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Research Article

The Reverse Hölder Inequality for the Solution to p-Harmonic Type System

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Some inequalities to A-harmonic equation $A(x, du) = d^*v$ have been proved. The A-harmonic equation is a particular form of p-harmonic type system $A(x, a + du) = b + d^*v$ only when $a = 0$ and $b = 0$. In this paper, we will prove the Poincaré inequality and the reverse Hölder inequality for the solution to the p-harmonic type system.

1. Introduction

Recently, amount of work about the A-harmonic equation for the differential forms has been done. In fact, the A-harmonic equation is an important generalization of the p-harmonic equation in $\mathbb{R}^n$, $p > 1$, and the p-harmonic equation is a natural extension of the usual Laplace equation (see [1] for the details). The reverse Hölder inequalities have been widely studied and frequently used in analysis and related fields, including partial differential equations and the theory of elasticity (see [2]). In 1999, Nolder gave the reverse Hölder inequality for the solution to the A-harmonic equation in [3], and different versions of Caccioppoli estimates have been established in [4–6]. In 2004, D'Onofrio and Iwaniec introduced the p-harmonic type system in [7], which is an important extension of the conjugate A-harmonic equation. In 2007, Ding proved the following inequality in [8].

Theorem A. *Let (u, v) be a pair of solutions to $A(x, g + du) = h + d^*v$ in a domain $\Omega \subset \mathbb{R}^n$. If $g \in L^p(B, \Lambda^L)$ and $h \in L^q(B, \Lambda^L)$, then $du \in L^p(B, \Lambda^L)$ if and only if $d^*v \in L^q(B, \Lambda^L)$. Moreover, there exist constants C_1, C_2 independent of u and v, such that*

$$\begin{aligned} \left\|d^*v\right\|_{q,B}^{q} &\le C_1\left(\|h\|_{q,B}^{q} + \|g\|_{p,B}^{p} + \|du\|_{p,B}^{p}\right), \\ \|du\|_{p,B}^{p} &\le C_2\left(\|h\|_{q,B}^{q} + \|g\|_{p,B}^{p} + \left\|d^*v\right\|_{q,B}^{q}\right) \quad \forall B \subset \sigma B \subset \Omega. \end{aligned} \tag{1.1}$$

In this paper, we will prove the Poincaré inequality (see Theorem 2.5) and the reverse Hölder inequality for the solution to the p-harmonic type system (see Theorem 3.5). Now let us see some notions and definitions about the p-harmonic type system.

Let $e_1, e_2, \ldots, e_n$ denote the standard orthogonal basis of $\mathbb{R}^n$. For $l = 0, 1, \ldots, n$, we denote by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ the linear space of all l-vectors, spanned by the exterior product $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$ corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$. The Grassmann algebra $\Lambda = \oplus \Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha_I e_I \in \Lambda$ and $\beta = \sum \beta_I e_I \in \Lambda$, then its inner product is obtained by

$$\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I, \tag{1.2}$$

with the summation over all $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. The Hodge star operator $*\colon \Lambda \to \Lambda$ is defined by the rule

$$\begin{gathered} *1 = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}, \\ \alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1) \quad \forall \alpha, \beta \in \Lambda. \end{gathered} \tag{1.3}$$

Hence, the norm of $\alpha \in \Lambda$ can be given by

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \Lambda_0 = \mathbb{R}. \tag{1.4}$$

Throughout this paper, $\Omega \subset \mathbb{R}^n$ is an open subset, for any constant $\sigma > 1$, Q denotes a cube such that $Q \subset \sigma Q \subset \Omega$, where σQ denotes the cube whose center is as same as Q and $\operatorname{diam}(\sigma Q) = \sigma \operatorname{diam} Q$. We say $\alpha = \sum \alpha_I e_I \in \Lambda$ is a differential l-form on Ω, if every coefficient α_I of α is Schwartz distribution on Ω. The space spanned by differential l-form on Ω is denoted by $D'(\Omega, \Lambda^l)$. We write $L^p(\Omega, \Lambda^l)$ for the l-form $\alpha = \sum \alpha_I dx_I$ on Ω with $\alpha_I \in L^p(\Omega)$ for all ordered l-tuple I. Thus $L^p(\Omega, \Lambda^l)$ is a Banach space with the norm

$$\|\alpha\|_{p,\Omega} = \left(\int_\Omega |\alpha|^p dx \right)^{1/p} = \left(\int_\Omega \left(\sum_I |\alpha_I|^2 \right)^{p/2} dx \right)^{1/p}. \tag{1.5}$$

Similarly $W^{k,p}(\Omega, \Lambda^l)$ denotes those l-forms on Ω with all coefficients in $W^{k,p}(\Omega)$. We denote the exterior derivative by

$$d : D'(\Omega, \Lambda^l) \longrightarrow D'(\Omega, \Lambda^{l+1}), \quad \text{for } l = 0, 1, 2, \ldots, n, \tag{1.6}$$

and its formal adjoint operator (the Hodge codifferential operator)

$$d^* : D'(\Omega, \Lambda^l) \longrightarrow D'(\Omega, \Lambda^{l-1}). \tag{1.7}$$

The operators d and d^* are given by the formulas

$$d\alpha = \sum_I d\alpha_I \wedge dx_I, \qquad d^* = (-1)^{nl+1} * d *. \tag{1.8}$$

The following two definitions appear in [7].

