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Research Article

A Cohen-Type Inequality for Jacobi-Sobolev Expansions

Bujar Xh. Fejzullahu

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Let μ be the Jacobi measure supported on the interval $[-1,\ 1]$. Let us introduce the Sobolev-type inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + Mf(1)g(1) + Nf'(1)g'(1)$, where $M,N \geq 0$. In this paper we prove a Cohen-type inequality for the Fourier expansion in terms of the orthonormal polynomials associated with the above Sobolev inner product. We follow Dreseler and Soardi (1982) and Markett (1983) papers, where such inequalities were proved for classical orthogonal expansions.

1. Introduction and main result

Let $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, $\alpha > -1$, $\beta > -1$, be the Jacobi measure supported on the interval $[-1,1]$. We will say that $f(x) \in L^p(d\mu)$ if $f(x)$ is measurable on $[-1,1]$ and $\|f\|_{L^p(d\mu)} < \infty$, where

$$\|f\|_{L^p(d\mu)} = \begin{cases} \left(\int_{-1}^{1} |f(x)|^p d\mu(x)\right)^{1/p} & \text{if } 1 \leq p \leq \infty, \\ \operatorname*{ess\,sup}_{-1<x<1} |f(x)| & \text{if } p = \infty. \end{cases} \tag{1.1}$$

Now let us introduce the Sobolev-type spaces

$$\begin{aligned} S_p &= \{f : \|f\|_{S_p}^p = \|f\|_{L^p(d\mu)}^p + M|f(1)|^p + N|f'(1)|^p < \infty\}, \quad 1 \leq p < \infty, \\ S_\infty &= \{f : \|f\|_{S_\infty} = \|f\|_{L^\infty(d\mu)} < \infty\}, \quad p = \infty. \end{aligned} \tag{1.2}$$

Let f and g function in S_2. We can introduce the discrete Sobolev-type inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + Mf(1)g(1) + Nf'(1)g'(1), \tag{1.3}$$

where $M \geq 0$, $N \geq 0$. We denote by $\{q_n^{(\alpha,\beta)}\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (1.3) (see [1, 2]). These polynomials are known in the literature as Jacobi-Sobolev-type polynomials. For $M = N = 0$, the classical Jacobi orthonormal polynomials appear. We will denote them by $\{p_n^{(\alpha,\beta)}\}_{n\geq 0}$.

For $f \in S_1$, the Fourier expansion in terms of Jacobi-Sobolev-type polynomials is

$$\sum_{k=0}^{\infty} \widehat{f}(k) q_k^{(\alpha,\beta)}(x), \tag{1.4}$$

where

$$\widehat{f}(k) = \left\langle f, q_k^{(\alpha,\beta)} \right\rangle. \tag{1.5}$$

The Cesàro means of order δ of the Fourier expansion (1.4) are defined by (see [3, pages 76-77])

$$\sigma_n^{\delta} f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^{\delta}}{A_n^{\delta}} \widehat{f}(k) q_k^{(\alpha,\beta)}(x), \tag{1.6}$$

where $A_k^{\delta} = \binom{k+\delta}{k}$.

For a function $f \in S_p$ and a given sequence $\{c_{k,n}\}_{k=0}^{n}, n \in N \cup \{0\}$, of complex numbers with $|c_{n,n}| > 0$, we define the operators $T_n^{\alpha,\beta,M,N}$ by

$$T_n^{\alpha,\beta,M,N}(f) = \sum_{k=0}^{n} c_{k,n} \widehat{f}(k) q_k^{(\alpha,\beta)}. \tag{1.7}$$

Let us denote $p_0 = (4\beta+4)/(2\beta+3)$ and its conjugate $q_0 = (4\beta+4)/(2\beta+1)$. Here is the main result.

