Research Article
Some Characteristic Quantities Associated with Homogeneous P-Type and M-Type Functions

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Several characteristic quantities associated with homogeneous P-type and M-type functions are introduced and studied in this paper. Further, the concepts of P-property and M-property for a couple of functions are introduced and some quantities for a pair of homogeneous functions having P-property and M-property are obtained, respectively. As an application, a bound for the solution of the homogeneous complementarity problem with a P-type function is derived.

1. Introduction

For any given P-matrix [1](see also [2–4]), $M \in \mathbb{R}^{n \times n}$, Mathias and Pang [5] introduced a quantity $\alpha(M)$ by

$$\alpha(M) = \min_{\|x\|_\infty = 1} \max_{1 \leq i \leq n} x_i (Mx)_i. \tag{1.1}$$

In terms of $\alpha(M)$, a bound for the solution of the linear complementarity problem LCP(M,q) (see [2–4]) with a P-matrix M is established in [5]. Recently, Xiu and Zhang [6] further gave some new properties of $\alpha(M)$ and introduced a new quantity $\beta(M)$, which is defined by

$$\beta(M) = \max_{\|x\|_\infty = 1} \max_{1 \leq i \leq n} x_i (Mx)_i. \tag{1.2}$$

Moreover, Xiu and Zhang [6] introduced a fundamental quantity $\alpha\{A, B\}$ associated with a pair $\{A, B\}$ having v-column P-property (see [2–4, 7]) by

$$\alpha\{A, B\} = \min_{\|x\|_\infty = 1} \max_{1 \le i \le n} (Ax)_i (Bx)_i, \tag{1.3}$$

where $A, B \in \mathbb{R}^{m \times n}$. They developed some characteristic quantities of $\alpha\{A, B\}$. By means of these quantities, Xiu and Zhang [6] established global error bounds for the vertical and horizontal linear complementarity problems.

Motivated by these works, in this paper, we introduce the concepts of P-type and M-type functions and give several quantities for homogeneous P-type and M-type functions. Furthermore, we give the concepts of P-property and M-property for a couple of functions, and obtain some quantities for homogeneous continuous pair with P-property and M-property, respectively. As an application, a bound of the solution to the homogeneous complementarity problem with a P-type function is obtained.

2. Characteristic quantities for P-Type and M-Type functions

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a function. We say that T is positively homogeneous with degree $\theta > 0$ if $T(\lambda x) = \lambda^\theta T(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. Define $\mathcal{H}$ by

$$\mathcal{H} = \{T \mid T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ is continuous and positively homogeneous}\}. \tag{2.1}$$

Given $T \in \mathcal{H}$, define

$$\|T\| = \max_{\|x\|=1} \|T(x)\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|^\theta}, \tag{2.2}$$

where $\theta > 0$ is the positively homogeneous degree of T and $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

Theorem 2.1. *Let $T, S : \mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Then the following conclusions hold:*
  (i) $\|T(x)\| \le \|T\| \cdot \|x\|^\theta$;
  (ii) *if the inverse T^{-1} in $\mathcal{H}$ exists, then T^{-1} is positively homogeneous with degree $1/\theta$;*
  (iii) $\|T \cdot S\| \le \|T\| \cdot \|S\|^\theta$.

Proof. (i) This follows directly from (2.2).
  (ii) Since $T^{-1} \in \mathcal{H}$, we suppose the degree of T^{-1} is θ'. It follows that

$$(T^{-1} \cdot T)(\lambda x) = \lambda x = T^{-1}(\lambda^\theta T(x)) = \lambda^{\theta\theta'} (T^{-1} \cdot T)(x) = \lambda^{\theta\theta'} x. \tag{2.3}$$

Hence $\theta' = 1/\theta$.

