Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 82910, 4 pages
doi:10.1155/2007/82910

Research Article

Linear Maps which Preserve or Strongly Preserve Weak Majorization

Ahmad Mohammad Hasani and Mohammad Ali Vali

Received 8 July 2007; Accepted 5 November 2007

Dedicated to Professor Mehdi Radjabalipour

Recommended by Jewgeni H. Dshalalow

For $x, y \in \mathbb{R}^n$, we say x is weakly submajorized (weakly supermajorized) by y, and write $x \prec_\omega y$ ($x \prec^\omega y$), if $\sum_1^k x_{[i]} \leq \sum_1^k y_{[i]}$, $k = 1,2,\ldots,n$ ($\sum_1^k x_{(i)} \geq \sum_1^k y_{(i)}$, $k = 1,2,\ldots,n$), where $x_{[i]}$ ($x_{(i)}$) denotes the ith component of the vector $x_\downarrow$ ($x^\uparrow$) whose components are a decreasing (increasing) rearrangment of the components of x. We characterize the linear maps that preserve (strongly preserve) one of the majorizations $\prec_\omega$ or $\prec^\omega$.

1. Introduction

The classical majorization and matrix majorization have received considerable attention by many authors. Recently, much interest has focused on the structure of linear preservers and strongly linear preservers of vector and matrix majorizations. Many nice results have been found by Beasley and S. G. Lee [1–4], Ando [5], Dahl [6], Li and Poon [7], and Hasani and Radjabalipour [8–10].

Marshal and Olkin's text [11] is the standard general reference for majorization. A matrix D with nonnegative entries is called doubly stochastic if the sum of each row of D and also the sum of each row of D^t are 1.

Let the following notations be fixed throughout the paper: M_{nm} (M_m) for the set of real $n \times m$ ($m \times m$) matrices, $\mathrm{DS}(n)$ for the set of all $n \times n$ doubly stochastic matrices, $P(n)$ for the set of all $n \times n$ permutation matrices, $\mathbb{R}^n$ for the set of all real $n \times 1$ (column) vectors (note that $\mathbb{R}^n = M_{n1}$), $\{e_1, e_2, \ldots, e_n\}$ for the standard basis for $\mathbb{R}^n$, $e = \sum_{j=1}^n e_j$, $J = ee^t$, the $n \times n$ matrix with all entries equal to 1, $\mathrm{tr}x$ for the trace of the vector x.

For $x, y \in \mathbb{R}^n$, we say x is weakly submajorized (weakly supermajorized) by y, and we write $x \prec_\omega y$ ($x \prec^\omega y$) if

$$\sum_{1}^{k} x_{[i]} \le \sum_{1}^{k} y_{[i]}, \quad k = 1,2,\ldots,n \quad \left(\sum_{1}^{k} x_{(i)} \ge \sum_{1}^{k} y_{(i)},\ k = 1,2,\ldots,n\right), \tag{1.1}$$

where $x_{[i]}$ ($x_{(i)}$) denotes the ith component of the vector $x_\downarrow$ ($x^\uparrow$) whose components are a decreasing (increasing) rearrangement of the components of x. If in addition to $x \prec_\omega y$ we also have $\sum_1^n x_j = \sum_1^n y_j$, we say x is majorized by y and write $x \prec y$. This definition $x \prec y$ is equivalent to $x = Dy$ for some $D \in \mathrm{DS}(n)$ [11].

Given $X, Y \in M_{n,m}$, we say X is multivariate majorized by Y (written $X \prec Y$) if $X = DY$ for some $D \in \mathrm{DS}(n)$. When $m = 1$, the definition of multivariate majorization reduces to the classical concept of majorization on $\mathbb{R}^n$. Let T be a linear map and let R be a relation on $\mathbb{R}^n$. We say T preserves R when $R(x,y)$ implies $R(Tx,Ty)$; if in addition $R(Tx,Ty)$ implies $R(x,y)$, we say T strongly preserves R.

We need the following interesting theorem in our work.

Theorem 1.1 (see [5]). *A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $Ax \prec Ay$ whenever $x \prec y$ if and only if one of the following holds:*

(i) *$Ax = (\mathrm{tr}\,x)a$ for some $a \in \mathbb{R}^n$,*

(ii) *$Ax = \alpha Px + \beta(\mathrm{tr}\,x)e = \alpha Px + \beta Jx$ for some $\alpha, \beta \in \mathbb{R}$ and $P \in P(n)$.*

2. Main results

Now we are ready to state and prove our main results.

