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Research Article

On Singular Solutions of Linear Functional Differential Equations with Negative Coefficients

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The problem on solutions with specified growth for linear functional differential equations with negative coefficients is treated by using two-sided monotone iterations. New theorems on the existence and localisation of such solutions are established.

1. Problem setting and introduction

In this paper, we are concerned with the problem on the existence and localisation of solutions of singular linear functional differential equations. The class of equations under examination comprises, in particular, the differential equation with argument deviations

$$u'(t) = \sum_{k=0}^{n} p_k(t)u\big(\omega_k(t)\big) + f(t), \quad t \in (a,b], \tag{1.1}$$

considered together with the additional condition

$$\sup_{t\in(a,b]} h(t)\big|u(t)\big| < +\infty, \tag{1.2}$$

where $h : (a,b] \to \mathbb{R}$ is a certain given continuous nondecreasing function such that $h(t) > 0$ for $t \in (a,b]$ and

$$\lim_{t \to a+} h(t) = 0. \tag{1.3}$$

The argument deviations in (1.1) are given by arbitrary Lebesgue measurable functions ω_k, $k = 0, 1, \ldots, n$, transforming the interval $(a, b]$ to itself (this setting does not lead one to any loss of generality, see, e.g., [1]).

We are interested in conditions guaranteeing the existence of solutions with property (1.2) in the case where the coefficients of (1.1) are nonpositive.

The class of solutions of (1.1) determined by condition (1.2) includes, in particular, those possessing the property

$$\lim_{t \to a+} h(t)u(t) = c, \tag{1.4}$$

where $c \in \mathbb{R}$. Problem (1.1), (1.4) with h satisfying (1.3) is referred to as the singular Cauchy problem [2]. It is reduced, in a natural way, to the classical regular Cauchy problem if h is equal identically to a nonzero constant. Regular and singular Cauchy problems for various classes of functional differential equations are treated, in particular, in [2–11]; a problem on regular solutions possessing properties of type (1.4) is studied in [12].

By a *solution* of problem (1.1), (1.2) we mean a locally absolutely continuous function $u : (a, b] \to \mathbb{R}$ such that $hu' \in L_1((a, b], \mathbb{R})$ which satisfies (1.1) almost everywhere on the interval $(a, b]$ and possesses property (1.2). One says that a function $u : (a, b] \to \mathbb{R}$ is *locally absolutely continuous* if its restriction $u|_{[a+\varepsilon, b]}$ to the interval $[a + \varepsilon, b]$ is absolutely continuous for any $\varepsilon \in (0, b - a)$.

A solution of (1.1), (1.2) may have a nonintegrable singularity at the point a. For example, the function $u(t) = \lambda t^{-4}$, $t \in (0, 1]$ for any real λ and $\varepsilon \in (0, +\infty)$ is a solution of the homogeneous problem

$$\begin{gathered} u'(t) = -\frac{4}{t^3} u\left(t^{1/2}\right), \quad t \in (0, 1], \\ \sup_{\xi \in (0,1]} \xi^{4+\varepsilon} \left|u(\xi)\right| < +\infty \end{gathered} \tag{1.5}$$

in the sense of the above definition. It should be noted that the property of the uniqueness of a solution is not typical for such problems; in other words, the resonance usually takes place.

2. Notation

The following notation is used throughout the paper.

(1) $\mathbb{R} := (-\infty, \infty)$.

(2) If $-\infty < a < b < \infty$ and $A \subseteq (a, b]$ is a measurable set, then $L_1(A, \mathbb{R})$ is the Banach space of all the Lebesgue integrable functions $u : A \to \mathbb{R}$ with the standard norm

$$L_1(A, \mathbb{R}) \ni u \longmapsto \int_A \left|u(t)\right| dt. \tag{2.1}$$

(3) $B((a, b], \mathbb{R})$ is the Banach space of all the bounded functions $u : (a, b] \to \mathbb{R}$ with the standard norm

$$B\left((a, b], \mathbb{R}\right) \ni u \longmapsto \sup_{t \in (a,b]} \left|u(t)\right|. \tag{2.2}$$

(4) $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ is the set of functions $u : (a,b] \to \mathbb{R}$ such that $u|_{[a+\varepsilon,b]} \in L_1([a+\varepsilon,b],\mathbb{R})$ for any $\varepsilon \in (0,b-a)$.

(5) $AC_{\mathrm{loc}}((a,b],\mathbb{R})$ is the set of all the locally absolutely continuous functions $u : (a,b] \to \mathbb{R}$.

(6) $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ is the set of all the locally absolutely continuous functions $u : (a,b] \to \mathbb{R}$ such that $hu' \in L_1((a,b],\mathbb{R})$ and $\sup_{t\in(a,b]} h(t)|u(t)| < +\infty$.