Theorem 1.1. *Let $\beta \geq \alpha \geq -1/2$, $\beta > -1/2$, and $1 \leq p \leq \infty$. There exists a positive constant c, independent of n, such that*

$$\left\| T_n^{\alpha,\beta,M,N} \right\|_{[S_p]} \geq c|c_{n,n}| \begin{cases} n^{(2\beta+2)/p-(2\beta+3)/2} & \text{if } 1 \leq p < p_0, \\ (\log n)^{(2\beta+1)/(4\beta+4)} & \text{if } p = p_0, p = q_0, \\ n^{(2\beta+1)/2-(2\beta+2)/p} & \text{if } q_0 \leq p < \infty, \end{cases} \tag{1.8}$$

where by $[S_p]$ one denotes the space of all bounded, linear operators from the space S_p into itself, with the usual operator norm $\|\cdot\|_{[S_p]}$.

Corollary 1.2. *Let α, β, and p be as in Theorem 1.1. For $c_{k,n} = 1$, $k = 0,\dots,n$, and for p outside the Pollard interval (p_0, q_0),*

$$\|S_n\|_{[S_p]} \longrightarrow \infty, \quad n \longrightarrow \infty, \tag{1.9}$$

where S_n denotes the nth partial sum of the expansion (1.4).

For $c_{k,n} = A^{\rho}_{n-k}/A^{\rho}_{n}$, $0 \le k \le n$, Theorem 1.1 yields the following.

Corollary 1.3. *Let α, β, p, and δ be given numbers such that $\beta > -1/2$,*

$$\begin{gathered} -\frac{1}{2} \le \alpha \le \beta, \\ 1 \le p \le \infty, \\ 0 \le \delta < \frac{2\beta+2}{p} - \frac{2\beta+3}{2} \quad \text{if } 1 \le p < p_0, \\ 0 \le \delta < \frac{2\beta+1}{2} - \frac{2\beta+2}{p} \quad \text{if } q_0 < p \le \infty. \end{gathered} \tag{1.10}$$

Then, for $p \notin [p_0, q_0]$,

$$\|\sigma^{\delta}_{n}\|_{[S_p]} \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{1.11}$$

2. Preliminaries

We summarize some properties of Jacobi-Sobolev-type polynomials that we will need in the sequel (cf. [1]). Throughout this paper, positive constants are denoted by $c, c_1, \dots$ and they may vary at every occurrence. The notation $u_n \sim v_n$ means $c_1 \le u_n/v_n \le c_2$ for n large enough, and by $u_n \cong v_n$, we mean that the sequence u_n/v_n converges to 1.

The representation of the polynomials $q^{(\alpha,\beta)}_n$ in terms of the Jacobi orthonormal polynomials $p^{(\alpha,\beta)}_n$ is

$$q^{(\alpha,\beta)}_n(x) = A_n p^{(\alpha,\beta)}_n(x) + B_n(x-1) p^{(\alpha+2,\beta)}_{n-1}(x) + C_n(x-1)^2 p^{(\alpha+4,\beta)}_{n-2}(x), \tag{2.1}$$

where

(a) if $M > 0$ and $N > 0$, then $A_n \cong -cn^{-2\alpha-2}$, $B_n \cong cn^{-2\alpha-2}$, $C_n \cong 1$,

(b) if $M = 0$ and $N > 0$, then $A_n \cong -1/(\alpha+2)$, $B_n \cong 1$, $C_n \cong 1/(\alpha+2)$,

(c) if $M > 0$ and $N = 0$, then $A_n \cong cn^{-2\alpha-2}$, $B_n \cong 1$, $C_n \cong 0$.

The maximum of $q^{(\alpha,\beta)}_n$ on $[-1,1]$ is

$$\max_{x\in[-1,1]} \left| q^{(\alpha,\beta)}_n(x) \right| \sim n^{\beta+1/2} \quad \text{if } \beta \ge \alpha \ge -\frac{1}{2}. \tag{2.2}$$

The polynomials $q_n^{(\alpha,\beta)}$ satisfy the estimate

$$\left| q_n^{(\alpha,\beta)}(\cos\theta) \right| = \begin{cases} O\left(\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}\right) & \text{if } \dfrac{c}{n} \le \theta \le \pi - \dfrac{c}{n}, \\ O\left(n^{\alpha+1/2}\right) & \text{if } 0 \le \theta \le \dfrac{c}{n}, \\ O\left(n^{\beta+1/2}\right) & \text{if } \pi - \dfrac{c}{n} \le \theta \le \pi, \end{cases} \tag{2.3}$$

for $\alpha \ge -1/2$, $\beta \ge -1/2$, and $n \ge 1$.