(iii) It is easy to see that $T \cdot S$ is positively homogeneous with degree $\theta\rho$. By (2.2),

$$\|T \cdot S\| = \sup_{x \neq 0} \frac{\|T(S(x))\|}{\|x\|^{\theta\rho}} \leq \sup_{x \neq 0} \frac{\|T\| \cdot \|S(x)\|^{\theta}}{\|x\|^{\theta\rho}}$$

$$\leq \sup_{x \neq 0} \frac{\|T\| \cdot \|S\|^{\theta} \cdot \|x\|^{\theta\rho}}{\|x\|^{\theta\rho}} = \|T\| \cdot \|S\|^{\theta}. \tag{2.4}$$

This completes the proof. □

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a function. Recall that T is a P-function (see [3, 4]) if

$$\max_{1 \leq i \leq n} (x_i - y_i)(T(x) - T(y))_i > 0 \tag{2.5}$$

for all $x \neq y$.

We now introduce the concepts of M-type and P-type functions as follows.

Definition 2.2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a function. T is said to be
    (i) M-type if

$$\min_{1 \leq i \leq n} x_i(T(x))_i > 0, \quad \forall x \neq 0; \tag{2.6}$$

    (ii) P-type if

$$\max_{1 \leq i \leq n} x_i(T(x))_i > 0, \quad \forall x \neq 0. \tag{2.7}$$

Note that a P-function T with $T(0) = 0$ is P-type and a function $T : \mathbb{R}^n \to \mathbb{R}^n$ is M-type if $T(0) = 0$ and $T_i : \mathbb{R}^n \to \mathbb{R}$ is strictly monotone for each i, where $T_i(x) = [T(x)]_i$.

For any given P-type and positively homogeneous function T with degree $\theta > 0$, we define $\alpha(T)$ and $\beta(T)$ by

$$\alpha(T) = \min_{\|x\|_{\infty}=1} \max_{1 \leq i \leq n} x_i(T(x))_i = \inf_{x \neq 0} \frac{\max_{1 \leq i \leq n} x_i(T(x))_i}{\|x\|_{\infty}^{\theta+1}}, \tag{2.8}$$

$$\beta(T) = \max_{\|x\|_{\infty}=1} \max_{1 \leq i \leq n} x_i(T(x))_i = \sup_{x \neq 0} \frac{\max_{1 \leq i \leq n} x_i(T(x))_i}{\|x\|_{\infty}^{\theta+1}}, \tag{2.9}$$

where $\|x\|_{\infty} = \max_{1 \leq i \leq n}\{|x_i|\}$. In addition, if T is M-type, we can further define $\alpha'(T)$ and $\beta'(T)$ by

$$\alpha'(T) = \max_{\|x\|_{\infty}=1} \min_{1 \leq i \leq n} x_i(T(x))_i = \sup_{x \neq 0} \frac{\min_{1 \leq i \leq n} x_i(T(x))_i}{\|x\|_{\infty}^{\theta+1}}, \tag{2.10}$$

$$\beta'(T) = \min_{\|x\|_{\infty}=1} \min_{1 \leq i \leq n} x_i(T(x))_i = \inf_{x \neq 0} \frac{\min_{1 \leq i \leq n} x_i(T(x))_i}{\|x\|_{\infty}^{\theta+1}}. \tag{2.11}$$

Obviously, $\alpha(T)$, $\beta(T)$, $\alpha'(T)$, and $\beta'(T)$ are well defined, finite, and positive.

Remarks 2.3. The definitions of $\alpha(T)$, $\beta(T)$ associated with a *P*-type positively homogeneous function T generalize the definitions of $\alpha(M)$, $\beta(M)$ associated with a *P*-matrix in [5, 6], respectively.

By (2.8)–(2.11), we can obtain the following proposition.

