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Research Article

Exact Values of Bernstein n-Widths for Some Classes of Convolution Functions

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We consider some classes of 2π-periodic convolution functions $\widetilde{B}_p$, and $\widetilde{K}_p$, which include the classical Sobolev class as a special case. With the help of the spectra of nonlinear integral equations, we determine the exact values of Bernstein n-width of the classes $\widetilde{B}_p$, $\widetilde{K}_p$ in the space L^p for $1 < p < \infty$.

1. Introduction and main results

Let X be a normed linear space and let A be a subset of X. Assume that A is closed, convex, and centrally symmetric (i.e., $x \in A$ implies $-x \in A$). The Bernstein n-width, which was originally introduced by Tikhomirov [1], of A in X is given by

$$b_n(A; X) = \sup_{X_{n+1}} \sup \{\lambda : \lambda S(X_{n+1}) \subseteq A\}, \tag{1.1}$$

where $S(X_{n+1}) = \{x : x \in X_{n+1}, \|x\| \le 1\}$ and X_{n+1} is taken over all subspaces of X of dimension at least $n+1$. Let $\mathbb{T} := [0, 2\pi)$ be the torus, and as usual, let $L^q := L^q[0, 2\pi]$ be the classical Lebesgue integral space of 2π-periodic real-valued functions with the usual norm $\|\cdot\|_q$, $1 \le q \le \infty$.

Denote by W_p^r the classical Sobolev class of real functions f whose $(r-1)$th derivative is absolutely continuous and whose rth derivative satisfies the condition $\|f^{(r)}\|_q \le 1$. The concept of Bernstein n-width for the Sobolev classes W_p^r was originally introduced by Tikhomirov [1]. He considered $b_n(W_p^r; L^q)$, $1 \le p$, $q \le \infty$, and found the exact value of $b_{2n-1}(W_\infty^r; L^\infty)$. Pinkus [2] obtained the exact value of $b_{2n-1}(W_1^r; L^1)$. Later, Magaril-Il'yaev [3] obtained the exact value of $b_{2n-1}(W_p^r; L^p)$, $1 < p < \infty$. The latest contribution to this field is due to Buslaev et al. [4] who found the exact values of $b_{2n-1}(W_p^r; L^q)$ for all $1 < p \le q < \infty$.

Definition 1.1 (see [2, page 129]). A real, 2π-periodic, continuous function G satisfies property B if for every choice of $0 \le t_1 < \cdots < t_m < 2\pi$ and each $m \in \mathbb{N}$, the subspace

$$X_m := \left\{ b + \sum_{j=1}^{m} b_j G(\cdot - t_j) : \sum_{j=1}^{m} b_j = 0 \right\} \tag{1.2}$$

is of dimension m, and is a weak Tchebycheff- (WT-) system (see [2, page 39]) for all m odd. A real, 2π-periodic, continuous function G is said to be B-kernel if G satisfies property B.

Definition 1.2 (see [2, pages 60, 126]). Assume that K is a real, continuous, 2π-periodic function. One says that K is a cyclic variation diminishing kernel of order $2m-1$ (CVD_{2m-1}) if there exist $\sigma_n \in \{-1, 1\}$, $n = 1, \ldots, m$, such that

$$\sigma_n \det \left(K(x_i - y_j) \right)_{i,j=1}^{2n-1} \ge 0, \tag{1.3}$$

for all $x_1 < \cdots < x_{2n-1} < x_1 + 2\pi$ and $y_1 < \cdots < y_{2n-1} < y_1 + 2\pi$. One will drop the subscript $2m-1$ from the acronyms CVD, if one assumes that these properties hold for all orders. One says that K is nondegenerate cyclic variation diminishing (NCVD) if K is nonnegative CVD and

$$\dim \operatorname{span}\{ K(x_1 - \cdot), \ldots, K(x_n - \cdot) \} = n, \tag{1.4}$$

for every choice of $0 \le x_1 < \cdots < x_n < 2\pi$ and all $n \in \mathbb{N}$.