3. Existence of solutions with restricted growth

Without significant loss of generality we can assume that the function $h : (a,b] \to \mathbb{R}$ involved in (1.2) has the following properties.

$$\text{The function } h : (a,b] \longrightarrow (0,+\infty) \text{ is absolutely continuous and nondecreasing, and the relation } \lim_{t\to a} h(t) = 0 \text{ holds.} \tag{3.1}$$

The following statement on problem (1.1), (1.2) is true.

Theorem 3.1. *Let h in condition* (1.2) *possess properties* (3.1)*, the functions ω_k, $k = 0,1,\ldots,n$, are Lebesgue measurable, and let the functions $p_k \in L_{1;\mathrm{loc}}((a,b],\mathbb{R})$, $k = 0,1,\ldots,n$, be such that*

$$p_k(t) \le 0, \quad t \in (a,b],\ k = 0,1,\ldots,n, \tag{3.2}$$

$$\max_{k=0,1,\ldots,n} \int_a^b \frac{h(s)}{h(\omega_k(s))} \left|p_k(s)\right| ds < +\infty. \tag{3.3}$$

Moreover, let there exist a nonnegative function $\varphi \in AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ such that

$$\varphi'(t) \le \sum_{k=0}^{n} p_k(t)\varphi(\omega_k(t)), \quad t \in (a,b]. \tag{3.4}$$

Then for an arbitrary $u_0 \in AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ and any function $f : (a,b] \to \mathbb{R}$ from $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ possessing the property

$$hf \in L_1((a,b],\mathbb{R}) \tag{3.5}$$

and satisfying the estimate

$$\varphi'(t) - \sum_{k=0}^{n} p_k(t)\varphi(\omega_k(t)) \le f(t) - u_0'(t) + \sum_{k=0}^{n} p_k(t)u_0(\omega_k(t)) \le 0, \quad t \in (a,b], \tag{3.6}$$

problem (1.1), (1.2) *has a solution $u : (a,b] \to \mathbb{R}$ such that*

$$0 \le u(t) - u_0(t) \le \varphi(t), \quad t \in (a,b]. \tag{3.7}$$

Remark 3.2. The set of functions f satisfying inequalities (3.6) is nonempty because φ is assumed to possess property (3.4). Of course, it makes sense to consider only those φ which do not satisfy the corresponding homogeneous equation.

The next corollary deals with the case where the solution in question is bounded from below.

Corollary 3.3. *Assume that h in condition* (1.2) *possesses properties* (3.1) *and the functions $p_k \in L_{1;\mathrm{loc}}((a,b],\mathbb{R})$, $k = 0,1,\dots,n$, satisfy* (3.2) *and* (3.3)*. Moreover, let the functional differential inequality* (3.4) *have a nonnegative solution $\varphi \in AC_{\mathrm{loc};h}((a,b],\mathbb{R})$.*

Then for an arbitrary real λ and any function $f : (a,b] \to \mathbb{R}$ from $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ possessing property (3.5) *and satisfying the estimate*

$$\varphi'(t) - \sum_{k=0}^{n} p_k(t)\varphi(\omega_k(t)) \leq f(t) + \lambda \sum_{k=0}^{n} p_k(t) \leq 0, \quad t \in (a,b], \tag{3.8}$$

problem (1.1), (1.2) *has a solution $u : (a,b] \to \mathbb{R}$ such that*

$$0 \leq u(t) - \lambda \leq \varphi(t), \quad t \in (a,b]. \tag{3.9}$$

For concrete classes of weight functions h, Theorem 3.1 allows one to obtain efficient results concerning the existence, localisation, and approximate construction of solutions of (1.1) possessing property (1.2). Let us consider the problem on the existence of solutions of (1.1) possessing the property

$$\sup_{t\in(a,b]} (t-a)^{\gamma} |u(t)| < +\infty, \tag{3.10}$$

where γ is a given positive constant.

Corollary 3.4. *Let the functions $p_k \in L_{1;\mathrm{loc}}((a,b],\mathbb{R})$, $k = 0,1,\dots,n$, be nonpositive almost everywhere on $(a,b]$. Moreover, let*

$$\max_{k=0,1,\dots,n} \int_a^b \left(\frac{s-a}{\omega_k(s)-a} \right)^{\gamma} |p_k(s)| ds < +\infty. \tag{3.11}$$

Then for arbitrary function $u_0 \in AC_{\mathrm{loc};h}((a,b],\mathbb{R})$, number $\delta \in (0,\gamma)$, and function $f : (a,b] \to \mathbb{R}$ from $L_{1;\mathrm{loc}}((a,b],\mathbb{R})$ possessing the properties (3.5) *and*

$$(-\gamma+\delta)(t-a)^{-\gamma+\delta-1} - \sum_{k=0}^{n} p_k(t)(\omega_k(t)-a)^{-\gamma+\delta} \leq f(t) - u_0'(t) + \sum_{k=0}^{n} p_k(t)u_0(\omega_k(t)) \leq 0, \quad t \in (a,b], \tag{3.12}$$

problem (1.1), (1.2) *has a solution $u : (a,b] \to \mathbb{R}$ such that*

$$0 \leq u(t) - u_0(t) \leq (t-a)^{-\gamma+\delta}, \quad t \in (a,b]. \tag{3.13}$$

Remark 3.5. Under condition (3.11), the coefficient p_k, $k = 0, 1, \ldots, n$, may have a nonintegrable singularity at the point a if the corresponding argument deviation ω_k has the property

$$\operatorname*{ess\,inf}_{t \in (a,b]} \frac{t-a}{\omega_k(t)-a} = 0 \tag{3.14}$$

and, in particular, if $\operatorname{ess\,inf}_{t \in (a,b]} \omega_k(t) > a$.