Definition 1.1. The Hodge system holds:

$$A(x, a + du) = b + d^*v, \tag{1.9}$$

where $a \in L^p(\Omega, \Lambda^l)$ and $b \in L^q(\Omega, \Lambda^l)$, is a p-harmonic type system if A is a mapping from $\Omega \times \Lambda^l$ to Λ^l satisfying

(1) $x \to A(x, \xi)$ is measurable in $x \in \Omega$ for every $\xi \in \Lambda^l$;

(2) $\xi \to A(x, \xi)$ is continuous in $\xi \in \Lambda^l$ for almost every $x \in \Omega$;

(3) $A(x, t\xi) = t^{p-1}A(x, \xi)$ for every $t \geq 0$;

(4) $K\langle A(x, \xi) - A(x, \zeta), \xi - \zeta\rangle \geq |\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}$;

(5) $|A(x, \xi) - A(x, \zeta)| \leq K|\xi - \zeta|(|\xi| + |\zeta|)^{p-2}$

for almost every $x \in \Omega$ and all $\xi, \zeta \in \Lambda^l$, where $K \geq 1$ is a constant. It should be noted that $A(x, *) : \Omega \times \Lambda^l \to \Lambda^l$ is invertible and its inverse denoted by A^{-1} satisfies similar conditions as A but with Hölder conjugate exponent q in place of p.

Definition 1.2. If (1.9) is a p-harmonic type system, then we say the equation

$$d^*A(x, a + du) = d^*b \tag{1.10}$$

is a p-harmonic type equation.

The following definition appears in [9].

Definition 1.3. A differential form u is a weak solution for (1.10) in Ω if u satisfies

$$\int_\Omega \langle A(x, a + du), d\varphi\rangle + \langle d^*b, \varphi\rangle \equiv 0 \tag{1.11}$$

for every $\varphi \in W^{k,p}(\Omega, \Lambda^{l-1})$ with compact support.

We can find that if we let $a = 0$ and $b = 0$, then the p-harmonic type system

$$A(x, a + du) = b + d^*v \tag{1.12}$$

becomes

$$A(x, du) = d^*v. \tag{1.13}$$

It is the conjugate A-harmonic equation, where the mapping $A : \Omega \times \Lambda^l \to \Lambda^l$ satisfies the following conditions:

$$\left|A(x, \xi)\right| \leq a|\xi|^{p-1}, \qquad \langle A(x, \xi), \xi\rangle \geq |\xi|^p. \tag{1.14}$$

If we let $A(x,\xi) = |\xi|^{p-2}\xi$, then the conjugate A-harmonic equation becomes the form

$$|du|^{p-2}du = d^*v. \tag{1.15}$$

It is the conjugate p-harmonic equation.

So we can see that the conjugate p-harmonic equation and the conjugate A-harmonic equation are the specific p-harmonic type system.

Remark 1.4. It should be noted that the mapping $A(x,*)$ in p-harmonic system $A(x, a+du) = b + d^*v$, is invertible. If we denote its inverse by $A^{-1}(x,*)$, then the mapping $A^{-1}(x,*) : \Lambda^l \to \Lambda^l$ satisfies similar conditions as A but with Hölder conjugate exponent q in place of p.

2. The Poincaré inequality

In this section, we will introduce the Poincaré inequality for the differential forms.

Now first let us see a lemma, which can be found in [9, Section 4] for the details.

Lemma 2.1. *Let $\mathbb{D}$ be a bounded, convex domain in $\mathbb{R}^n$. To each $y \in \mathbb{D}$ there corresponds a linear operator $K_y : C^\infty(\mathbb{D},\Lambda^l) \to C^\infty(\mathbb{D},\Lambda^{l-1})$ defined by*

$$(K_y\omega)(x;\xi_1,\ldots,\xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx+y-ty; x-y, \xi_1,\ldots,\xi_{l-1})dt, \tag{2.1}$$

and the decomposition

$$\omega = d(K_y\omega) + K_y(d\omega) \tag{2.2}$$

holds at any point $y \in \mathbb{D}$.

We construct a homotopy operator $T : C^\infty(\mathbb{D},\Lambda^l) \to C^\infty(\mathbb{D},\Lambda^{l-1})$ by averaging K_y over all points $y \in \mathbb{D}$:

$$T\omega = \int_{\mathbb{D}} \varphi(y)K_y\omega\, dy, \tag{2.3}$$

where φ form $C^\infty(\mathbb{D})$ is normalized so that $\int \varphi(y)dy = 1$. It is obvious that $\omega = d(K_y\omega) + K_y(d\omega)$ remains valid for the operator T :

$$\omega = d(T\omega) + T(d\omega). \tag{2.4}$$

We define the l-forms $\omega_{\mathbb{D}} \in D'(\mathbb{D},\Lambda^l)$ by $\omega_{\mathbb{D}} = |\mathbb{D}|^{-1}\int_{\mathbb{D}}\omega(y)dy$ for $l=0$ and $\omega_{\mathbb{D}} = d(T\omega)$ for $l = 1,2,\ldots,n$, and all $\omega \in W^{1,p}(D,\Lambda^l)$, $1 < p < \infty$.