Now, we introduce the classes of functions to be studied. Let K be a NCVD kernel [2] and let G be a B-kernel. The 2π-periodic convolution function classes $\widetilde{K}_p$ and $\widetilde{B}_p$ are defined as follows:

$$\begin{gathered} \widetilde{B}_p := \{ f : f(x) = (G * h)(x) + a,\ a \in \mathbb{R}, h \perp 1, \|h\|_p \le 1 \}, \\ \widetilde{K}_p := \{ f : f(x) = (K * h)(x),\ h \perp 1, \|h\|_p \le 1 \}, \end{gathered} \tag{1.5}$$

where

$$(g * h)(x) := \int_{\mathbb{T}} g(x - y) h(y) dy, \tag{1.6}$$

and $h \perp 1$ means $\int_{\mathbb{T}} h(y) dy = 0$.

The exact values of $b_n(\widetilde{B}_p; L^q)$ and $b_n(\widetilde{K}_p; L^q)$ are known for the cases $p = q = 1$, $p = q = \infty$, and n is odd (see [2] for more details). Chen [5] is the one who found the lower estimate of $b_{2n-1}(\widetilde{B}_p, L^p)$ and $b_{2n-1}(\widetilde{K}_p, L^p)$ for $1 < p < \infty$. In this paper, we will determine the exact constants of some classes of periodic convolution functions $\widetilde{B}_p$ with B-kernel (or NCVD-kernel) for $p \in (1, \infty)$, which include the classical Sobolev class as its special case.

Now, we are in a position to state our main results of this paper.

Theorem 1.3. *Let G be a B-kernel, and $n = 1, 2, \ldots$. Then*

$$b_{2n-1}(\widetilde{B}_p; L^p) = \lambda_n(p, p, G), \quad 1 < p < \infty, \tag{1.7}$$

$$s_{2n}(\widetilde{B}_p; L^p) = b_{2n-1}(\widetilde{B}_p; L^p) = \lambda_n(p, p, G), \tag{1.8}$$

where

$$\begin{gathered} D_n := \left\{h : h\left(x + \frac{\pi}{n}\right) = -h(x),\ h(x)\{\sin\} nx \geq 0, \|h\|_p \leq 1\right\}, \\ \lambda_n := \lambda_n(p, q, G) = \{\sup\}\{\|G*h\|_q : h \in D_n, \}, \quad 1 < q \leq p < \infty, \end{gathered} \tag{1.9}$$

and $s_n(\widetilde{B}_p; L^p)$ denotes any one of the three n-widths, Kolmogorov, Gel'fand and [2, pages 1; 7; 20].

Theorem 1.4. *Let K be a* {NCVD} *kernel and $n = 1, 2, \ldots$. Then*

$$\begin{gathered} b_{2n-1}(\widetilde{K}_p; L^p) = \lambda_n(p, p, K), \quad 1 < p < \infty, \\ s_{2n}(\widetilde{K}_p; L^p) = b_{2n-1}(\widetilde{K}_p; L^p) = \lambda_n(p, p, K), \\ \lambda_n(p, q, K) = \{\sup\}\{\|K*h\|_q : h \in D_n, \}, \quad 1 < q \leq p < \infty. \end{gathered} \tag{1.10}$$

We will only give the proof for the case of a B-kernel. As for the case of a NCVD kernel, the proof is similar and even more simple.

2. Nonlinear integral equation and its spectral couple

Before we prove Theorem 1.3, we need some results about nonlinear integral equations and their spectral couple. First, we introduce some definitions and notations.

Definition 2.1 (see [2, pages 45, 59]). Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$ be a real nontrivial vector.

(i) $S^-(x)$ indicates the number of sign changes in the sequence $x_1, \ldots, x_n$ with zero terms discarded. The number $S_c^-(x)$ of cyclic variations of sign of x is given by

$$S_c^-(x) := \max_i S^-(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_i) = S^-(x_k, \ldots, x_n, x_1, \ldots, x_k), \tag{2.1}$$

where k is some integer for which $x_k \neq 0$. Obviously, $S_c^-(x)$ is invariant under cyclic permutations, and $S_c^-(x)$ is always an even number.