The Mehler-Heine-type formula for Jacobi orthonormal polynomials is (see [4, Theorem 8.1.1] and [4, Formula (4.3.4)])

$$\lim_{n\to\infty} (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)}\left(\cos\left(\pi - \frac{z}{n}\right)\right) = 2^{-(\alpha+\beta)/2}\left(\frac{z}{2}\right)^{-\beta} J_\beta(z), \tag{2.4}$$

where α, β are real numbers, and $J_\beta(z)$ is the Bessel function. This formula holds uniformly for $|z| \le R$, for R a given positive real number.

From (2.4),

$$\lim_{n\to\infty} (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)}\left(\cos\left(\pi - \frac{z}{n+j}\right)\right) = 2^{-(\alpha+\beta)/2}\left(\frac{z}{2}\right)^{-\beta} J_\beta(z) \tag{2.5}$$

holds uniformly for $|z| \le R$, $R > 0$ fixed, and uniformly on $j \in N \cup \{0\}$.

Lemma 2.1. *Let $\alpha,\beta > -1$ and $M,N \ge 0$. There exists a positive constant c such that*

$$\lim_{n\to\infty} (-1)^n n^{-\beta-1/2} q_n^{(\alpha,\beta)}\left(\cos\left(\pi - \frac{z}{n}\right)\right) = c\left(\frac{z}{2}\right)^{-\beta} J_\beta(z), \tag{2.6}$$

uniformly for $|z| \le R$, $R > 0$ fixed.

Proof. Here we will only analyze the case when $M = 0$ and $N > 0$. The proof of the other cases can be done in a similar way. From (2.1), we have

$$\begin{aligned}(-1)^n n^{-\beta-1/2} q_n^{(\alpha,\beta)}\left(\cos\left(\pi - \frac{z}{n+j}\right)\right) &= A_n (-1)^n n^{-\beta-1/2} p_n^{(\alpha,\beta)}\left(\cos\left(\pi - \frac{z}{n+j}\right)\right) \\ &\quad - B_n\left(\cos\left(\pi - \frac{z}{n+j}\right) - 1\right)(-1)^{n-1} \\ &\quad \times n^{-\beta-1/2} p_{n-1}^{(\alpha+2,\beta)}\left(\cos\left(\pi - \frac{z}{n+j}\right)\right) \\ &\quad + C_n\left(\cos\left(\pi - \frac{z}{n+j}\right) - 1\right)^2 (-1)^{n-2} \\ &\quad \times n^{-\beta-1/2} p_{n-2}^{(\alpha+4,\beta)}\left(\cos\left(\pi - \frac{z}{n+j}\right)\right),\end{aligned} \tag{2.7}$$

where $j \in N \cup \{0\}$.

Finally, if $n\to\infty$ and using (2.1) and (2.5), we get

$$\begin{aligned}
&\lim_{n\to\infty}(-1)^n n^{-\beta-1/2} q_n^{(\alpha,\beta)}\left(\cos\left(\pi-\frac{z}{n+j}\right)\right) \\
&\quad=\left(-\frac{1}{\alpha+2}2^{-(\alpha+\beta)/2}+2\cdot 2^{-(\alpha+\beta+2)/2}+\frac{1}{\alpha+2}4\cdot 2^{-(\alpha+\beta+4)/2}\right)\left(\frac{z}{2}\right)^{-\beta} J_\beta(z) \qquad (2.8)\\
&\quad=2^{-(\alpha+\beta)/2}\left(\frac{z}{2}\right)^{-\beta} J_\beta(z).
\end{aligned}$$

□

We also need to know the S_p norms for Jacobi-Sobolev-type polynomials

$$\left\|q_n^{(\alpha,\beta)}\right\|_{S_p}^p=\int_{-1}^1\left|q_n^{(\alpha,\beta)}(x)\right|^p d\mu(x)+M\left|q_n^{(\alpha,\beta)}(1)\right|^p+M\left|\left(q_n^{(\alpha,\beta)}\right)'(1)\right|^p, \qquad (2.9)$$

where $1\le p<\infty$. Hence, it is sufficient to estimate the $L^p(d\mu)$ norms for $q_n^{(\alpha,\beta)}$. For $M=N=0$, the calculation of these norms is given in [4, page 391, Exercise 91] (see also [5, Formula (2.2)]).