The following definition can be found in [9, page 34].

Definition 2.2. For $\omega \in D'(\mathbb{D},\Lambda^l)$, the vector valued differential form

$$\nabla\omega = \left(\frac{\partial\omega}{\partial x_1},\ldots,\frac{\partial\omega}{\partial x_n}\right) \tag{2.5}$$

consists of differential forms $\partial\omega/\partial x_i \in D'(\mathbb{D}, \Lambda^l)$, where the partial differentiation is applied to coefficients of ω.

The proof of [9, Proposition 4.1] implies the following inequality.

Lemma 2.3. *For any $\omega \in L^p(\mathbb{D}, \Lambda^l)$, it holds that*

$$\|\nabla T\omega\|_{p,\mathbb{D}} \le C(n,p)\|\omega\|_{p,\mathbb{D}} \tag{2.6}$$

for any ball or cube $\mathbb{D} \in \mathbb{R}^n$.

The following Poincaré inequality can be found in [2].

Lemma 2.4. *If $u \in W_0^{1,p}(\Omega)$, then there is a constant $C = C(n,p) > 0$ such that*

$$\left(\frac{1}{|B|}\int_B |u|^{p\chi} dx\right)^{1/p\chi} \le Cr\left(\frac{1}{|B|}\int_B |\nabla u|^p dx\right)^{1/p}, \tag{2.7}$$

whenever $B = B(x_0, r)$ is a ball in $\mathbb{R}^n$, where $n \ge 2$ and $\chi = 2$ for $p \ge n$, $\chi = np/(n-p)$ for $p < n$.

Theorem 2.5. *Let $u \in D'(\mathbb{D}, \Lambda^l)$, and $du \in L^p(\mathbb{D}, \Lambda^{l+1})$. Then, $u - u_{\mathbb{D}}$ is in $L^{\chi p}(\mathbb{D}, \Lambda^l)$ and*

$$\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |u - u_{\mathbb{D}}|^{p\chi} dx\right)^{1/p\chi} \le C(n,p,l)\operatorname{diam}(\mathbb{D})\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |du|^p dx\right)^{1/p} \tag{2.8}$$

for any ball or cube $\mathbb{D} \in \mathbb{R}^n$, where $\chi = 2$ for $p \ge n$ and $\chi = np/(n-p)$ for $1 < p < n$.

Proof. We know $T(du) = u - u_{\mathbb{D}}$. Now we suppose $u - u_Q = T(du) = \sum_I u_I dx_I$, where $I = (i_1, \dots, i_{l+1})$ take over all $l+1$-tuples. So we have

$$\nabla T(du) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = \left(\sum_I \frac{\partial u_I}{\partial x_1} dx_I, \dots, \sum_I \frac{\partial u_I}{\partial x_n} dx_I\right). \tag{2.9}$$

So we have

$$\begin{aligned}\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |u - u_{\mathbb{D}}|^{p\chi} dx\right)^{1/p\chi} &= \left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \left|\sum_I u_I dx_I\right|^{p\chi} dx\right)^{1/p\chi} \\ &= \left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} \left(\sum_I |u_I|^2\right)^{p\chi/2} dx\right)^{1/p\chi}.\end{aligned} \tag{2.10}$$

By the inequality

$$\left(\sum_{i=1}^{n}(a_i)^2\right)^{1/2} \le \sum_{i=1}^{n} a_i \le n^{1/2}\left(\sum_{i=1}^{n}(a_i)^2\right)^{1/2} \tag{2.11}$$

for any $a_i \ge 0$, and the Minkowski inequality, we have

$$\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}\left(\sum_I |u_I|^2\right)^{p\chi/2} dx\right)^{1/p\chi} \le \sum_I \left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |u_I|^{p\chi} dx\right)^{1/p\chi}. \tag{2.12}$$

According to the Poincaré inequality, we have

$$\sum_I \left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |u_I|^{p\chi} dx\right)^{1/p\chi} \le C_1(n,p)\operatorname{diam}(\mathbb{D})\sum_I \left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |\nabla u_I|^{p} dx\right)^{1/p}. \tag{2.13}$$

Combining (2.10), (2.12), and (2.13), we can obtain

$$\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |u - u_{\mathbb{D}}|^{p\chi} dx\right)^{1/p\chi} \le C_1(n,p)\operatorname{diam}(\mathbb{D})\sum_I \left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |\nabla u_I|^{p} dx\right)^{1/p}. \tag{2.14}$$