(ii) $S^+(x)$ counts the maximum number of sign changes in the sequence $x_1, \ldots, x_n$ where zero terms are arbitrarily assigned values +1 or −1. The number $S_c^+(x)$ of maximum cyclic variations of sign of x is defined by

$$S_c^+(x) := \max_i S^+(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_i). \tag{2.2}$$

Let f be a piecewise continuous, 2π-periodic, real-valued function on $\mathbb{R}$. One assumes that $f(x) = [f(x+) + f(x-)]/2$ for all x and

$$S_c(f) := \sup S_c^-((f(x_1), \dots, f(x_m))), \tag{2.3}$$

where the supremum is taken over all $x_1 < \cdots < x_m < x_1 + 2\pi$ and all $m \in \mathbb{N}$.

Moreover, one needs further counts of zeros of a function. Suppose that f is a continuous, 2π-periodic, real-valued function on $\mathbb{R}$. One defines

$$\widetilde{Z}_c(f) := \sup S_c^+((f(x_1), \dots, f(x_m))), \tag{2.4}$$

where the supremum runs over all $x_1 < \cdots < x_m < x_1 + 2\pi$ and all $m \in \mathbb{N}$. Assume that f is a 2π-periodic, real-valued function on $\mathbb{R}$ for which f is sufficiently smooth. The number of zeros of f on a period, counting multiplicities, is denoted by $Z_c^\star(f)$.

Clearly, $S_c(f)$ denotes the number of sign changes of f on a period, and $\widetilde{Z}_c(f)$ denotes the number of zeros of f on a period, where the zeros which are sign changes are counted once and zeros which are not sign changes are counted twice. Moreover, we have

$$S_c(f) \le \widetilde{Z}_c(f) \le Z_c^\star(f). \tag{2.5}$$

We define Q_p to be the nonlinear transformation:

$$(Q_p f)(t) := |f(t)|^{p-1} \operatorname{sign} f(t), \quad 1 < p < \infty. \tag{2.6}$$

Since the function $F(y) := |y|^{p-1}$ sign y is continuous and strictly increasing, $Q_p f$ is continuous if and only if f is. Moreover, since $F(y)$ is uniformly continuous on every compact interval, $Q_p f$ is a continuous operator from $C(\mathbb{T})$ to $C(\mathbb{T})$. It is clear that if $f \in L^p$, $1 < p < \infty$, then $Q_p f \in L^{p'}$, $p' = p/(p-1)$, and $Q_{p'} Q_p f = f$ for every f. For $1 \le q$, $p < \infty$, (f, λ^q) is called a spectral couple, and f is called a spectral function if

$$\begin{gathered} \|h\|_p = 1, \quad f(x) = (G * h)(x) + \beta, \\ (Q_p h)(y) = \lambda^{-q} \int_{\mathbb{T}} G(x-y)(Q_q f)(x)\,dx, \end{gathered} \tag{2.7}$$

where β satisfies the condition

$$\inf_{c \in \mathbb{R}} \|(G * h) + c\|_q = \|(G * h) + \beta\|_q, \tag{2.8}$$

when $\int_{\mathbb{T}} G(x)dx = 0$. It is well known that if $1 < q < \infty$, then β is unique. The set of all spectral couples is denoted by $\Gamma(p, q, G)$, and the spectral class $\Gamma_{2n}(p, q, G)$ is given by

$$\Gamma_{2n}(p, q, G) := \{(f, \lambda^q) \in \Gamma(p, q, G) : S_c(f) = 2n\}. \tag{2.9}$$

Lemma 2.2 (see [2, page 177]). *Let ϕ be a real piecewise continuous 2π-periodic function satisfying $\phi \perp \mathbf{1}$ and set $\psi(x) := a + (G * \phi)(x)$. If G satisfies property B, then*

$$\widetilde{Z}_c(\psi) \le S_c(\phi). \tag{2.10}$$

Lemma 2.3. *For $1 < p,\ q < \infty$, if $(f, \lambda^q) \in \Gamma(p,q,G)$ with $S_c(h) < \infty$. Then, f has a finite number of zeros, and all its zeros are simple.*

Proof. By (2.7) and Lemma 2.2, we have $S_c(f) \le \widetilde{Z}_c(f) \le S_c(h) \le \widetilde{Z}_c(Q_p h) \le S_c(Q_q f) = S_c(f)$. Obviously, $S_c(f) = S_c(h) = \widetilde{Z}_c(f)$. Therefore, f has a finite number of zeros, and all its zeros are simple. □