Lemma 2.2. *Let $M,\ N\ge 0$ and $\gamma>-1/p$. For $\beta\ge -1/2$,*

$$\int_{-1}^0 (1+x)^\gamma\left|q_n^{(\alpha,\beta)}(x)\right|^p dx\sim\begin{cases} c & \text{if } 2\gamma>p\beta-2+\frac{p}{2},\\ \log n & \text{if } 2\gamma=p\beta-2+\frac{p}{2},\\ n^{p\beta+p/2-2\gamma-2} & \text{if } 2\gamma<p\beta-2+\frac{p}{2}.\end{cases} \qquad (2.10)$$

Proof. From (2.3), for $p\beta+p/2-2\gamma-2\ne 0$, we have

$$\begin{aligned}
\int_{-1}^0(1+x)^\gamma\left|q_n^{(\alpha,\beta)}(x)\right|^p dx&=O(1)\int_{\pi/2}^{\pi}(\pi-\theta)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}(\cos\theta)\right|^p d\theta\\
&=O(1)\int_{\pi/2}^{\pi-1/n}(\pi-\theta)^{2\gamma+1}(\pi-\theta)^{-p\beta-p/2}d\theta\\
&\quad+O(1)\int_{\pi-1/n}^{\pi}(\pi-\theta)^{2\gamma+1}n^{p\beta+p/2}d\theta\\
&=O\left(n^{p\beta+p/2-2\gamma-2}\right)+O(1);
\end{aligned} \qquad (2.11)$$

and for $(p\beta+p/2-2\gamma-2)=0$, we have

$$\int_{-1}^0(1+x)^\gamma\left|q_n^{(\alpha,\beta)}(x)\right|^p dx=O(\log n). \qquad (2.12)$$

On the other hand, according to Lemma 2.1, we have

$$\begin{aligned}\int_{\pi/2}^{\pi}(\pi-\theta)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}(\cos\theta)\right|^p d\theta &> \int_{\pi-1/n}^{\pi}(\pi-\theta)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}(\cos\theta)\right|^p d\theta\\ &= \int_0^1\left(\frac{z}{n}\right)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}\left(\cos\left(\pi-\frac{z}{n}\right)\right)\right|^p n^{-1}dz\\ &\cong c\int_0^1\left(\frac{z}{n}\right)^{2\gamma+1}n^{p\beta+p/2}\left|\left(\frac{z}{2}\right)^{-\beta}J_\beta(z)\right|^p n^{-1}dz\\ &\sim n^{p\beta+p/2-2\gamma-2}.\end{aligned} \tag{2.13}$$

Using a similar argument as above, for $2\gamma = p\beta - 2 + p/2$, we have

$$\begin{aligned}\int_{\pi/2}^{\pi}(\pi-\theta)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}(\cos\theta)\right|^p dx &> \int_{\pi-n^{-1/2}}^{\pi}(\pi-\theta)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}(\cos\theta)\right|^p dx\\ &\cong c\int_0^{n^{1/2}} z^{2\gamma+1}\left|\left(\frac{z}{2}\right)^{-\beta}J_\beta(z)\right|^p dz \sim n^{\gamma+1} \ge c\log n.\end{aligned} \tag{2.14}$$

Finally, from [1, Theorem 5], we get

$$\int_{\pi/2}^{\pi}(\pi-\theta)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}(\cos\theta)\right|^p d\theta > \int_{\pi/2}^{3\pi/4}(\pi-\theta)^{2\gamma+1}\left|q_n^{(\alpha,\beta)}(\cos\theta)\right|^p d\theta \sim c. \tag{2.15}$$

□

Notice that some of the above results appear in [6].