By (2.9) we have

$$\begin{aligned}
\|\nabla T du\|_{p,\mathbb{D}} &= \left\|\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)\right\|_{p,\mathbb{D}} \\
&= \left(\int_{\mathbb{D}}\left|\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)\right|^p dx\right)^{1/p} \\
&= \left(\int_{\mathbb{D}}\left(\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2\right)^{p/2} dx\right)^{1/p} \\
&= \left(\int_{\mathbb{D}}\left(\sum_{i=1}^{n}\sum_I\left|\frac{\partial u_I}{\partial x_i}\right|^2\right)^{p/2} dx\right)^{1/p} \\
&= \left(\int_{\mathbb{D}}\left(\sum_I\sum_{i=1}^{n}\left|\frac{\partial u_I}{\partial x_i}\right|^2\right)^{p/2} dx\right)^{1/p}.
\end{aligned} \tag{2.15}$$

Combining (2.11) and (2.15), then we have

$$
\begin{aligned}
\|\nabla T du\|_{p,\mathbb{D}} &= \left(\int_{\mathbb{D}}\left(\sum_I\sum_{i=1}^n\left|\frac{\partial u_I}{\partial x_i}\right|^2\right)^{p/2}dx\right)^{1/p}\\
&\geq \left(C_n^{(l+1)}\right)^{-1/2}\left(\int_{\mathbb{D}}\left(\sum_I\left(\sum_{i=1}^n\left|\frac{\partial u_I}{\partial x_i}\right|^2\right)^{1/2}\right)^p dx\right)v^{1/p}\\
&\geq \left(C_n^{(l+1)}\right)^{-1/2}\left(\int_{\mathbb{D}}\sum_I\left(\sum_{i=1}^n\left|\frac{\partial u_I}{\partial x_i}\right|^2\right)^{p/2}dx\right)^{1/p}\\
&\geq \left(C_n^{(l+1)}\right)^{-1/2}\left(C_n^{(l+1)}\right)^{-(p-1)/p}\sum_I\left(\int_{\mathbb{D}}\left(\sum_{i=1}^n\left|\frac{\partial u_I}{\partial x_i}\right|^2\right)^{p/2}dx\right)^{1/p}\\
&\geq \left(C_2(n,p,l)\right)^{-1}\sum_I\left(\int_{\mathbb{D}}\left|\nabla u_I\right|^p dx\right)^{1/p},
\end{aligned}
\tag{2.16}
$$

where $C_2(n,p,l) = (C_n^{(l+1)})^{1/2+(p-1)/p}$. Now combining (2.14), (2.16), and (2.6), we can get

$$
\begin{aligned}
\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}|u-u_{\mathbb{D}}|^{p\chi}dx\right)^{1/p\chi} &\leq C_1(n,p)\operatorname{diam}(\mathbb{D})\sum_I\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}\left|\nabla u_I\right|^p dx\right)^{1/p}\\
&\leq C_1(n,p,l)C_2(n,p,l)\left(\frac{1}{|\mathbb{D}|}\right)^{1/p}\|\nabla T du\|_{p,\mathbb{D}}\\
&\leq C_3(n,p,l)\operatorname{diam}(\mathbb{D})\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}}|du|^p dx\right)^{1/p}.
\end{aligned}
\tag{2.17}
$$

□

3. The reverse Hölder inequality

In this section, we will prove the reverse Hölder inequality for the solution of the p-harmonic type system. Before we prove the reverse Hölder inequality, let us first see some lemmas.

Lemma 3.1. *If $f, g \geq 0$ and for any nonnegative $\eta \in C_0^\infty(\Omega)$, it holds*

$$\int_\Omega \eta f\,dx \leq \int_\Omega g\,dx, \tag{3.1}$$

then for any $h \geq 0$:

$$\int_\Omega \eta f h\,dx \leq \int_\Omega gh\,dx. \tag{3.2}$$

Proof. Let μ be a measure in X, f be a nonnegative μ-measurable function in a measure space X, using the standard representation theorem, we have

$$\int_X f^q \, d\mu = q\int_0^\infty t^{q-1}\mu(x : f(x) > t)dt \tag{3.3}$$

for any $0 < t < q$. Now, we let $\mu(E) = \int_E \eta f \, dx$ and $\nu(E) = \int_E g \, dx$ then, we can obtain

$$\int_\Omega \eta f h \, dx = \int_0^\infty \int_{h>t} \eta f \, dx \, dt \le \int_0^\infty \int_{h>t} g \, dx \, dt = \int_\Omega gh \, dx. \tag{3.4}$$

So Lemma 3.1 is proved. □

Lemma 3.2. *If (u, v) is a pair of solution to the p-harmonic type system* (1.9), *then it holds*

$$\int_\Omega |\eta \, da|^p dx \le C\int_\Omega |(a + du)d\eta|^p dx \tag{3.5}$$

for any nonnegative $\eta \in C_0^\infty(\Omega)$ and where $C = (C_n^{l+1})^p$.