Lemma 2.4. (a) *If $1 < q < p < \infty$, and f_1 and f_2 are two spectral functions, then*

$$S_c(f_1 + f_2) \le \max\{S_c(f_1), S_c(f_2)\} < \infty. \tag{2.11}$$

(b) *If $1 < q \le p < \infty$, and f_1 and f_2 correspond to the same spectral value and $f_1 \ne f_2$, then all the zeros of $f_1 + f_2$ are with sign changes.*

Proof. Suppose that (f_1, λ_1^q) and (f_2, λ_2^q) are spectral couples and, say $0 < \lambda_1 \le \lambda_2$. For $\varepsilon > 0$, let $\sigma(\varepsilon) := S_c(f_1 + \varepsilon f_2)$. For all sufficiently small ε, we have $\sigma(\varepsilon) = S_c(f_1) = \widetilde{Z}_c(f_1) =: N$. Indeed, let $t_1, \ldots, t_N$ be the zeros of f_1. Then, by the continuity, there exist neighborhoods $V_{t_1}, V_{t_2}, \ldots, V_{t_N}$ for all small ε, so that $f_1 + \varepsilon f_2$ has exactly one zero in each V_{t_i}. On the other hand, $f_1 + \varepsilon f_2 \ne 0$ if $t \in T \setminus \bigcup_i (V_{t_i})$ and $\varepsilon > 0$ is sufficiently small. By using (2.5)–(2.7), Lemma 2.2, and the identity $\operatorname{sign}(a+b) = \operatorname{sign}(|a|^{p-1}\operatorname{sign} a + |b|^{p-1}\operatorname{sign} b)$, we have

$$\begin{aligned}
\sigma(\varepsilon) &= S_c(f_1 + \varepsilon f_2) \le \widetilde{Z}_c(f_1 + \varepsilon f_2) \le S_c(h_1 + \varepsilon h_2) \\
&= S_c(Q_p h_1 + Q_p(\varepsilon h_2)) = S_c(Q_p h_1 + \varepsilon^{p-1}(Q_p h_2)) \\
&\le S_c(\lambda_1^{-q} Q_q f_1 + \varepsilon^{p-1} \lambda_2^{-q} Q_q f_2) \\
&= S_c\left(Q_q f_1 + Q_q\left(\varepsilon^{(p-1)/(q-1)} (\lambda_1/\lambda_2)^{q/(q-1)} f_2\right)\right) \\
&= S_c\left(f_1 + \varepsilon^{(p-1)/(q-1)} (\lambda_1/\lambda_2)^{q/(q-1)} f_2\right) \\
&= \sigma\left(\varepsilon^{(p-1)/(q-1)} (\lambda_1/\lambda_2)^{q/(q-1)}\right).
\end{aligned} \tag{2.12}$$

Iterating this inequality for $0 < \varepsilon < 1$, we obtain $\sigma(\varepsilon) \le \sigma(\varepsilon_0)$, where ε_0 can be made arbitrarily close to zero (due to $1 < q < p < \infty$), so that we may assume that $\sigma(\varepsilon_0) = N$. Consequently, $\sigma(\varepsilon) \le N$ for $0 < \varepsilon < 1$. But then also $\sigma(1) = S_c(f_1 + f_2) \le N$ for otherwise one can choose $\varepsilon < 1$ so close to 1 that $\sigma(\varepsilon) > N$.

Now, we turn to prove part (b). Taking $\lambda_1 = \lambda_2$, $\varepsilon = 1$ in (2.12), we get $S_c(f_1 + f_2) = \widetilde{Z}_c(f_1 + f_2)$. Lemma 2.4 is proved. □

For a spectral function f, let $t_1 < t_2 < \cdots < t_m$ be all its zeros on $\mathbb{T}$, and let $s_k := (t_k + t_{k+1})/2$, $k = 1, \ldots, m$, $t_{m+1} = t_1 + 2\pi$ be the midpoints of the intervals between them.

Lemma 2.5. *For $1 < q \le p < \infty$, a spectral function f is odd with respect to each of its zeros t_k, that is, $f(t_k - t) = -f(t_k + t)$, and is even with respect to each s_k. Moreover, the number of zeros is even, $m = 2n$, and the points t_k are equidistant on $\mathbb{T}$. The f is periodic with period $2\pi/n$.*

Proof. Let $(f, \lambda^q) \in \Gamma(p,q,G)$. Then by [6], $\lambda = \|f\|_q$, and for each k, $f(t_k \pm t)$ is also a spectral function with the same λ. Therefore, $F(t) = f(t_k - t) + f(t_k + t)$ has a zero at $t = 0$ without sign change. By (b) of Lemma 2.4, this function $F(t)$ must be zero.