Theorem 2.1. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. The following are equivalent:*

(i) *A preserves $\prec_\omega$;*

(ii) *A preserves $\prec^\omega$;*

(iii) *A is nonnegative and preserves $\prec$.*

Proof. The proof of (i)⇔(ii) is obvious from the fact that $x \prec_\omega y$ if and only if $-x \prec^\omega -y$.

(i)⇒(iii) First we show that if $x = Py$ for some $P \in P(n)$, then $Ax = QAy$ for some $Q \in P(n)$.

Now $x = Py$ if and only if $x \prec_\omega y \prec_\omega x$. By hypothesis, $Ax \prec_\omega Ay \prec_\omega Ax$, hence $Ax = QAy$ for some $Q \in P(n)$.

Let $x \prec y$. Then $x = Dy$ for some doubly stochastic matrix D. Since $D = \sum_i L_i P_i$, $0 \le L_i \le 1$, $P_i \in P(n)$, $i = 1,2,\ldots,n_0$, for some $n_0 \in N$. So we have

$$Ax = \sum_i L_i A P_i y = \sum_i L_i Q_i A y = D' A y, \quad D' \in \mathrm{DS}(n). \tag{2.1}$$

Hence $Ax \prec Ay$.

The nonnegativity of A follows from the fact that $-e_i \prec_\omega 0$, $i = 1,2,\ldots,n$, implies $A(e_i) \prec^\omega 0 = A(0)$. Hence $\min\{a_{ij},\ i = 1,\ldots,n,\ s = 1,\ldots,n\} \ge 0$, where a_{ij} is the ijth entry of matrix A.

(iii)⇒(i) Let $x \prec_\omega y$. There exists $\varepsilon \ge 0$ such that

$$(x_{[1]}, x_{[2]}, \ldots, x_{[n]}) \prec (y_{[1]}, y_{[2]}, \ldots, y_{[n]}) - \varepsilon e_n. \tag{2.2}$$

By hypothesis, $(Ax)_\downarrow \prec (Ay)_\downarrow - \varepsilon A e_n$, which implies that $Ax \prec_\omega Ay$, because Ae_n has nonnegative components. □

Lemma 2.2. *Let $A:\mathbb{R}^n\to\mathbb{R}^n$ be a linear map that strongly preserves one of the weak majorizations $\prec_\omega$ or $\prec^\omega$. Then A is invertible.*

Proof. Let $Ax = 0$. Then $A(0) \prec_\omega Ax \prec_\omega A0$ implies $0 \prec_\omega x \prec_\omega 0$. Hence $x = 0$. □

Theorem 2.3. *A linear map $A:\mathbb{R}^n\to\mathbb{R}^n$ strongly preserves one of the weak majorizations $\prec_\omega$ or $\prec^\omega$ if and only if it has the form*

$$x \longmapsto rPx \tag{2.3}$$

for some positive real number r and some $P \in P(n)$.

Proof. By Theorem 2.1, A preserves the majorization relation $\prec$, and A is nonnegative. By Theorem 1.1, A has one of the following forms:

(1) $Ax = (\operatorname{tr}x)\alpha$ for some $a \in \mathbb{R}^n$, or

(2) $Ax = (rP+sJ)x$ for some $r,s \in \mathbb{R}$ and $P \in P(n)$.

By Lemma 2.2, A is invertible and hence has only the form

$$Ax = (rP+sJ)x = P(rI+sJ)x. \tag{2.4}$$

It follows from $(rI+sJ)e = (r+ns)e$ that $r+ns$ needs to be nonzero, because $(rI+sJ)$ is invertible. Also r needs to be nonzero for $(rI+sJ)$ to be invertible. Now if $x \prec_\omega y$, then $A(A^{-1}x) \prec_\omega A(A^{-1}y)$, and by hypothesis, $A^{-1}x \prec_\omega A^{-1}y$. By Theorem 2.1, A^{-1} preserves the majorization relation $\prec$, and A^{-1} is nonnegative and so has the form

$$A^{-1}x = (r'P+s'J)x \quad \text{for some } r', s' \in \mathbb{R},\ P \in P(n). \tag{2.5}$$

Using $AA^{-1} = I_{n\times n}$, we conclude that $r' = 1/r$ and $s' = -s/r(r+ns)$.