In particular, in the case of the problem on finding solutions of the equation

$$u'(t) = \sum_{k=0}^{n} p_k(t) u\left(t^{\beta_k}\right) + f(t), \quad t \in (0,1], \tag{3.15}$$

satisfying the additional condition

$$\sup_{t \in (0,1]} t^{\gamma} \left| u(t) \right| < +\infty, \tag{3.16}$$

where $\gamma > 0$, $\beta_k \geq 0$, $k = 0, 1, \ldots, n$, we have the following.

Corollary 3.6. *Let the functions* $p_k \in L_{1;\mathrm{loc}}((0,1], \mathbb{R})$, $k = 0, 1, \ldots, n$, *be nonpositive almost everywhere on* $(0,1]$ *and let the relation*

$$\sum_{k=0}^{n} \int_0^1 t^{(1-\beta_k)\gamma} \left| p_k(t) \right| dt < +\infty \tag{3.17}$$

be satisfied. Moreover, assume that there exists some $\delta \in (0, \gamma)$ *such that*

$$\operatorname*{ess\,sup}_{t \in (0,1]} \sum_{k=0}^{n} \left| p_k(t) \right| t^{(\gamma-\delta)(1-\beta_k)+1} \leq \gamma - \delta. \tag{3.18}$$

Then for an arbitrary $u_0 \in AC_{\mathrm{loc};h}((0,1], \mathbb{R})$ *and any function* $f : (0,1] \to \mathbb{R}$ *from* $L_{1;\mathrm{loc}}((0,1], \mathbb{R})$ *possessing the properties*

$$\begin{gathered} hf \in L_1((0,1], \mathbb{R}), \\ (-\gamma+\delta) t^{-\gamma+\delta-1} - \sum_{k=0}^{n} p_k(t) t^{\beta_k(-\gamma+\delta)} \leq f(t) - u_0'(t) + \sum_{k=0}^{n} p_k(t) u_0\left(t^{\beta_k}\right) \leq 0, \quad t \in (0,1], \end{gathered} \tag{3.19}$$

problem (3.15), (3.16) *has a solution* $u : (0,1] \to \mathbb{R}$ *such that*

$$0 \leq u(t) - u_0(t) \leq t^{-\gamma+\delta}, \quad t \in (0,1]. \tag{3.20}$$

Remark 3.7. Under the conditions of Corollary 3.6, the coefficient p_k, $k = 0, 1, \ldots, n$, of (3.15) may be nonintegrable if the corresponding exponent β_k satisfies the inequalities $0 < \beta_k < 1$, that is, if there is an advance of argument in the kth term.

The statements formulated above are consequences of the general theorem which will be established in Section 4.

4. A general theorem

Let us consider the functional differential equation

$$u'(t) = (lu)(t) + f(t), \quad t \in [a, b], \tag{4.1}$$

where $l : AC_{\mathrm{loc}}((a, b], \mathbb{R}^n) \to L_{1;\mathrm{loc}}((a, b], \mathbb{R}^n)$ is a linear operator and $f \in L_{1;\mathrm{loc}}((a, b], \mathbb{R}^n)$ is a given locally integrable function. Assuming implicitly conditions (3.1) on the function h, we pose the problem on finding solutions of (4.1) possessing property (1.2).

Definition 4.1. An operator $l : AC_{\mathrm{loc}}((a, b], \mathbb{R}^n) \to L_{1;\mathrm{loc}}((a, b], \mathbb{R}^n)$ is said to be *pointwise negative* if $(lu)(t) \le 0$ for a.e. $t \in (a, b]$ whenever $\inf_{t \in (a,b]} u(t) \ge 0$ and $hu' \in L_1((a, b], \mathbb{R}^n)$.

The following theorem holds.

