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left[\pi - \frac{7}{5(\sqrt{n}+3)}\right]a_n^2 \sum_{n=0}^{\infty}\left[\pi - \frac{7}{5(\sqrt{n}+3)}\right]b_n^2\right\}^{1/2} \tag{1.2}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} &< \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{\ln 2 - C}{(2n+1)^{1+1/p}}\right]a_n^p\right\}^{1/p} \\ &\quad\times\left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{\ln 2 - C}{(2n+1)^{1+1/q}}\right]b_n^q\right\}^{1/q},\end{aligned} \tag{1.3}$$

where $\ln 2 - C = 0.1159315^{+}$ (C is the Euler constant),

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p\right\}^{1/p} \times\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^q\right\}^{1/q}, \tag{1.4}$$

where the constant $B((p+\lambda-2)/p,(q+\lambda-2)/q)$ is the best possible $(2-\min\{p,q\}<\lambda\le 2)$,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1-\lambda}a_n^p\right\}^{1/p} \times\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q-1-\lambda}b_n^q\right\}^{1/q}, \tag{1.5}$$

where the constant $B(\lambda/p,\lambda/q)$ is the best possible $(0<\lambda\le\min\{p,q\})$,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} < B\left(\frac{(r-2)t+\lambda}{r},\frac{(s-2)t+\lambda}{s}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p(1-t+(2t-\lambda)/r)-1}a_n^p\right\}^{1/p} \times\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q(1-t+(2t-\lambda)/s)-1}b_n^q\right\}^{1/q}, \tag{1.6}$$

where the constant $B(((r-2)t+\lambda)/r,((s-2)t+\lambda)/s)$ is the best possible ($r>1$, $1/r+1/s=1$, $t\in[0,1]$, $2-\min\{r,s\}t<\lambda\le 2-\min\{r,s\}t+\min\{r,s\}$).

For the reverse Hardy-Hilbert inequality, recently, Yang [7] gave a reverse form of inequalities (1.4), (1.5), and (1.6) for $\lambda=2$. The main objective of this paper is to establish an extension of the above Yang's work for $1.5<\lambda<3$, by estimating the weight coefficient.

For this, we need the following expression of the β function $B(p,q)$ (see [8]):

$$B(p,q)=B(q,p)=\int_0^{\infty}\frac{1}{(1+u)^{p+q}}u^{p-1}du,\quad (p,q>0) \tag{1.7}$$

and the following inequality [3]:

$$\int_0^{\infty}f(x)dx+\frac{1}{2}f(0)<\sum_{m=0}^{\infty}f(m)<\int_0^{\infty}f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f'(0), \tag{1.8}$$

where $f(x)\in C^3[0,\infty)$, and $\int_0^{\infty}f(x)dx<\infty$, $(-1)^n f^{(n)}(x)>0$, $f^{(n)}(\infty)=0$ $(n=0,1,2,3)$.

2. Main results

Lemma 2.1. *Let N_0 be the set of nonnegative integers, N the set of positive integers, and $\mathbb{R}$ the set of real numbers. The weight coefficient $\omega_\lambda(n)$ is defined by*

$$\omega_\lambda(n) = \sum_{m=0}^{\infty} \frac{1}{(m+n+1)^\lambda}, \quad n \in N_0,\ 1.5 \le \lambda < 3. \tag{2.1}$$

Then

$$\frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)}\left[1 - \frac{(\lambda-1)^2}{4(n+1)^2}\right] < \omega_\lambda(n) < \frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)}. \tag{2.2}$$

Proof. If $n \in N_0$, let $f(x) = 1/(m+n+1)^\lambda$, $x \in [0,\infty)$. By (1.8), we obtain

$$\begin{aligned} \omega_\lambda(n) &> \int_0^\infty \frac{dx}{(x+n+1)^\lambda} + \frac{1}{2(n+1)^\lambda} = \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda}, \\ \omega_\lambda(n) &< \int_0^\infty \frac{1}{(x+n+1)^\lambda}dx + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}} \\ &= \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}}. \end{aligned} \tag{2.3}$$