Proof. Since (u, v) is a pair of solutions to $A(x, a + du) = b + d^*v$, it is also the solution to $A^{-1}(x, b + d^*v) = a + du$, where $A^{-1}(x, *)$ is the inverse $A(x, *)$. Now, we suppose that $da = \sum_I \omega_I \, dx_I$ and let $\varphi_1 = -\sum_I \eta \operatorname{sign}(\omega_I)dx_I$. By using $\varphi = \varphi_1$ and $d\varphi_1 = \sum_I \operatorname{sign}(\omega_I)d\eta \wedge dx_I$ in (1.11), we can obtain

$$\int_\Omega \langle A^{-1}(x, b + d^*v), d\varphi_1\rangle + \langle da, \varphi_1\rangle dx \equiv 0. \tag{3.6}$$

That is,

$$\int_\Omega \left\langle da, \sum_I \eta \operatorname{sign}(\omega_I)dx_I \right\rangle dx = \int_\Omega \left\langle A^{-1}(x, b + d^*v), -\sum_I \operatorname{sign}(\omega_I)d\eta \wedge dx_I \right\rangle dx. \tag{3.7}$$

In other words,

$$\int_\Omega \sum_I \eta|\omega_I| dx = \int_\Omega \left\langle A^{-1}(x, b + d^*v), -\sum_I \operatorname{sign}(\omega_I)d\eta \wedge dx_I \right\rangle dx. \tag{3.8}$$

By the elementary inequality

$$\left(\sum_{i=1}^n {a_i}^2\right)^{1/2} \le \sum_{i=1}^n |a_i|, \tag{3.9}$$

we have

$$\begin{aligned}\int_\Omega \eta|da|dx &= \int_\Omega \eta\left(\sum_I {\omega_I}^2\right)^{1/2} dx \leq \int_\Omega \sum_I \eta|\omega_I|dx \\ &= \int_\Omega \left\langle A^{-1}(x, b + d^* v), -\sum_I \operatorname{sign}(\omega_I) d\eta \wedge dx_I \right\rangle dx.\end{aligned} \tag{3.10}$$

Using the inequality

$$|\langle a, b\rangle| \leq |a||b|, \tag{3.11}$$

(3.10) becomes

$$\begin{aligned}\int_\Omega \eta|da|dx &\leq \int_\Omega \left|A^{-1}(x, b + d^* v)\right| \left|\sum_I \operatorname{sign}(\omega_I) d\eta \wedge dx_I\right| \\ &\leq \int_\Omega \left|A^{-1}(x, b + d^* v)\right| \sum_I \left|\operatorname{sign}(\omega_I)\right| |d\eta| dx \\ &= C_n^{l+1} \int_\Omega \left|A^{-1}(x, b + d^* v)\right| |d\eta| dx \\ &= C_n^{l+1} \int_\Omega |a + du||d\eta| dx,\end{aligned} \tag{3.12}$$

where I takes over all $(l+1)$-tuples for $d\eta \in \Lambda^{l+1}$, thus it has C_n^{l+1} numbers at most. Now we let $f = |da|$ and $g = C_n^{l+1}|a + du||d\eta|$. In the subset $\{x : f\eta = g\}$, we have

$$\int_{\{x: f\eta = g\}} |\eta da|^p dx \leq \int_{\{x: f\eta = g\}} |(a + du) d\eta|^p dx. \tag{3.13}$$

In the subset $\{x : f\eta \neq g\}$, let $h = (|f\eta|^p - |g|^p)/(f\eta - g)$, then we easily obtain $h > 0$. So by Lemma 3.1, we have

$$\int_{\{x: f\eta \neq g\}} h f \eta \, dx \leq \int_{\{x: f\eta \neq g\}} h g \, dx. \tag{3.14}$$

That is to say

$$\int_{\{x: f\eta \neq g\}} h(f\eta - g) dx \leq 0, \tag{3.15}$$

that is,

$$\int_{\{x:f\eta\neq g\}} |f\eta|^p dx \leq \int_{f\eta\neq g} |g|^p dx. \tag{3.16}$$

Combining (3.13) and (3.16), we have

$$\int_{\Omega} |f\eta|^p dx \leq \int_{\Omega} |g|^p dx, \tag{3.17}$$

that is,

$$\int_{\Omega} |\eta\, da|^p dx \leq \int_{\Omega} \left|C_n^{l+1}(a+du)d\eta\right|^p dx. \tag{3.18}$$

So Lemma 3.2 is proved. □

The following lemma appears in [2].