Theorem 4.2. *Let the operator* $l : AC_{\mathrm{loc}}((a, b], \mathbb{R}^n) \to L_{1;\mathrm{loc}}((a, b], \mathbb{R}^n)$ *in* (4.1) *be pointwise negative such that*

$$hl\left(\frac{1}{h}\right) \in L_1((a, b], \mathbb{R}). \tag{4.2}$$

Furthermore, let there exist some nonnegative absolutely continuous function $g : (a, b] \to \mathbb{R}$ *such that*

$$h\left(\frac{g}{h}\right)' \in L_1((a, b], \mathbb{R}), \tag{4.3}$$

and almost everywhere on $(a, b]$,

$$(h(t))^2 l\left(\frac{g}{h}\right)(t) \ge g'(t)h(t) - g(t)h'(t), \quad t \in (a, b]. \tag{4.4}$$

Then for an arbitrary $u_0 \in AC_{\mathrm{loc};h}((a, b], \mathbb{R})$ *and any function* $f : (a, b] \to \mathbb{R}$ *from* $L_{1;\mathrm{loc}}((a, b], \mathbb{R})$ *possessing the properties* (3.5) *and*

$$\frac{g'(t)h(t) - g(t)h'(t)}{(h(t))^2} - l\left(\frac{g}{h}\right)(t) \le f(t) - u_0'(t) + (lu_0)(t) \le 0, \quad t \in (a, b], \tag{4.5}$$

problem (4.1), (1.2) *has at least one solution* $u : (a,b] \to \mathbb{R}$ *such that*

$$0 \leq u(t) - u_0(t) \leq \frac{g(t)}{h(t)}, \quad t \in (a,b]. \tag{4.6}$$

It should be noted that the solution of problem (4.1), (1.2), the existence of which is stated in Theorem 4.2, can be found approximately by using a convergent monotone two-sided iteration procedure. Moreover, besides (4.6), the solution indicated admits the estimate

$$\frac{g'(t)h(t) - g(t)h'(t)}{\left(h(t)\right)^2} \leq u'(t) - u_0'(t) \leq 0, \quad t \in (a,b]. \tag{4.7}$$

5. Auxiliary statements

In the sequel, we need an abstract theorem on operators in partially ordered normed spaces [13, Theorem 4.1]. In order to state it, we first formulate definitions. We use [13] as the main reference (see also [14, 15]).

5.1. General notions

Let $\langle E, \|\cdot\| \rangle$ be a normed space over $\mathbb{R}$ and let K be a cone [13] in E, that is, a nonempty closed subset of E possessing the properties $K \cap (-K) = \{0\}$ and $\alpha_1 K + \alpha_2 K \subseteq K$ for all $\{\alpha_1, \alpha_2\} \subset [0, +\infty)$. A cone K generates a natural partial ordering of E. As usual, we will write $u \leqq_K v$ and $v \geqq_K u$ if and only if $v - u \in K$.

Definition 5.1 (see [13]). A sequence $\{u_k \mid k \geq 0\} \subset E$ is said to be *order-bounded* if there exists some $v \in E$ such that $u_k \leqq_K v$ for any $k \geq 0$. A sequence $\{u_k \mid k \geq 0\} \subset E$ is said to be *monotone* if $u_k \leqq_K u_{k+1}$ for any $k \geq 0$.

Definition 5.2 (see [13]). A cone $K \subset E$ is said to be *regular* if every order-bounded monotone sequence converges in E.

Definition 5.3 (see [13]). Given a cone $K \subset E$, a functional $l : E \to \mathbb{R}$ is said to be *positive* (with respect to K) if $l(u) \geq 0$ for any $u \geqq_K 0$. A positive functional $l : E \to \mathbb{R}$ is called *strictly nondecreasing* on the cone K if

$$\lim_{n \to \infty} l\left(u_1 + u_2 + \ldots + u_n\right) = +\infty \tag{5.1}$$

for any sequence $u_n \in K$, $n = 1, 2, \ldots$, possessing the property $\inf_{n \geq 1} \|u_n\| > 0$.

A linear functional $l : E \to \mathbb{R}$ is said to be *uniformly positive* on K if there exists a certain $\lambda \in (0, +\infty)$ such that

$$l(x) \geq \lambda \|x\| \quad (\forall x \in K). \tag{5.2}$$

A uniformly positive linear functional, in particular, is strictly nondecreasing.

Definition 5.4 (see [13]). One says that an operator $T : E \to E$ is *monotone* (with respect to K) if $Tu \geqq_K Tv$ for any u and v from E such that $u \geqq_K v$.

Theorem 5.5 (see [13]). *Let $T : E \to E$ be a monotone and continuous operator. Let there exist some elements $\{u_0, v_0\} \subset E$ such that $u_0 \leqq_K v_0$ and, moreover,*

$$Tu_0 \geqq_K u_0, \tag{5.3}$$

$$Tv_0 \leqq_K v_0. \tag{5.4}$$

Moreover, assume that the cone K is regular. Then the operator T has at least one fixed point u such that $u_0 \leqq_K u \leqq_K v_0$.

Remark 5.6. The assertion of Theorem 5.5 is established in [13] under the assumption that $T(K) \subseteq K$. The positivity of T, however, is not used in the proof.

5.2. The space $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$

We assume that the function h involved in the nonlocal condition (1.2) possesses properties (3.1).