PROPOSITION 2.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a positively homogeneous function with degree θ. Then the following conclusions hold:*
  (i) *if T is P-type, then*

$$\alpha(T)\|x\|_\infty^{\theta+1} \le \max_{1 \le i \le n} x_i(T(x))_i \le \beta(T)\|x\|_\infty^{\theta+1}; \tag{2.12}$$

  (ii) *if T is M-type, then*

$$\beta'(T)\|x\|_\infty^{\theta+1} \le \min_{1 \le i \le n} x_i(T(x))_i \le \alpha'(T)\|x\|_\infty^{\theta+1}, \tag{2.13}$$

$$\beta'(T) \le \alpha'(T) \le \alpha(T) \le \beta(T). \tag{2.14}$$

THEOREM 2.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a P-type and positively homogeneous function with degree θ and have inverse T^{-1} in $\mathcal{H}$. Then the following conclusions hold:*
  (a) $\beta(T) \le \|T\|_\infty$;
  (b) $\alpha(T) \le 1/\|T^{-1}\|_\infty^\theta$;
  (c) $1/\|T^{-1}\|_\infty^{\theta+1} \le \beta(T)/\beta(T^{-1})$, $\alpha(T)/\alpha(T^{-1}) \le \|T\|_\infty^{1+1/\theta}$.

Proof. For any nonzero $x \in \mathbb{R}^n$, we know

$$x_i(T(x))_i \le \|x\|_\infty \cdot \|T(x)\|_\infty \le \|T\|_\infty \cdot \|x\|_\infty^{\theta+1}, \quad i = 1, 2, \ldots, n. \tag{2.15}$$

By (2.9), we obtain $\beta(T) \le \|T\|_\infty$. Hence (a) is true.

From (2.2) and Theorem 2.1,

$$\|T^{-1}\|_\infty = \sup_{x \ne 0} \frac{\|T^{-1}(x)\|_\infty}{\|x\|_\infty^{1/\theta}} = \sup_{y \ne 0} \frac{\|y\|_\infty}{\|T(y)\|_\infty^{1/\theta}} = \sup_{y \ne 0} \frac{\left(\|y\|_\infty^{1+\theta}\right)^{1/\theta}}{\left(\|T(y)\|_\infty \cdot \|y\|_\infty\right)^{1/\theta}}. \tag{2.16}$$

Since $\|T(y)\|_\infty \cdot \|y\|_\infty \ge \max_{1 \le i \le n} y_i(T(y))_i$, we have

$$\begin{aligned}
\|T^{-1}\|_\infty &\le \sup_{y \ne 0} \left[\frac{\|y\|_\infty^{1+\theta}}{\max_{1 \le i \le n} y_i(T(y))_i} \right]^{1/\theta} \\
&= \left[\sup_{y \ne 0} \frac{\|y\|_\infty^{1+\theta}}{\max_{1 \le i \le n} y_i(T(y))_i} \right]^{1/\theta} = \left[\frac{1}{\alpha(T)} \right]^{1/\theta}
\end{aligned} \tag{2.17}$$

and so

$$\alpha(T) \le \frac{1}{\|T^{-1}\|_\infty^\theta}. \tag{2.18}$$

Hence (b) is true.

From (2.8), (2.9), and Theorem 2.1, we know

$$
\begin{aligned}
\beta(T^{-1}) &= \sup_{x \neq 0} \frac{\max_{1 \leq i \leq n} x_i(T^{-1}(x))_i}{\|x\|_{\infty}^{1+1/\theta}} = \sup_{y \neq 0} \frac{\max_{1 \leq i \leq n} y_i(T(y))_i}{\|T(y)\|_{\infty}^{1+1/\theta}} \\
&\geq \sup_{y \neq 0} \frac{\max_{1 \leq i \leq n} y_i(T(y))_i}{\|T\|_{\infty}^{1+1/\theta} \cdot \|y\|_{\infty}^{1+\theta}} = \frac{\beta(T)}{\|T\|_{\infty}^{1+1/\theta}}, \\
\alpha(T^{-1}) &= \inf_{x \neq 0} \frac{\max_{1 \leq i \leq n} x_i(T^{-1}(x))_i}{\|x\|_{\infty}^{1+1/\theta}} = \inf_{y \neq 0} \frac{\max_{1 \leq i \leq n} y_i(T(y))_i}{\|T(y)\|_{\infty}^{1+1/\theta}} \\
&\geq \inf_{y \neq 0} \frac{\max_{1 \leq i \leq n} y_i(T(y))_i}{\|T\|_{\infty}^{1+1/\theta} \cdot \|y\|_{\infty}^{1+\theta}} = \frac{\alpha(T)}{\|T\|_{\infty}^{1+1/\theta}},
\end{aligned}
\tag{2.19}
$$

which yields the second inequality in (c).