Since we find

$$\begin{aligned} &\left[\frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda}\right]\left[2(n+1)^{\lambda-1} - (\lambda-1)(n+1)^{\lambda-2}\right] = \frac{2}{\lambda-1} - \frac{\lambda-1}{2(n+1)^2}, \\ &\left[\frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}}\right]\left[2(n+1)^{\lambda-1} - (\lambda-1)(n+1)^{\lambda-2}\right] \\ &\quad = \frac{2}{\lambda-1} - \frac{2\lambda-3}{6(n+1)^2} - \frac{\lambda(\lambda-1)}{12(n+1)^3}. \end{aligned} \tag{2.4}$$

Then we obtain

$$\begin{aligned} \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} &= \frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)}\left[1 - \frac{(\lambda-1)^2}{4(n+1)^2}\right], \\ \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}} &= \frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)} \\ &\quad \times \left[1 - \frac{(2\lambda-3)(\lambda-1)}{12(n+1)^2} - \frac{\lambda(\lambda-1)^2}{24(n+1)^3}\right]. \end{aligned} \tag{2.5}$$

Since for $1.5 \le \lambda < 3$, $2(n+1)^{2-\lambda}/(\lambda-1)(2n+3-\lambda) > 0$, $(2\lambda-3)(\lambda-1)/12(n+1)^2 \ge 0$, $\lambda(\lambda-1)^2/24(n+1)^3 > 0$, then we have (2.2). The lemma is proved. □

Theorem 2.2. *Let $0 < p < 1$, $1/p + 1/q = 1$, $1.5 \le \lambda < 3$, and $a_n \ge 0$, $b_n > 0$, such that $0 < \sum_{n=0}^{\infty}((n+1)^{2-\lambda}a_n^p/(2n+3-\lambda)) < \infty$, $0 < \sum_{n=0}^{\infty}((n+1)^{2-\lambda}b_n^q/(2n+3-\lambda)) < \infty$. Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} > \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p\right\}^{1/p} \times\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}b_n^q\right\}^{1/q}. \tag{2.6}$$

Proof. By the reverse Hölder inequality [9], we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda/p}}\cdot\frac{b_n}{(m+n+1)^{\lambda/q}}\\ &\ge \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m^p}{(m+n+1)^{\lambda}}\right\}^{1/p}\cdot\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{b_n^q}{(m+n+1)^{\lambda}}\right\}^{1/q}\\ &= \left\{\sum_{m=0}^{\infty}\omega_\lambda(m)a_m^p\right\}^{1/p}\cdot\left\{\sum_{n=0}^{\infty}\omega_\lambda(n)b_n^q\right\}^{1/q}.\end{aligned} \tag{2.7}$$

Since $0 < p < 1$ and $q < 0$, then by (2.2), we obtain (2.6). The theorem is proved. □

In Theorem 2.2, for $\lambda = 2$, we have the following corollary.

Corollary 2.3. *Let $0 < p < 1$, $1/p + 1/q = 1$, and $a_n \ge 0$, $b_n > 0$, such that $0 < \sum_{n=0}^{\infty}(a_n^p/(2n+1)) < \infty$, $0 < \sum_{n=0}^{\infty}(b_n^q/(2n+1)) < \infty$. Then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^2} > 2\left\{\sum_{n=0}^{\infty}\left[1-\frac{1}{4(n+1)^2}\right]\frac{a_n^p}{2n+1}\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\frac{b_n^q}{2n+1}\right\}^{1/q}. \tag{2.8}$$

Remark 2.4. Inequality (2.8) is inequality [7, Inquality (8)]. Hence, inequality (2.6) is an extension of Yang's inequality [7, Inquality (8)] for $1 < \lambda < 3$.