Lemma 5.7. *The set $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ is a Banach space with respect to the norm*

$$AC_{\mathrm{loc};h}((a,b],\mathbb{R}) \ni u \longmapsto \|u\| := \int_a^b h(s)\left|u'(s)\right|ds + \sup_{\xi\in(a,b]} h(\xi)\left|u(\xi)\right|. \tag{5.5}$$

Proof. Let us assume that $\{u_m : m \geq 1\}$ is a Cauchy sequence in the space $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$. Then, in view of (5.5), the sequence $\{hu_m' : m \geq 1\}$ is a Cauchy sequence in $L_1((a,b],\mathbb{R})$ and, therefore, there exists some Lebesgue integrable function $v : (a,b] \to \mathbb{R}$ such that

$$\lim_{m\to+\infty} \int_a^b \left|h(s)u_m'(s) - v(s)\right|ds = 0. \tag{5.6}$$

According to the definition of the set $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$, every function $hu_m : (a,b] \to \mathbb{R}$, $m = 1,2,\ldots$, is bounded in a neighbourhood of the point a, whence, in view of the assumption, we find that the sequence $\{hu_m : m \geq 1\}$ is a Cauchy sequence in the complete space $B((a,b],\mathbb{R})$. Therefore, one can specify a bounded function $w : (a,b] \to \mathbb{R}$ such that

$$\lim_{m\to+\infty} \max_{t\in[a,b]} \left|h(s)u_m(s) - w(s)\right| = 0. \tag{5.7}$$

Let us put

$$\tilde{u} := \frac{w}{h}, \tag{5.8}$$

fix an arbitrary $\delta \in (0, b-a)$, and construct the sequence of functions

$$\begin{aligned} z_{m,\delta} := -\int_{a+\delta}^{b} h'(s)\big(u_m(s) - \widetilde{u}(s)\big)ds + h(b)\big(u_m(b) - \widetilde{u}(b)\big) \\ - h(a+\delta)\big(u_m(a+\delta) - \widetilde{u}(a+\delta)\big), \quad m = 1, 2, \ldots . \end{aligned} \tag{5.9}$$

Since the number δ is positive, we see from assumption (3.1) that $h'/h \in L_1([a+\delta, b], \mathbb{R})$, whence, in view of (5.7) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{m \to +\infty} \int_{a+\delta}^{b} h'(s)\big(u_m(s) - \widetilde{u}(s)\big)ds = 0. \tag{5.10}$$

By virtue of assumption (3.1), $h(t) \neq 0$ for $a < t \leq b$. It then follows from (5.7) that $\lim_{m \to +\infty} u_m(t) = \widetilde{u}(t)$ for any $t \in (a, b]$. By virtue of (5.10), this yields

$$\lim_{m \to +\infty} z_{m,\delta} = 0 \quad \big(\forall \delta \in (0, b-a)\big). \tag{5.11}$$

On the other hand, it follows immediately from (5.9) that

$$z_{m,\delta} = \int_{a+\delta}^{b} \big(h'(s)\widetilde{u}(s) + h(s)u_m'(s)\big)ds - h(b)\widetilde{u}(b) + h(a+\delta)\widetilde{u}(a+\delta), \tag{5.12}$$

whence, in view of (5.6),

$$\int_{a+\delta}^{b} v(s)ds = -\int_{a+\delta}^{b} h'(s)\widetilde{u}(s)ds + h(b)\widetilde{u}(b) - h(a+\delta)\widetilde{u}(a+\delta) \tag{5.13}$$

for any $0 < \delta < b - a$. Recalling that $v \in L_1((a, b], \mathbb{R})$ and $h\widetilde{u} \in B((a, b], \mathbb{R})$, using the relation $-\infty < \lim_{\delta \to 0^+} \int_{a+\delta}^{b} v(s)ds = \int_a^b v(s)ds < +\infty$, and passing to the limit as $\delta \to 0^+$ in (5.13), we find that

$$\int_a^b h'(s)\widetilde{u}(s)ds = -\int_a^b v(s)ds + h(b)\widetilde{u}(b) - \lim_{\xi \to a^+} h(\xi)\widetilde{u}(\xi) \tag{5.14}$$

and, in particular, $h'\widetilde{u} \in L_1((a, b], \mathbb{R})$.

It follows from (5.13) that the functions v and $\widetilde{u}$ satisfy the relation

$$h(t)\widetilde{u}(t) = -\int_t^b \big(v(s) + h'(s)\widetilde{u}(s)\big)ds + h(b)\widetilde{u}(b) \quad \big(\forall t \in (a, b]\big). \tag{5.15}$$

Moreover, the product $h\tilde{u}$ has a derivative almost every where on $(a,b]$. Indeed, it follows from (5.14) that, for almost every $t \in (a,b]$ and any $\eta \in (a-t, b-t]$,

$$\begin{aligned}\frac{1}{\eta}\left(h(t+\eta)\tilde{u}(t+\eta) - h(t)\tilde{u}(t)\right) &= -\frac{1}{\eta}\left(\int_{t+\eta}^{b}\left(v(s) + h'(s)\tilde{u}(s)\right)ds - \int_{t}^{b}\left(v(s) + h'(s)\tilde{u}(s)\right)ds\right)\\ &= \frac{1}{\eta}\int_{t}^{t+\eta}\left(v(s) + h'(s)\tilde{u}(s)\right)ds.\end{aligned} \tag{5.16}$$