3. Proof of Theorem 1.1

For the proof of Theorem 1.1, we will use the test functions

$$g_n^{\alpha,\beta,j}(x) = \left(1-x^2\right)^j p_n^{(\alpha+j,\beta+j)}(x), \tag{3.1}$$

where $\beta \ge \alpha \ge -1/2$, $\beta > -1/2$, and $j \in \mathbf{N} \setminus \{\mathbf{1}\}$. By applying the operators $T_n^{\alpha,\beta,M,N}$ to the test functions $g_n^{\alpha,\beta,j}$, for some $j > \beta + 1/2 - (2\beta+2)/p$, we get

$$T_n^{\alpha,\beta,M,N}\left(g_n^{\alpha,\beta,j}\right) = \sum_{k=0}^{n} c_{k,n}\left(g_n^{\alpha,\beta,j}\right)^{\wedge}(k)q_k^{(\alpha,\beta)}, \tag{3.2}$$

where

$$\left(g_n^{\alpha,\beta,j}\right)^{\wedge}(k) = \left\langle g_n^{\alpha,\beta,j}, q_k^{(\alpha,\beta)}\right\rangle, \quad k = 0,1,\dots,n. \tag{3.3}$$

From (2.1), we have

$$\begin{aligned}
\left(g_n^{\alpha,\beta,j}\right)^{\wedge}(k) &= \int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)q_k^{(\alpha,\beta)}(x)d\mu(x)\\
&= A_k\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)p_k^{(\alpha,\beta)}(x)d\mu(x)\\
&\quad + B_k\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)(x-1)p_{k-1}^{(\alpha+2,\beta)}(x)d\mu(x)\\
&\quad + C_k\int_{-1}^{1}(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x)(x-1)^2 p_{k-2}^{(\alpha+4,\beta)}(x)d\mu(x)\\
&= I_1^{k,n}+I_2^{k,n}+I_3^{k,n},
\end{aligned} \tag{3.4}$$

where $0 \le k \le n$, and it is assumed that $p_i^{(\gamma,\rho)}(x)=0$ for $i=-1,-2$.

According to [5, Formula (2.8)] and [4, Formula (4.3.4)], we get

$$(1-x^2)^j p_n^{(\alpha+j,\beta+j)}(x) = \left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n)\left\{h_{n+m}^{\alpha,\beta}\right\}^{1/2} p_{n+m}^{(\alpha,\beta)}(x). \tag{3.5}$$

Taking into account (3.5)

$$I_1^{k,n} = A_k\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n)\left\{h_{n+m}^{\alpha,\beta}\right\}^{1/2}\times\int_{-1}^{1}p_{n+m}^{(\alpha,\beta)}(x)p_k^{(\alpha,\beta)}(x)d\mu(x). \tag{3.6}$$

Thus

$$\begin{gathered}
I_1^{k,n}=0,\quad 0\le k\le n-1,\\
I_1^{n,n}=A_n\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\left\{h_n^{\alpha,\beta}\right\}^{1/2} b_{0,j}(\alpha,\beta,n),\quad n\ge 0,\ m=0.
\end{gathered} \tag{3.7}$$

Again, according to [5, Formula (2.8)] and [4, Formula (4.3.4)],

$$\begin{aligned}
I_2^{k,n} &= B_k\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\left\{h_{k-1}^{\alpha+2,\beta}\right\}^{-1/2}\\
&\quad\times\int_{-1}^{1}(1-x^2)^j P_n^{(\alpha+j,\beta+j)}(x)(x-1)P_{k-1}^{(\alpha+2,\beta)}(x)d\mu(x)\\
&= B_k\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\left\{h_{k-1}^{\alpha+2,\beta}\right\}^{-1/2}\sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n)\\
&\quad\times\int_{-1}^{1}P_{n+m}^{(\alpha,\beta)}(x)(x-1)P_{k-1}^{(\alpha+2,\beta)}(x)d\mu(x).
\end{aligned} \tag{3.8}$$