The proof of Lemma 2.5 is complete. □

Lemma 2.6 (see [6]). *Let G be a B-kernel, $n \in \mathbb{N}$, $1 < p, q < \infty$. Then, $\Gamma_{2n}(p,q,G) \neq \emptyset$. Moreover, if $(f, \lambda^q) \in \Gamma_{2n}(p,q,G)$, then the function $f := (G*h) + \beta$ satisfies the following conditions:*

$$f\left(x + \frac{\pi}{n}\right) = -f(x), \quad \forall x \in [0, 2\pi), \tag{2.13}$$

with $\beta = 0$, and the simple zeros of f are equidistant on T, and

$$h\left(t + \frac{\pi}{n}\right) = -h(t), \quad \forall t \in [0, 2\pi). \tag{2.14}$$

Lemma 2.7. *Let G be a B-kernel. For $n \in \mathbb{N}$, $1 < q \leq p < \infty$, if $(f, \lambda^q) \in \Gamma_{2n}(p,q,G)$. Then, there exists $h \in D_n$, such that $\lambda = \|f\|_q = \|G*h\|_q$.*

Proof. For $(f, \lambda^q) \in \Gamma_{2n}(p,q,G)$, by (2.7), and Lemma 2.6, we have $f = (G*h)(x)$. We choose $h(x_0) \geq 0$, $x_0 \in [0, \pi/n)$, then $h(x_0)\sin nx_0 \geq 0$, $x_0 \in [0, \pi/n)$. For $x \in \mathbb{T}$, there exists a i, $i = 1, \ldots, 2n$, such that $x \in [(i-1)\pi/n, i\pi/n)$. Since $h(x + \pi/n) = -h(x)$. Thus

$$h(x)\sin nx = h\left(x_0 + \frac{(i-1)\pi}{n}\right)\sin\left(n\left(x_0 + \frac{(i-1)\pi}{n}\right)\right) = h(x_0)\sin nx_0 \geq 0. \tag{2.15}$$

Combining (2.14), we get $h \in D_n$, and $\lambda = \|f\|_q = \|G*h\|_q$. The proof of Lemma 2.7 is complete. □

3. Upper estimate of Bernstein n-width

Following some ideas of Buslaev [4], Tikhomirov [1], Chen and Li [7], and Chen [5], the proofs of our main results are based on some iteration process which starts with an arbitrary function $h_0 \in L^p$ with mean value zero and produces a sequence of functions h_k, and then a subsequence of their integrals f_k converges to a spectral function f.

First, we take some $h_0 \in L^p$ such that $\|h_0\|_p = 1$, $h_0 \perp 1$. Let

$$f_0(x) = (G*h_0)(x) + \beta_0, \tag{3.1}$$

where β_0 satisfies the condition:

$$\inf_{c\in\mathbb{R}}\|(G*h_0) + c\|_q = \|(G*h_0) + \beta_0\|_q, \quad 1 < q < \infty. \tag{3.2}$$

Next, we construct the sequences of functions $\{h_k\}$ and $\{f_k\}$ as follows:

$$f_k(x) = (G*h_k)(x) + \beta_k, \quad k = 1, 2, \ldots, \tag{3.3}$$

$$(Q_p h_{k+1})(y) = \mu_{k+1}^{-q}\int_{\mathbb{T}} G(x-y)(Q_q f_k)(x)dx, \quad k = 0, 1, 2, \ldots, \tag{3.4}$$

where β_k is uniquely determined by the condition

$$\|f_{k+1}\|_q = \inf_{c\in\mathbb{R}}\|(G*h_{k+1}) + c\|_q = \|(G*h_{k+1}) + \beta_{k+1}\|_q, \quad 1 < q < \infty, \tag{3.5}$$

and $\mu_{k+1} > 0$ is determined by the condition $\|h_{k+1}\|_p = 1, 1 < p < \infty$.