By the same arguments, we can prove

$$
\beta(T) \geq \frac{\beta(T^{-1})}{\|T^{-1}\|_{\infty}^{1+\theta}}, \qquad \alpha(T) \geq \frac{\alpha(T^{-1})}{\|T^{-1}\|_{\infty}^{1+\theta}},
\tag{2.20}
$$

which yields the first inequality in (c). This completes the proof. $\square$

Similarly, we can obtain the following results.

THEOREM 2.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an M-type and positively homogeneous function with degree θ and have inverse T^{-1} in $\mathcal{H}$. Then*
  (i) $\beta'(T) \leq 1/\|T^{-1}\|_{\infty}^{\theta}$;
  (ii) $1/\|T^{-1}\|_{\infty}^{\theta+1} \leq \beta'(T)/\beta'(T^{-1})$, $\alpha'(T)/\alpha'(T^{-1}) \leq \|T\|_{\infty}^{1+1/\theta}$.

THEOREM 2.7. *Let $T, S : \mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with the same degree θ. Then the following conclusions hold:*
  (1) *if both T and S are P-type, then $\beta(T + S) \leq \beta(T) + \beta(S)$;*
  (2) *if T is P-type and S is M-type, then $\alpha(T + S) \geq \alpha(T)$, $\beta(T + S) \geq \beta(T)$;*
  (3) *if both T and S are M-type, then*

$$
\begin{aligned}
\beta'(T + S) &\geq \beta'(T) + \beta'(S), \qquad \alpha'(T + S) \geq \max\{\alpha'(T), \alpha'(S)\}, \\
\beta'(T + S) &\geq \max\{\beta'(T), \beta'(S)\}.
\end{aligned}
\tag{2.21}
$$

Proof. The facts directly follow from the definitions of $\alpha, \beta, \alpha', \beta'$, and simple arguments.
$\square$

Remarks 2.8. Theorems 2.5–2.7 generalize partly Theorems 2.1 and 2.5 of Xiu and Zhang [6].

3. Extensions

In this section, we introduce the definitions of *P*-property and *M*-property for a pair $\{T, S\}$ and generalize some results for a function T in Section 2 to a pair $\{T, S\}$.

Definition 3.1. Let $T, S: \mathbb{R}^n \to \mathbb{R}^n$ be two functions. Say that $\{T, S\}$ has

    (1) *P*-property if for any nonzero $x \in \mathbb{R}^n$,

$$\max_{1 \leq i \leq n} (T(x))_i (S(x))_i > 0; \tag{3.1}$$

    (ii) *M*-property if for any nonzero $x \in \mathbb{R}^n$,

$$\min_{1 \leq i \leq n} (T(x))_i (S(x))_i > 0. \tag{3.2}$$

Let $T, S \in \mathcal{H}$ with positively homogeneous degrees θ and ρ, respectively, and $\{T, S\}$ have *P*-property. Define $\alpha\{T, S\}$ and $\beta\{T, S\}$ as follows:

$$\alpha\{T, S\} = \min_{\|x\|_\infty = 1} \max_{1 \leq i \leq n} (T(x))_i (S(x))_i = \inf_{x \neq 0} \frac{\max_{1 \leq i \leq n} (T(x))_i (S(x))_i}{\|x\|_\infty^{\theta + \rho}}, \tag{3.3}$$

$$\beta\{T, S\} = \max_{\|x\|_\infty = 1} \max_{1 \leq i \leq n} (T(x))_i (S(x))_i = \sup_{x \neq 0} \frac{\max_{1 \leq i \leq n} (T(x))_i (S(x))_i}{\|x\|_\infty^{\theta + \rho}}. \tag{3.4}$$

Remarks 3.2. The definitions of $\alpha\{T, S\}$, $\beta\{T, S\}$ associated with a positively homogeneous function pair $\{T, S\}$ having *P*-property generalize the definitions of $\alpha\{M, N\}$, $\beta\{M, N\}$ associated with a matrix pair having ν-column *P*-property in [2, 6, 7].