Since (see [4, Formula (4.5.4)])

$$(x-1)P_{k-1}^{(\alpha+2,\beta)}(x) = \frac{2k}{2k+\alpha+\beta+1}P_k^{(\alpha+1,\beta)}(x) - \frac{2(k+\alpha+1)}{2k+\alpha+\beta+1}P_{k-1}^{(\alpha+1,\beta)}(x), \tag{3.9}$$

and $\deg P_{k-1}^{(\alpha+1,\beta)} \le n-1$, we have

$$\begin{aligned} I_2^{k,n} &= \frac{2k\,B_k}{2k+\alpha+\beta+1}\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\left\{h_{k-1}^{\alpha+2,\beta}\right\}^{-1/2} \\ &\quad\times \sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n)\int_{-1}^{1} P_{n+m}^{(\alpha,\beta)}(x)P_k^{(\alpha+1,\beta)}(x)\,d\mu(x). \end{aligned} \tag{3.10}$$

Formula 16.4 (11) in [7, page 285] shows that

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_n^{(\alpha+1,\beta)}(x)\,d\mu(x) = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\alpha+\beta+n+2)} = \frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1}h_n^{\alpha,\beta}. \tag{3.11}$$

This formula can also be proved by using [4, page 257, Identity (9.4.3)].

Thus

$$\begin{aligned} I_2^{k,n} &= 0, \quad 0 \le k \le n-1, \\ I_2^{n,n} &= \frac{2nB_n}{n+\alpha+\beta+1}\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\left\{h_{n-1}^{\alpha+2,\beta}\right\}^{-1/2} \\ &\quad\times h_n^{\alpha,\beta} b_{0,j}(\alpha,\beta,n), \quad n \ge 1,\ m = 0. \end{aligned} \tag{3.12}$$

In a similar way,

$$\begin{aligned} I_3^{k,n} &= C_k\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\left\{h_{k-2}^{\alpha+4,\beta}\right\}^{-1/2}\sum_{m=0}^{2j} b_{m,j}(\alpha,\beta,n) \\ &\quad\times \int_{-1}^{1} P_{n+m}^{(\alpha,\beta)}(x)(x-1)^2 P_{k-2}^{(\alpha+4,\beta)}(x)\,d\mu(x). \end{aligned} \tag{3.13}$$

Again, as applications of [4, Formula (4.5.4)] and [4, Formula (9.4.3)], we point out the following formulas:

$$(x-1)^2 P_{k-2}^{(\alpha+4,\beta)}(x) = \frac{4k(k-1)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}P_k^{(\alpha+2,\beta)} + Q_{k-1}(x), \tag{3.14}$$

where $\deg Q_{k-1} \le n-1$, and

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_n^{(\alpha+2,\beta)}(x)\,d\mu(x) = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{(n+\alpha+\beta+1)(n+\alpha+\beta+2)}h_n^{\alpha,\beta}. \tag{3.15}$$

Thus

$$\begin{aligned} I_3^{k,n} &= 0, \quad 0 \le k \le n-1, \\ I_3^{n,n} &= \frac{4n(n-1)C_n}{(n+\alpha+\beta+1)(n+\alpha+\beta+2)}\left\{h_n^{\alpha+j,\beta+j}\right\}^{-1/2}\left\{h_{n-2}^{\alpha+4,\beta}\right\}^{-1/2} \\ &\quad\times h_n^{\alpha,\beta} b_{0,j}(\alpha,\beta,n), \quad n \ge 2,\ m = 0. \end{aligned} \tag{3.16}$$

In order to estimate $(g_n^{\alpha,\beta,j})^\wedge(k)$, we will distinguish the following three cases.

(1) $M > 0$, $N > 0$, then

$$I_1^{n,n} \cong -2^j c n^{-2\alpha-2}, \qquad I_2^{n,n} \cong 2^j c_1 n^{-2\alpha-2}, \qquad I_3^{n,n} \cong 2^j. \tag{3.17}$$

Thus

$$\left(g_n^{\alpha,\beta,j}\right)^\wedge(n) = I_1^{n,n} + I_2^{n,n} + I_3^{n,n} \cong 2^j. \tag{3.18}$$

(2) $M = 0$, $N > 0$, then

$$I_1^{n,n} \cong \frac{-2^j}{\alpha+2}, \qquad I_2^{n,n} \cong 2^j, \qquad I_3^{n,n} \cong \frac{-2^j}{\alpha+2}. \tag{3.19}$$