Therefore, $\lim_{\eta\to 0}(1/\eta)(h(t+\eta)\tilde{u}(t+\eta) - h(t)\tilde{u}(t)) = v(t) + h'(t)\tilde{u}(t)$, which means that

$$\left(h(t)\tilde{u}(t)\right)' = v(t) + h'(t)\tilde{u}(t) \tag{5.17}$$

for a.e. $t \in (a,b]$. Relation (5.17) implies, in particular, that $\tilde{u}'$ exists almost everywhere on $(a,b]$ and, moreover,

$$h(t)\tilde{u}'(t) = v(t) \tag{5.18}$$

for a.e. $t \in (a,b]$. Using (5.5) and (5.18), we get

$$\left\|u_m - \tilde{u}\right\| = \int_a^b h(s)\left|u_m(s) - \tilde{u}'(s)\right|ds + \sup_{\xi\in(a,b]} h(\xi)\left|u_m(\xi) - \tilde{u}(\xi)\right| \longrightarrow 0 \tag{5.19}$$

as $m \to +\infty$, that is, the sequence $\{u_m : m \geq 1\}$ converges to $\tilde{u}$ in $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$. □

Lemma 5.8. *The set*

$$K := \left\{u \in AC_{\mathrm{loc};h}\left((a,b],\mathbb{R}\right) : \inf_{t\in(a,b]} u(t) \geq 0,\ \operatorname{ess\,sup}_{t\in(a,b]} u'(t) \leq 0\right\} \tag{5.20}$$

is a regular cone in the space $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$.

Proof. The set (5.20) is obviously closed, and the axioms of a cone (see (5.1)) are verified directly. If a monotonous strictly nondecreasing functional is defined on the cone K, then the cone K is regular (see, e.g., [13, Theorem 1.11]). In our case, the functional

$$\varphi(u) := -\int_a^b h(s)u'(s)ds + \sup_{\xi\in(a,b]} h(\xi)u(\xi) \quad \left(u \in AC_{\mathrm{loc};h}\left((a,b],\mathbb{R}\right)\right) \tag{5.21}$$

is uniformly positive on the cone (5.20). It then follows that the cone K is regular. □

Lemma 5.9. *If the operator l and function h satisfy condition* (4.2) *and h is nondecreasing, then*

$$\sup_{t\in(a,b]} h(t)\int_t^b \left| l\left(\frac{1}{h}\right)(s)\right| ds < +\infty. \tag{5.22}$$

Proof. Property (5.22) is a consequence of the estimate

$$h(t)\int_t^b \left| l\left(\frac{1}{h}\right)(s)\right| ds \le \int_t^b h(s)\left| l\left(\frac{1}{h}\right)(s)\right| ds, \tag{5.23}$$

which is valid for any $t \in (a,b]$ because h is nondecreasing. □

5.3. The operator T

Let us fix a certain locally integrable function $f : (a,b] \to \mathbb{R}$ with properties (3.5) and put

$$(Tu)(t) := u(b) - \int_t^b \big((lu)(s) + f(s)\big) ds, \quad t \in (a,b], \tag{5.24}$$

for any u from $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$.

Lemma 5.10. *If l is pointwise negative and* (3.1) *holds, then*

(1) *T is a well-defined mapping from $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ to $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$;*

(2) *T is continuous with respect to norm* (5.5)*;*

(3) *T is monotone with respect to cone* (5.20).

Proof. (1) Let u be an arbitrary function from $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$. Then there exists a constant $\mu_u > 0$ such that

$$|u(t)| \le \frac{\mu_u}{h(t)}, \tag{5.25}$$

and hence

$$-\frac{\mu_u}{h(t)} \le -|u(t)| \le u(t) \le |u(t)| \le \frac{\mu_u}{h(t)}, \quad t \in (a,b]. \tag{5.26}$$

Relation (5.26), due to the pointwise negativity (see Definition 4.1) and linearity of l, yields

$$l\left(-\frac{\mu_u}{h}\right)(t) \ge l\big(-|u|\big)(t) \ge (lu)(t) \ge \big(l|u|\big)(t) \ge l\left(\frac{\mu_u}{h}\right)(t), \tag{5.27}$$

$$-\mu_u l\left(\frac{1}{h}\right)(t) \ge -l\big(|u|\big)(t) \ge (lu)(t) \ge \big(l|u|\big)(t) \ge \mu_u l\left(\frac{1}{h}\right)(t), \quad t \in (a,b]. \tag{5.28}$$

Thus, we have the estimate

$$\left|(lu)(t)\right| \leq \mu_u \left| l\left(\frac{1}{h}\right)(t)\right|, \quad t \in (a,b], \tag{5.29}$$

and therefore, due to assumption (4.2),

$$\lim_{t\to a+}\int_t^b h(s)\left|(lu)(s)\right|ds < \infty. \tag{5.30}$$

Since f is assumed to satisfy (3.5), this guarantees that the function hw', where $w := Tu$, belongs to $L_1((a,b],\mathbb{R})$.