Lemma 3.3. *Suppose that $0 < q < p < s \leq \infty$, $\xi \in \mathbb{R}$, and that $B = B(x_0, r)$ is a ball. If a nonnegative function $v \in L^p(B, d\mu)$ satisfies*

$$\left(\frac{1}{\mu(\lambda B')}\int_{\lambda B'} v^s\, d\mu\right)^{1/s} \leq C(1-\lambda)^{\xi}\left(\frac{1}{\mu(B')}\int_{B'} v^p\, d\mu\right)^{1/p} \tag{3.19}$$

for each ball $B' = B(x_0, r')$ with $r' \leq r$ and for all $0 < \lambda < 1$, then

$$\left(\frac{1}{\mu(\lambda B)}\int_{\lambda B} v^s\, d\mu\right)^{1/s} \leq C(1-\lambda)^{\xi/\theta}\left(\frac{1}{\mu(B)}\int_{B} v^q\, d\mu\right)^{1/q} \quad \forall 0 < \lambda < 1. \tag{3.20}$$

Here $C > 0$ is a constant depending on p, q, s and $\theta \in (0,1)$ is such that $1/p = \theta/q + (1-\theta)/s$.

The following lemma appears in [10].

Lemma 3.4. *Let (u, v) be a pair of solutions of the p-harmonic type system on domain Ω, then we have a constant C only depending on K, n, p, and l, such that*

$$\|\eta\, du\|_{p,\Omega} \leq C\left(\|(u-c)d\eta\|_{p,\Omega} + \|\eta a\|_{p,\Omega}\right), \tag{3.21}$$

where c is any closed form (i.e., $dc = 0$) and for any $\eta \in C_0^\infty(\Omega)$. Also we have a constant C' only depending on $K,\ n,\ q$, such that

$$\|\eta\, d^* v\|_{q,\Omega} \le C'\left(\|(v-c')d\eta\|_{q,\Omega} + \|\eta b\|_{q,\Omega}\right), \tag{3.22}$$

where c' is any coclosed form (i.e., $d^*c' = 0$) and q is the conjugate exponent of p.

Theorem 3.5. *If (u,v) is a pair of solutions to the p-harmonic type system, then there exists a constant $C>0$ dependent on $K,\ p,\ n$, and l, such that*

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q\left(|u-u_Q| + \|a\|_{\infty,Q}\right)^s dx\right)^{1/s} &\le C\left(1-\sigma^{-1}\right)^{-t\chi/p(\chi-1)}(\operatorname{diam} Q+1)^{\chi/(\chi-1)} \\ &\quad\times\left(\frac{1}{|\sigma Q|}\int_{\sigma Q}\left(|u-u_{\sigma Q}| + \|a\|_{\infty,\sigma Q}\right)^t dx\right)^{1/t}\end{aligned} \tag{3.23}$$

for any $0<s,\ t<\infty$, $\sigma>1$ and all cubes with $Q\subset \sigma Q\subset\Omega$, where $\chi>1$ is the Poincaré constant.

Proof. Suppose that the center of Q is x_0 and $\operatorname{diam} Q = r$, $0<\lambda=\sigma^{-1}<1$. Let

$$r_m = \lambda + (1-\lambda)2^{-m}, \quad m = 0,1,2,\ldots. \tag{3.24}$$

Then r_m is decreasing and $\lambda < r_m < 1$. So we have $u_Q|_{r_mQ} = u_{r_mQ}$, for any $m\in 0,1,2,\ldots$. Let $\eta_m \in C_0^\infty(r_mQ)$ be a nonnegative function such that $\eta_m = 1$ in $r_{m+1}Q$, $0\le\eta_m\le 1$ in $r_mQ - r_{m+1}Q$. $|d\eta_m| \le (1-\lambda)^{-1}2^m r^{-1}$. Given any $t\ge 0$ and let $\omega_m = (|u-u_Q| + \|a\|_{\infty,Q})^{1+t/p}\eta_m$, then we have

$$du_m = \left(1+\frac{t}{p}\right)\left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{t/p}\eta_m d|u-u_Q + |\left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{1+t/p}d\eta_m. \tag{3.25}$$

By the Minkowski inequality, we can obtain

$$\begin{aligned}\left(\int_{r_mQ}|du_m|^p dx\right)^{1/p} &\le \left(\int_{r_mQ}\left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t}|d\eta_m|^p dx\right)^{1/p} \\ &\quad+\frac{(p+t)}{p}\left(\int_{r_mQ}|d|u-u_Q||^p\left(|u-u_Q| + \|a\|_{\infty,Q}\right)^t|\eta_m|^p dx\right)^{1/p}.\end{aligned} \tag{3.26}$$

We assume that $u-u_Q = \sum_I a_I dx_I$, then we have $|u-u_Q| = (\sum_I a_I^2)^{1/2}$. If $u-u_Q$ is zero, then we have $|d|u-u_Q|| = 0 = |\nabla T(du)|$. If $u-u_Q$ is not equal zero, and the proof of (2.15) implies that $|\nabla T du| = (\sum_I \sum_{i=1}^{n} |\partial a_I/\partial x_i|^2)^{1/2}$