Since A and A^{-1} have nonnegative entries, we must have $r+s \geq 0$, $r'+s' \geq 0$, $s \geq 0$, $s' = -s/r(r+ns) \geq 0$, which implies that $r(r+ns) < 0$ if $s > 0$. Also from $r'+s' = (r+(n-1)s)/r(r+ns) \geq 0$, we have $r(r+ns) > 0$, which is impossible unless $s = 0$, and hence $s' = 0$.

So $r > 0$, and the form of A is

$$x \longmapsto rPx, \tag{2.6}$$

where $r > 0$ and $P \in P(n)$. Also A^{-1} has the form

$$x \longmapsto r^{-1}P^t x. \tag{2.7}$$

Clearly, the linear map $x\to rPx$, for $r > 0$ and $P \in P(n)$, strongly preserves weak majorizations $\prec_\omega$ and $\prec^\omega$. □

Remark 2.4. Fumio Hiai in [12, Section 3] gives the noncommutative version of our main results, where linear maps from the set of $n \times n$ Hermitian matrices to themselves, which preserve majorization and weak majorization relations on spectrum, are characterized. Also it is shown that such a linear map preserves weak majorization of the spectrum if and only if it is positive and preserves majorization of the spectrum. Our result is a commutative version of Hial's result.

Acknowledgment

The authors would like to thank the referees for their valuable comments that helped them improve this paper.

References

[1] L. B. Beasley and S.-G. Lee, "Linear operators preserving multivariate majorization," *Linear Algebra and Its Applications*, vol. 304, no. 1–3, pp. 141–159, 2000.
[2] L. B. Beasley, S.-G. Lee, and Y.-H. Lee, "Linear operators strongly preserving multivariate majorization with $T(I) = I$," *Kyungpook Mathematical Journal*, vol. 39, no. 1, pp. 191–194, 1999.
[3] L. B. Beasley, S.-G. Lee, and Y.-H. Lee, "Resolution of the conjecture on strong preservers of multivariate majorization," *Bulletin of the Korean Mathematical Society*, vol. 39, no. 2, pp. 283–287, 2002.
[4] L. B. Beasley, S.-G. Lee, and Y.-H. Lee, "A characterization of strong preservers of matrix majorization," *Linear Algebra and Its Applications*, vol. 367, pp. 341–346, 2003.
[5] T. Ando, "Majorization, doubly stochastic matrices, and comparison of eigenvalues," *Linear Algebra and Its Applications*, vol. 118, pp. 163–248, 1989.
[6] G. Dahl, "Matrix majorization," *Linear Algebra and Its Applications*, vol. 288, no. 1–3, pp. 53–73, 1999.
[7] C.-K. Li and E. Poon, "Linear operators preserving directional majorization," *Linear Algebra and Its Applications*, vol. 325, no. 1–3, pp. 141–146, 2001.
[8] A. M. Hasani and M. Radjabalipour, "Linear preserver of matrix majorization," *International Journal of Pure and Applied Mathematics*, vol. 32, no. 4, pp. 475–482, 2006.
[9] A. M. Hasani and M. Radjabalipour, "On linear preservers of (right) matrix majorization," *Linear Algebra and Its Applications*, vol. 423, no. 2-3, pp. 255–261, 2007.
[10] A. M. Hasani and M. Radjabalipour, "The structure of linear operators strongly preserving majorizations of matrices," *Electronic Journal of Linear Algebra*, vol. 15, pp. 260–268, 2006.
[11] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.
[12] F. Hiai, "Similarity preserving linear maps on matrices," *Linear Algebra and Its Applications*, vol. 97, pp. 127–139, 1987.

Ahmad Mohammad Hasani: Department of Mathematics, Shahid Bahonar University of Kerman, Kerman 76169-14111, Iran
Email address: mohamad.h@graduate.uk.ac.ir

Mohammad Ali Vali: Department of Mathematics, Shahid Bahonar University of Kerman, Kerman 76169-14111, Iran
Email address: mohamadali_35@yahoo.com