Furthermore, using condition (3.5) and arguing similarly to the proof of Lemma 5.9, we find that

$$\sup_{t\in(a,b]} h(t)\int_t^b \left|f(s)\right|ds < +\infty. \tag{5.31}$$

Then, by virtue of (3.1) and (5.27),

$$\begin{aligned}
\sup_{t\in(a,b]} h(t)\left|(Tu)(t)\right| &= \sup_{t\in(a,b]} h(t)\left|u(b) - \int_t^b ((lu)(s) + f(s))ds\right| \\
&\leq h(b)u(b) + \sup_{t\in(a,b]} h(t)\int_t^b \left(\left|(lu)(s)\right| + \left|f(s)\right|\right)ds \\
&\leq h(b)u(b) + \mu_u \sup_{t\in(a,b]} h(t)\int_t^b \left| l\left(\frac{1}{h}\right)(t)\right| ds + \sup_{t\in(a,b]} h(t)\int_t^b \left|f(s)\right|ds,
\end{aligned} \tag{5.32}$$

whence, in view of assumptions (3.1), (4.2), (5.31) and Lemma 5.9, it follows that $\sup_{t\in(a,b]} h(t)|(Tu)(t)| < +\infty$. Thus, T is well defined and maps the space $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ into itself.

(2) The continuity of T follows from the properties of l. Indeed, let u and v be arbitrary functions from $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ and let $\varepsilon \in (0,+\infty)$ be given. According to (5.5) and (5.24), in view of the linearity of l, we have

$$\begin{aligned}
\|Tv - Tu\| &= \int_a^b h(t)\left|l(v-u)(t)\right|dt + \sup_{t\in(a,b]} h(t)\left|v(b) - u(b) - \int_t^b l(v-u)(s)ds\right| \\
&\leq \int_a^b h(t)\left|l(v-u)(t)\right|dt + \sup_{t\in(a,b]} h(t)\left|v(b) - u(b)\right| + \sup_{t\in(a,b]} h(t)\left|\int_t^b l(v-u)(s)ds\right|.
\end{aligned} \tag{5.33}$$

Since u and v belong to $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$, it follows that

$$-\frac{\beta_{u,v}}{h(t)} \le v(t)-u(t) \le \frac{\beta_{u,v}}{h(t)}, \quad t\in(a,b], \tag{5.34}$$

where $\beta_{u,v} := \sup_{t\in(a,b]} h(t)|v(t)-u(t)| < +\infty$. By assumption, l is pointwise negative, and therefore

$$\left|l(v-u)(t)\right| \le -\beta_{u,v} l\left(\frac{1}{h}\right)(t), \quad t\in(a,b]. \tag{5.35}$$

By virtue of condition (4.2), it follows from (5.35) that

$$\int_a^b h(t)\left|l(v-u)(t)\right|dt \ \le \beta_{u,v}C, \tag{5.36}$$

where $C := -\int_a^b h(t)l(1/h)(t)dt < +\infty$ (note that C is positive).

Due to Lemma 5.9, assumption (4.2) ensures the validity of relation (5.22). Using (5.22) and (5.36), taking estimate (5.23) from the proof of Lemma 5.9 into account, and arguing analogously, we obtain

$$\begin{aligned}\sup_{t\in(a,b]} h(t)\left|\int_t^b l(v-u)(s)ds\right| &\le \sup_{t\in(a,b]}\left|\int_t^b h(s)l(v-u)(s)ds\right| \\ &\le \sup_{t\in(a,b]}\int_t^b h(s)\left|l(v-u)(s)\right|ds \le \beta_{u,v}C.\end{aligned} \tag{5.37}$$

Since $\|v-u\| \ge \beta_{u,v}$, we see from (5.33), (5.36), and (5.37) that $\|Tv-Tu\| < \varepsilon$ whenever $\|v-u\| < \varepsilon(1+2C)^{-1}$. This, in view of the arbitrariness of ε, proves the continuity of the mapping T.

(3) Let u and v be arbitrary elements of $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ such that $v \geqq_K u$ with respect to cone (5.20), that is,

$$(-1)^i\left(v^{(i)}(t)-u^{(i)}(t)\right) \ge 0, \quad t\in(a,b],\ i=0,1. \tag{5.38}$$

Since l is assumed to be pointwise negative, in view of (5.38), we have

$$\begin{gathered}(Tv)(t)-(Tu)(t) = v(b)-u(b)-\int_t^b \left(l(v-u)\right)(s)ds \ge 0, \quad t\in(a,b], \\ (Tv)'(t)-(Tu)'(t) = \left(l(v-u)\right)(t) \le 0, \quad t\in(a,b],\end{gathered} \tag{5.39}$$

which means that $Tv \geqq_K Tu$. □

Lemma 5.11. *A function $u:(a,b]\to\mathbb{R}$ from $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ is a solution of problem* (4.1), (1.2) *if and only if it is a fixed point of mapping* (5.24).