In addition, if $\{T, S\}$ has *M*-property, we can define $\alpha'\{T, S\}$ and $\beta'\{T, S\}$ by

$$\alpha'\{T, S\} = \max_{\|x\|_\infty = 1} \min_{1 \leq i \leq n} (T(x))_i (S(x))_i = \sup_{x \neq 0} \frac{\min_{1 \leq i \leq n} (T(x))_i (S(x))_i}{\|x\|_\infty^{\theta + \rho}}, \tag{3.5}$$

$$\beta'\{T, S\} = \min_{\|x\|_\infty = 1} \min_{1 \leq i \leq n} (T(x))_i (S(x))_i = \inf_{x \neq 0} \frac{\min_{1 \leq i \leq n} (T(x))_i (S(x))_i}{\|x\|_\infty^{\theta + \rho}}. \tag{3.6}$$

By the definitions of $\alpha\{T, S\}, \beta\{T, S\}, \alpha'\{T, S\}$, and $\beta'\{T, S\}$, we can obtain the following proposition.

PROPOSITION 3.3. *Let $T, S: \mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Then the following conclusions hold:*

    (i) *if $\{T, S\}$ has P-property, then*

$$\alpha\{T, S\} \|x\|_\infty^{\theta + \rho} \leq \max_{1 \leq i \leq n} (T(x))_i (S(x))_i \leq \beta\{T, S\} \|x\|_\infty^{\theta + \rho}; \tag{3.7}$$

    (ii) *if $\{T, S\}$ has M-property, then*

$$\beta'\{T, S\} \|x\|_\infty^{\theta + \rho} \leq \min_{1 \leq i \leq n} (T(x))_i (S(x))_i \leq \alpha'\{T, S\} \|x\|_\infty^{\theta + \rho}, \tag{3.8}$$

$$\beta'\{T, S\} \leq \alpha'\{T, S\} \leq \alpha\{T, S\} \leq \beta\{T, S\}. \tag{3.9}$$

Note that, if T^{-1} exists, then the condition that $\{T, S\}$ has P-property (M-property) is equivalent to the condition that ST^{-1} is P-type (M-type).

THEOREM 3.4. *Let $T, S: \mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Suppose that $\{T, S\}$ has P-property and T has inverse T^{-1} in $\mathcal{H}$. Then*

the following conclusions hold:

(a) $\beta\{T,S\} \leq \|T\|_\infty \cdot \|S\|_\infty$;

(b) $\alpha\{T,S\} \leq \|S\|_\infty / \|T^{-1}\|_\infty^\theta$;

(c) $1/\|T^{-1}\|_\infty^{\theta+\rho} \leq \beta\{T,S\}/\beta(ST^{-1})$, $\alpha\{T,S\}/\alpha(ST^{-1}) \leq \|T\|_\infty^{1+(\rho/\theta)}$.

Proof. (a) For any nonzero $x \in \mathbb{R}^n$, it follows from (i) of Theorem 2.1 that

$$(T(x))_i (S(x))_i \leq \|T(x)\|_\infty \cdot \|S(x)\|_\infty \leq \|T\|_\infty \cdot \|S\|_\infty \cdot \|x\|_\infty^{\theta+\rho}, \quad i = 1,2,\ldots,n. \tag{3.10}$$

By (3.4),

$$\beta\{T,S\} \leq \|T\|_\infty \cdot \|S\|_\infty. \tag{3.11}$$

(b) From (2.2) and (i) of Theorem 2.1,

$$\|T^{-1}\|_\infty = \sup_{x\neq 0} \frac{\|T^{-1}(x)\|_\infty}{\|x\|_\infty^{1/\theta}} = \sup_{y\neq 0} \frac{\|y\|_\infty}{\|T(y)\|_\infty^{1/\theta}} = \sup_{y\neq 0} \frac{\left(\|y\|_\infty^\theta \cdot \|S(y)\|_\infty\right)^{1/\theta}}{\left(\|T(y)\|_\infty \cdot \|S(y)\|_\infty\right)^{1/\theta}}. \tag{3.12}$$