Thus

$$\left(g_n^{\alpha,\beta,j}\right)^\wedge(n) \cong 2^j. \tag{3.20}$$

(3) $M > 0$, $N = 0$, then

$$I_1^{n,n} \cong -2^j c n^{-2\alpha-2}, \qquad I_2^{n,n} \cong 2^j, \qquad I_3^{n,n} = 0. \tag{3.21}$$

Thus

$$\left(g_n^{\alpha,\beta,j}\right)^\wedge(n) \cong 2^j. \tag{3.22}$$

As a conclusion,

$$\begin{aligned} &\left(g_n^{\alpha,\beta,j}\right)^\wedge(k) = 0, \quad 0 \le k \le n-1, \\ &\left(g_n^{\alpha,\beta,j}\right)^\wedge(n) \cong 2^j. \end{aligned} \tag{3.23}$$

On the other hand, for $1 \le p < \infty$,

$$\begin{aligned} \left\|g_n^{\alpha,\beta,j}\right\|_{S_p}^p &= \left\|g_n^{\alpha,\beta,j}\right\|_{L^p(d\mu)}^p \\ &= \int_{-1}^{1} (1-x)^{jp+\alpha}(1+x)^{jp+\beta} \left| p_n^{(\alpha+j,\beta+j)}(x)\right|^p dx \\ &\le c_1 \int_{-1}^{0} (1+x)^{jp+\alpha} \left| p_n^{(\beta+j,\alpha+j)}(x)\right|^p dx \\ &\quad + c_2 \int_{-1}^{0} (1+x)^{jp+\beta} \left| p_n^{(\alpha+j,\beta+j)}(x)\right|^p dx. \end{aligned} \tag{3.24}$$

Taking $M = N = 0$ in lemma, we have

$$\left\|g_n^{\alpha,\beta,j}\right\|_{S_p}^p \le c \tag{3.25}$$

for $j > \beta + 1/2 - (2\beta+2)/p > \alpha + 1/2 - (2\alpha+2)/p$ and $q_0 \le p < \infty$.

It is well known (see, e.g., [8, Theorem 1]) that

$$\left| p_n^{(\alpha+j,\beta+j)}(x) \right| \leq c(1-x)^{-j/2-\alpha/2-1/4}(1+x)^{-j/2-\beta/2-1/4} \tag{3.26}$$

for $\alpha,\beta \geq -1/2$, and $x \in (-1,1)$. Therefore,

$$\left\| g_n^{\alpha,\beta,j} \right\|_{S_\infty} = \left\| g_n^{\alpha,\beta,j} \right\|_{L^\infty(d\mu)} \leq c(1-x)^{1/2(j-\alpha-1/2)}(1+x)^{1/2(j-\beta-1/2)} \leq c, \tag{3.27}$$

for $j > \beta + 1/2 \geq \alpha + 1/2$.

Now, we will prove our main result.

Proof of Theorem 1.1. Let $\beta \geq \alpha \geq -1/2$ and $\beta > -1/2$. By duality, it is enough to assume that $q_0 \leq p \leq \infty$. From (3.2), (3.23), (3.25), and (3.27), we have

$$\left\| T_n^{\alpha,\beta,M,N} \right\|_{[S_p]} \geq \left[\left\| g_n^{\alpha,\beta,j} \right\|_{S_p} \right]^{-1} \left\| T_n^{\alpha,\beta,M,N}\left(g_n^{\alpha,\beta,j} \right) \right\|_{S_p} \geq c \left| c_{n,n} \right| \left\| q_n^{(\alpha,\beta)} \right\|_{S_p}. \tag{3.28}$$

On the other hand, from (2.9) [1, Theorem 2], and lemma, we have

$$\left\| q_n^{(\alpha,\beta)} \right\|_{S_p} \geq c \begin{cases} (\log n)^{1/p} & \text{if } p = q_0, \\ n^{(2\beta+1)/2-(2\beta+2)/p} & \text{if } q_0 < p < \infty. \end{cases} \tag{3.29}$$

From this expression, taking into account (2.2) and (3.28), the statement of the theorem follows. □

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Bujar Xh. Fejzullahu: Faculty of Mathematics and Sciences, University of Prishtina, Mother Teresa 5, 10000 Prishtina, Kosovo, Serbia
Email address: bujarfej@uni-pr.edu