$$
\begin{aligned}
\left|d\left|u-u_Q\right|\right| &= \left|\nabla\left|u-u_Q\right|\right| = \left|\left(\frac{\partial\left|u-u_Q\right|}{\partial x_1},\ldots,\frac{\partial\left|u-u_Q\right|}{\partial x_n}\right)\right| \\
&= \left(\sum_{i=1}^{n}\left|\frac{\partial\left|u-u_Q\right|}{\partial x_i}\right|^2\right)^{1/2} = \left(\sum_{i=1}^{n}\left|\frac{\partial\left|u-u_Q\right|}{\partial x_i}\right|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\left|\frac{\partial(\sum_I a_I^2)^{1/2}}{\partial x_i}\right|^2\right)^{1/2} = \left(\sum_{i=1}^{n}\frac{1}{\sum_I a_I^2}\left|\sum_I a_I\frac{\partial a_I}{\partial x_i}\right|^2\right)^{1/2} \\
&\le \left(\sum_{i=1}^{n}\frac{1}{\sum_I a_I^2}\sum_I a_I^2\sum_I\left(\frac{\partial a_I}{\partial x_i}\right)^2\right)^{1/2} = \left(\sum_{i=1}^{n}\sum_I\left(\frac{\partial a_I}{\partial x_i}\right)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}\sum_I\left|\frac{\partial a_I}{\partial x_i}\right|^2\right)^{1/2} = \left|\nabla T(du)\right| = \left|\nabla(u-u_Q)\right|.
\end{aligned}
\tag{3.27}
$$

So we have

$$
\left|d\left|u-u_Q\right|\right| \le \left|\nabla T(du)\right|. \tag{3.28}
$$

For any $\eta \in C_0^{\infty}(\Omega)$, according to (2.6), we have

$$
\|\eta\nabla T\, d\omega\|_{p,\mathbb{D}} \le C(n,p)\max_{x\in\mathbb{D}}(\eta)\|d\omega\|_{p,\mathbb{D}}. \tag{3.29}
$$

By the similar method as Lemma 3.1, we can prove the following inequality:

$$
\begin{aligned}
&\left(\int_{r_mQ}\left|d\left|u-u_Q\right|\right|^p\left(\left|u-u_Q\right|+\|a\|_{\infty,Q}\right)^t\left|\eta_m\right|^p dx\right)^{1/p} \\
&\quad\le \left(\int_{r_mQ}\left|\eta_m\right|^p\left|\nabla T(du)\right|^p\left(\left|u-u_Q\right|+\|a\|_{\infty,Q}\right)^t dx\right)^{1/p} \\
&\quad\le C(n,p)\max_{x\in\mathbb{D}}(\eta_m^p)\left(\int_{r_mQ}\left|\eta_m\right|^p|du|^p\left(\left|u-u_Q\right|+\|a\|_{\infty,Q}\right)^t dx\right)^{1/p}
\end{aligned}
\tag{3.30}
$$

for any $\eta \in C_0^\infty(\Omega)$. By Lemma 3.1 and (3.21), we can obtain

$$\begin{aligned}
&\left(\int_{r_mQ} |\eta_m|^p |du|^p \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^t dx\right)^{1/p} \\
&\quad \le 2C\left(\int_{r_mQ} |\eta_m|^p |a|^p \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^t dx\right)^{1/p} \\
&\qquad + 2C\left(\int_{r_mQ} |d\eta_m|^p \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t} dx\right)^{1/p} \\
&\quad \le 2C\left(\int_{r_mQ} |\eta_m|^p \|a\|_{\infty,Q}^p \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^t dx\right)^{1/p} \\
&\qquad + 2C\left(\int_{r_mQ} |d\eta_m|^p \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t} dx\right)^{1/p} \\
&\quad \le 2C\left(\int_{r_mQ} |\eta_m|^p \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t} dx\right)^{1/p} \\
&\qquad + 2C\left(\int_{r_mQ} |d\eta_m|^p \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t} dx\right)^{1/p}.
\end{aligned} \tag{3.31}$$

Combining (3.26), (3.30), and (3.31), by the values of η_m, we have

$$\left(\int_{r_mQ} |du_m|^p dx\right)^{1/p} \le C_1(p+t)\left(1+(1-\lambda)^{-1}2^m r^{-1}\right)\left(\int_{r_mQ} \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t} dx\right)^{1/p}. \tag{3.32}$$

For $\eta_m = 1$ in $r_{m+1}Q$ and $0 \le \eta_m \le 1$ in $r_mQ - r_{m+1}Q$, and as we have $|r_m|/r_{m+1} = |\lambda + (1-\lambda)2^{-m}|/(\lambda + (1-\lambda)2^{-m-1}) \le 2$, so we have $|r_mQ|/|r_{m+1}Q| \le 2^n$. By the Poincaré inequality, we know