Since $\|T(y)\|_\infty \cdot \|y\|_\infty \geq \max_{1\leq i\leq n} y_i(T(y))_i$ and $\|S(y)\|_\infty \leq \|S\|_\infty \cdot \|y\|_\infty^\rho$, we have

$$\|T^{-1}\|_\infty \leq \sup_{y\neq 0} \left[\frac{\|S\|_\infty \cdot \|y\|_\infty^{\rho+\theta}}{\max_{1\leq i\leq n}(T(y))_i(S(y))_i} \right]^{1/\theta}$$
$$= \|S\|_\infty^{1/\theta} \cdot \left[\sup_{y\neq 0} \frac{\|y\|_\infty^{\rho+\theta}}{\max_{1\leq i\leq n}(T(y))_i(S(y))_i} \right]^{1/\theta} = \left[\frac{\|S\|_\infty}{\alpha\{T,S\}} \right]^{1/\theta}. \tag{3.13}$$

This implies that

$$\alpha\{T,S\} \leq \frac{\|S\|_\infty}{\|T^{-1}\|_\infty^\theta}. \tag{3.14}$$

(c) It follows from (2.9) that

$$\beta(ST^{-1}) = \sup_{x\neq 0} \frac{\max_{1\leq i\leq n} x_i((ST^{-1})(x))_i}{\|x\|_\infty^{1+\rho/\theta}} = \sup_{y\neq 0} \frac{\max_{1\leq i\leq n}(T(y))_i(S(y))_i}{\|T(y)\|_\infty^{1+\rho/\theta}}$$
$$\geq \sup_{y\neq 0} \frac{\max_{1\leq i\leq n}(T(y))_i(S(y))_i}{\|T\|_\infty^{1+\rho/\theta} \cdot \|y\|_\infty^{\rho+\theta}} = \frac{\beta\{T,S\}}{\|T\|_\infty^{1+\rho/\theta}}. \tag{3.15}$$

By (3.4) and Theorem 2.1,

$$\|T^{-1}(y)\|_\infty \leq \|T^{-1}\|_\infty \cdot \|y\|_\infty^{1/\theta},$$
$$\beta\{T,S\} = \sup_{x\neq 0} \frac{\max_{1\leq i\leq n}(T(x))_i(S(x))_i}{\|x\|_\infty^{\rho+\theta}} = \sup_{y\neq 0} \frac{\max_{1\leq i\leq n} y_i((ST^{-1})(y))_i}{\|T^{-1}(y)\|_\infty^{\rho+\theta}}. \tag{3.16}$$

It follows that

$$\beta\{T,S\} \geq \sup_{y\neq 0} \frac{\max_{1\leq i\leq n} y_i((ST^{-1})(y))_i}{\|T^{-1}\|_{\infty}^{\theta+\rho} \cdot \|y\|_{\infty}^{1+\rho/\theta}} = \frac{\beta(ST^{-1})}{\|T^{-1}\|_{\infty}^{\theta+\rho}}. \tag{3.17}$$

Hence

$$\frac{1}{\|T^{-1}\|_{\infty}^{\theta+\rho}} \leq \frac{\beta\{T,S\}}{\beta(ST^{-1})} \leq \|T\|_{\infty}^{1+\rho/\theta}. \tag{3.18}$$

By similar arguments, we can prove that

$$\frac{1}{\|T^{-1}\|_{\infty}^{\theta+\rho}} \leq \frac{\alpha\{T,S\}}{\alpha(ST^{-1})} \leq \|T\|_{\infty}^{1+\rho/\theta}. \tag{3.19}$$

This completes the proof. □

Remarks 3.5. Theorem 3.4 generalizes and improves Theorem 2.7 of Xiu and Zhang [6].