Theorem 2.5. *Let $0<p<1$, $1/p + 1/q = 1$, $1.5 \le \lambda < 3$, and $a_n \ge 0$, such that $0 < \sum_{n=0}^{\infty}((n+1)^{2-\lambda}a_n^p/(2n+3-\lambda)) < \infty$. Then*

$$\sum_{n=0}^{\infty}\left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p > \left(\frac{2}{\lambda-1}\right)^p\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p. \tag{2.9}$$

Inequalities (2.9) and (2.6) are equivalent.

Proof. Let

$$b_n = \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^{p-1}, \quad n \in N_0. \tag{2.10}$$

By (2.6), we have

$$\begin{aligned}\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda}\right\}^p &= \left\{\sum_{n=0}^{\infty}\left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p\right\}^p \\ &= \left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}\right\}^p \\ &\geq \left(\frac{2}{\lambda-1}\right)^p\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p \\ &\quad\times\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda}\right\}^{p-1}.\end{aligned} \tag{2.11}$$

Then we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda} &= \sum_{n=0}^{\infty}\left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p \\ &\geq \left(\frac{2}{\lambda-1}\right)^p\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p.\end{aligned} \tag{2.12}$$

If $\sum_{n=0}^{\infty}((n+1)^{2-\lambda}b_n^q/(2n+3-\lambda)) = \infty$, then in view of

$$0 < \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}a_n^p}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right] \leq \sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}a_n^p}{2n+3-\lambda} < \infty \tag{2.13}$$

and (2.12), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p &> \left(\frac{2}{\lambda-1}\right)^p \\ &\times\sum_{n=0}^{\infty}\frac{(n+1)^{\lambda-2}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p;\end{aligned} \tag{2.14}$$

if $0 < \sum_{n=0}^{\infty}((n+1)^{\lambda-2}b_n^q/(2n+3-\lambda)) < \infty$, then by (2.6), we find

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p &> \left(\frac{2}{\lambda-1}\right)^p \\ &\times\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]a_n^p.\end{aligned} \tag{2.15}$$

Hence we obtain (2.9).

On the other hand, by the reverse Hölder inequality [9], we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} &= \left[\sum_{n=0}^{\infty}(n+1)^{(\lambda-2)/q}(2n+3-\lambda)^{1/q}\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right] \\ &\quad\times\left[\frac{b_n}{(n+1)^{(\lambda-2)/q}(2n+3-\lambda)^{1/q}}\right] \\ &\geq \left\{\sum_{n=0}^{\infty}\left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda}\right]^{1-p}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^{p}\right\}^{1/p} \\ &\quad\times\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda}\right\}^{1/q}.\end{aligned} \tag{2.16}$$

Hence by (2.9), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} &> \frac{2}{\lambda-1}\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}a_n^p}{2n+3-\lambda}\left[1-\frac{(\lambda-1)^2}{4(n+1)^2}\right]\right\}^{1/p} \\ &\quad\times\left\{\sum_{n=0}^{\infty}\frac{(n+1)^{2-\lambda}b_n^q}{2n+3-\lambda}\right\}^{1/q}.\end{aligned} \tag{2.17}$$

Then, (2.9) and (2.6) are equivalent. The theorem is proved. □

In (2.9), for $\lambda = 2$, we have the following corollary.

Corollary 2.6. *Let* $0 < p < 1$, $1/p + 1/q = 1$, $a_n \geq 0$, $0 < \sum_{n=0}^{\infty}(a_n^p/(2n+1)) < \infty$, *Then*

$$\sum_{n=0}^{\infty}(2n+1)^{p-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^2}\right]^p > 2^p\sum_{n=0}^{\infty}\left[1-\frac{1}{4(n+1)^2}\right]\frac{a_n^p}{2n+1}. \tag{2.18}$$

Inequalities (2.18) and (2.8) are equivalent.

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