$$\begin{aligned}
&\left(\frac{1}{|r_{m+1}Q|}\int_{r_{m+1}Q} \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{\chi(p+t)} dx\right)^{1/p\chi} \\
&\quad \le \frac{1}{|r_{m+1}Q|}\int_{r_mQ} \left(\eta_m^{p\chi}|u-u_Q| + \|a\|_{\infty,Q}\right)^{\chi(p+t)} dx)^{1/p\chi} \\
&\quad \le \left(\frac{1}{|r_{m+1}Q|}\int_{r_mQ} |u_m|^{p\chi} dx\right)^{1/p\chi} \\
&\quad \le 2^n\left(\frac{1}{|r_mQ|}\int_{r_mQ} |u_m|^{p\chi} dx\right)^{1/p\chi} \\
&\quad \le C_2 r_m r\left(\frac{1}{|r_mQ|}\int_{r_mQ} |du_m|^p dx\right)^{1/p} \\
&\quad \le C_3 r_m r(p+t)\left(1+(1-\lambda)^{-1}2^m r^{-1}\right)\left(\int_{r_mQ} \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t} dx\right)^{1/p} \\
&\quad \le C_3(p+t)(1-\lambda)^{-1}2^m(1+r)\left(\int_{r_mQ} \left(|u-u_Q| + \|a\|_{\infty,Q}\right)^{p+t} dx\right)^{1/p}.
\end{aligned} \tag{3.33}$$

Now we set $\kappa = p + t$, then by computation, we obtain

$$\left(\frac{1}{|r_{m+1}Q|}\int_{r_{m+1}Q}\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{\kappa\chi}dx\right)^{1/\kappa\chi} \le (C_3)^{p/\kappa}\kappa^{p/\kappa}(1-\lambda)^{-p/\kappa}2^{pm/\kappa}(r+1)^{p/\kappa} \times\left(\frac{1}{|r_mQ|}\int_{r_mQ}\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{\kappa}dx\right)^{1/\kappa}. \tag{3.34}$$

Since this inequality holds for all $\kappa > p$, it can be applied with $\kappa = \kappa_m = p\chi^m$. And we can easily prove $((1/|Q|)\int_Q|f|^p dx)^{1/p}$ is increasing with p and its limit is $\operatorname{ess\,sup}_Q|f|$. So by iterating we arrive at the desired inequality for $q = p$:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\lambda Q}\left(|u-u_Q|+\|a\|_{\infty,Q}\right)\\
&\quad\le \lim_{m\to\infty}\left(\frac{1}{|r_mQ|}\int_{r_mQ}\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{\kappa_m\chi}dx\right)^{1/\kappa_m\chi}\\
&\quad\le C_4\left((1-\lambda)^{-1}(1+r)\right)^{\sum_{i=0}^{\infty}\chi^{-m}}\prod_{m=0}^{\infty}2^{m\chi^{-m}}\prod_{m=0}^{\infty}\left(p\chi^m\right)^{\chi^{-m}}\\
&\qquad\times\left(\frac{1}{|Q|}\int_Q\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{p}dx\right)^{1/p}\\
&\quad\le C_5(1-\lambda)^{-\chi/(\chi-1)}(r+1)^{\chi/(\chi-1)}\left(\frac{1}{|Q|}\int_Q\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{p}dx\right)^{1/p}.
\end{aligned} \tag{3.35}$$

We can observe that the constants C_5 and χ are independent of x_0 and r in (3.35), thus (3.35) holds not only in the cube $Q = Q(x_0, r)$ but also in each ball inside Q. By Lemma (3.5) we can obtain

$$\left(\frac{1}{|\lambda Q|}\int_{\lambda Q}\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{s}dx\right)^{1/s} \le C_5(1-\lambda)^{-\theta\chi/(\chi-1)}(r+1)^{\chi/(\chi-1)} \times\left(\frac{1}{|Q|}\int_Q\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{t}dx\right)^{1/t} \tag{3.36}$$

for any $0 < t < p < s \le \infty$, where $\theta = t(s-p)/p(s-t)$. So we have $\theta \le t/p$ for any $0 < t < p < s \le \infty$. Since $((1/|Q|)\int_Q|f|^p dx)^{1/p}$ is increasing with p,

$$\left(\frac{1}{|\lambda Q|}\int_{\lambda Q}\left(|u-u_Q|+\|a\|_{\infty,Q}\right)^{s}dx\right)^{1/s} \le C_5(1-\lambda)^{-t\chi/p(\chi-1)}(r+1)^{\chi/(\chi-1)} \times\left(\frac{1}{|Q|}\int_Q\left(|u-u_Q|+\|a\|_{\infty,\sigma Q}\right)^{t}dx\right)^{1/t} \tag{3.37}$$

for any $0 < s < \infty$ and $1 < p < t < \infty$. Combining (3.36) and (3.37), we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q \left(\left|u-u_Q\right| + \|a\|_{\infty,Q}\right)^s dx\right)^{1/s} \le{}& C_6(1-\lambda)^{-t\chi/p(\chi-1)}(r+1)^{\chi/(\chi-1)} \\ &\times \left(\frac{1}{|\sigma Q|}\int_{\sigma Q}\left(|u-u_{\sigma Q}| + \|a\|_{\infty,\sigma Q}\right)^t dx\right)^{1/t}\end{aligned} \tag{3.38}$$

for any $0 < s,\ t < \infty$ and $\sigma > 1$ such that $\sigma Q \subset \Omega$. Theorem 3.5 is proved. □

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