Similarly, we can obtain the following result.

THEOREM 3.6. *Let $T,S : \mathbb{R}^n \to \mathbb{R}^n$ be two positively homogeneous functions with degrees θ and ρ, respectively. Suppose that $\{T,S\}$ has M-property and T has inverse T^{-1} in $\mathcal{H}$. Then the following conclusions hold:*
(1) $\beta'\{T,S\} \leq \|S\|_{\infty}/\|T^{-1}\|_{\infty}^{\theta}$;
(2) $1/\|T^{-1}\|_{\infty}^{\theta+\rho} \leq \alpha'\{T,S\}/\alpha'(ST^{-1})$, $\beta'\{T,S\}/\beta'(ST^{-1}) \leq \|T\|_{\infty}^{1+(\rho/\theta)}$.

4. An application

In this section, we give a bound for the solution of the homogeneous complementarity problem, denoted by HCP(T,q), which consists of finding $x \in \mathbb{R}^n$ such that

$$x \geq 0, \qquad T(x) + q \geq 0, \qquad x^T(T(x) + q) = 0, \tag{4.1}$$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a P-type and positively homogenous function and $q \in \mathbb{R}^n$.

THEOREM 4.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a P-type and positively homogeneous function with degree θ. Suppose that T has inverse T^{-1} in $\mathcal{H}$ and x is the unique solution of HCP(T,q). Then*

$$[\alpha(T^{-1})]^{\theta}\|(-q)_+\|_{\infty} \leq \|x\|_{\infty}^{\theta} \leq \frac{\|(-q)_+\|_{\infty}}{\alpha(T)}, \tag{4.2}$$

where $(-q)_+$ denotes the nonnegative part of $-q$.

Proof. If $x = 0$, then $(-q)_+ = 0$. The conclusion holds trivially. In the sequel we always suppose that $x \neq 0$, equivalently, q is not nonnegative. Since x solves HCP(T,q), by Proposition 2.4, one has

$$\alpha(T)\|x\|_{\infty}^{\theta+1} \leq \max_{1\leq i\leq n} x_i(T(x))_i = \max_{1\leq i\leq n} x_i(-q)_i$$
$$\leq \max_{1\leq i\leq n} x_i((-q)_+)_i \leq \|x\|_{\infty} \cdot \|(-q)_+\|_{\infty}. \tag{4.3}$$

This implies that

$$\|x\|_\infty^\theta \leq \frac{\|(-q)_+\|_\infty}{\alpha(T)}. \tag{4.4}$$

It follows from (2.12) that

$$\alpha(T^{-1})\|y\|_\infty^{1+1/\theta} \leq \max_{1\leq i\leq n} y_i(T^{-1}(y))_i. \tag{4.5}$$

Thus we have

$$\alpha(T^{-1})\|T(x)\|_\infty^{1+1/\theta} \leq \max_{1\leq i\leq n} x_i(T(x))_i. \tag{4.6}$$

Since $T(x) \geq -q$, we know that $|T(x)| \geq (T(x))_+ \geq (-q)_+$ and so

$$\|T(x)\|_\infty \geq \|(-q)_+\|_\infty. \tag{4.7}$$

By (4.6), (4.7), and the fact that $x_i(T(x)+q)_i = 0$, we know

$$\begin{aligned}
\alpha(T^{-1})\|(-q)_+\|_\infty^{1+1/\theta} &\leq \alpha(T^{-1})\|T(x)\|_\infty^{1+1/\theta} \leq \max_{1\leq i\leq n} x_i(T(x))_i \\
&= \max_{1\leq i\leq n} x_i(-q)_i \leq \|x\|_\infty \cdot \|(-q)_+\|_\infty.
\end{aligned} \tag{4.8}$$

Hence

$$\left[\alpha(T^{-1})\right]^\theta \|(-q)_+\|_\infty \leq \|x\|_\infty^\theta. \tag{4.9}$$

This completes the proof. □

Acknowledgments

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