Proof. To obtain this assertion, it is sufficient to take into account (5.24) and the definition of the set $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$. □

6. Proofs

Let us now turn to the proofs of the statements formulated in Sections 3 and 4.

6.1. Proof of Theorem 3.1

It is clear that (1.1) can be represented in the form of (4.1), where the operator l is defined by the equality

$$(lu)(t) = \sum_{k=0}^{n} p_k(t)u(\omega_k(t)), \quad t \in (a,b]. \tag{6.1}$$

Since the functions $p_k \in L_{1;\mathrm{loc}}((a,b],\mathbb{R})$, $k = 0,1,\ldots,n$ are nonpositive, it follows that the operator l is pointwise negative. Moreover, in view of property (3.3), the operator l satisfies condition (4.2). Furthermore, a nonnegative function $\varphi \in AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ can be written in the form

$$\varphi = \frac{g}{h}, \tag{6.2}$$

where g is a nonnegative absolutely continuous function on $(a,b]$. Then it follows from (3.4) and (3.6) that inequalities (4.4) and (4.5) are true.

Thus, all the conditions of Theorem 4.2 hold, and therefore problem (1.1), (1.2) has a solution $u : (a,b] \to \mathbb{R}$ belonging to $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ and satisfying estimate (3.9).

6.2. Proof of Corollary 3.3

The statement is an immediate consequence of Theorem 3.1 with $u_0(t) := \lambda$, $t \in (a,b]$.

6.3. Proof of Corollary 3.4

It is sufficient to apply Theorem 3.1 in the case where h has the form $h(t) = (t-a)^{\gamma}$, $t \in (a,b]$, with $\gamma > 0$, and the function φ is given by the formula $\varphi(t) = (t-a)^{-\gamma+\delta}$, $t \in (a,b]$, where $\delta \in (0,\gamma)$.

6.4. Proof of Corollary 3.6

It is sufficient to apply Corollary 3.4 with $a = 0$, $b = 1$, and $\omega_k(t) = t^{\beta_k}$, $t \in (0,1]$, $k = 0,1,2\ldots,n$.

6.5. Proof of Theorem 4.2

We are going to use Theorem 5.5. Lemmas 5.7 and 5.8 guarantee that the set K given by (5.20) is a regular cone in the Banach space $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$. By Lemma 5.10, the operator $T: AC_{\mathrm{loc};h}((a,b],\mathbb{R}) \to AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ defined by formula (5.24) is continuous and monotone with respect to the cone K. Therefore, in order to be able to apply Theorem 5.5, we need to specify a pair of elements u_0 and v_0 in $AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ such that $u_0 \leqq_K v_0$ and (5.3), (5.4) hold.

Let us put $v_0 := u_0 + \varphi$, where

$$\varphi := \frac{g}{h} \tag{6.3}$$

and u_0 is the function from the formulation of the theorem. Note that $\varphi \in AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ in view of assumption (4.3) and the absolute continuity of g. Moreover, $(-1)^i \varphi^{(i)}(t) \geq 0$, $t \in (a,b]$, $i = 0,1$, because, by assumption, g is a nonnegative function and the functional differential inequality (4.4) is satisfied. Hence, $\varphi \in K$.

The functions indicated satisfy relations (5.3) and (5.4) with respect to cone (5.20), that is,

$$(-1)^i \left((Tu_0)^{(i)}(t) - u_0^{(i)}(t) \right) \geq 0, \quad t \in (a,b],\ i = 0,1, \tag{6.4}$$

$$(-1)^i \left((Tv_0)^{(i)}(t) - v_0^{(i)}(t) \right) \leq 0, \quad t \in (a,b],\ i = 0,1. \tag{6.5}$$

Indeed, it follows immediately from (5.24) that relations (6.4) and (6.5) with $i = 1$ are satisfied if and only if

$$-u_0'(t) - \varphi'(t) + (lu_0)(t) + (l\varphi)(t) \geq -f(t) \geq -u_0'(t) + (lu_0)(t), \quad t \in (a,b]. \tag{6.6}$$

Therefore, it suffices to notice that (6.6) holds in view of (4.5) and (6.3), whereas relations (6.4) and (6.5) with $i = 0$ are obtained from (6.6) by integrating from $t \in (a,b]$ to b.

In view of (4.4), the function (6.3) satisfies the functional differential inequality

$$\varphi'(t) \leq (l\varphi)(t), \quad t \in (a,b], \tag{6.7}$$

and hence the set of functions $f \in AC_{\mathrm{loc};h}((a,b],\mathbb{R})$ satisfying condition (6.6) is nonempty. Applying Theorem 5.5 and using Lemma 5.11, we complete the proof of Theorem 4.2.

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