Lemma 3.1. *Let $1 < p, q < \infty$. Then*

$$\|f_k\|_q \leq \mu_{k+1} \leq \|f_{k+1}\|_q, \quad k = 1, 2, \ldots. \tag{3.6}$$

Proof. By the Hölder's inequality, (2.7), and $\|Q_p g\|_{p'} = \|g\|_p^{p-1}$, we have

$$1 = \|h_{k+1}\|_p^{p-1} \cdot \|h_k\|_p \geq \langle Q_p h_{k+1}, h_k \rangle \geq \mu_{k+1}^{-q} \|f_k\|_q^q, \tag{3.7}$$

which proves the first inequality in (3.6). We now use this first inequality and similarly prove the second inequality:

$$\begin{aligned} 1 = \|h_{k+1}\|_p^p &= \langle Q_p h_{k+1}, h_{k+1} \rangle = \mu_{k+1}^{-q} \langle G * Q_q f_k, h_{k+1} \rangle \\ &\leq \mu_{k+1}^{-q} \|f_{k+1}\|_q \cdot \|Q_q f_k\|_{q'} = \mu_{k+1}^{-q} \|f_{k+1}\|_q \cdot \|f_k\|_q^{q-1} \leq \mu_{k+1}^{-1} \|f_{k+1}\|_q. \end{aligned} \tag{3.8}$$

The proof of Lemma 3.1 is complete. □

It follows from Lemma 3.1 that the construction of the sequence $\{f_k\}_{k=1}^{\infty}$ is unambiguous. Moreover, it follows from (3.6) that $\{\mu_{k+1}\}_{k=1}^{\infty}$ is monotonic nondecreasing sequence and tends to some number μ. It is clear that

$$\mu := \lim_{k \to \infty} \mu_k = \lim_{k \to \infty} \|f_k\|_q > 0. \tag{3.9}$$

Lemma 3.2. *For each starting function $h_0 \neq 0, h_0 \perp 1$, the sequence $\{h_k\}_{k=1}^{\infty}$ of (3.4) contains a subsequence $\{h_{k_i}\}_{i=1}^{\infty}$ for which $\{f_{k_i}(x) = (G * h_{k_i})(x) + \beta_{k_i}\}_{i=1}^{\infty}$ converges uniformly to a spectral function f (with a spectral value $\lambda = \mu$).*

Proof. By using the weak compactness of the unit ball of the space L^p, $1 < p < \infty$, one can choose a subsequence $\{h_{k_i}\}_{i=1}^{\infty}$ converging weakly to some h with $\|h\|_p = 1$, with $\{f_{k_i}\}_{i=1}^{\infty}$ converging uniformly to $f := (G * h) + \beta$. It follows from (3.4) that $\{Q_p h_{k_i+1}\}_{i=1}^{\infty}$ converges uniformly because the operator Q_p, $1 < p < \infty$, preserves uniform convergence. Consequently, $\{Q_{p'} Q_p h_{k_i+1} = h_{k_i+1}\}_{i=1}^{\infty}$ converges uniformly to some v with $\|v\|_p = 1$, where $1/p' + 1/p = 1$. Let $k \to \infty$ in (3.4) and with μ in (3.9). Then, we can obtain

$$(Q_p v)(y) = \mu^{-q} \int_{\mathbb{T}} G(x - y)(Q_q f)(x) dx. \tag{3.10}$$

Now, we turn to prove that (f, μ) is a spectral couple. Since in the following inequality, $Q_p h_{k_i+1} \to Q_p v$ uniformly and $h_{k_i} \to h$ weakly in L^p,

$$\langle Q_p h_{k_i+1}, h_{k_i} \rangle = \mu_{k_i+1}^{-q} \langle Q_q f_{k_i}, h_{k_i} \rangle \geq \mu_{k_i+1}^{-q} \|f_{k_i}\|_q^q \longrightarrow \mu^{-q} \cdot \mu^q = 1, \tag{3.11}$$

which implies $\langle Q_p v, h \rangle \geq 1$. On the other hand, by the Hölder's inequality, and $\|v\|_p = \|h\|_p = 1$, we get

$$\langle Q_p v, h \rangle \leq \|v\|_p^{p-1} \cdot \|h\|_p = 1. \tag{3.12}$$

Therefore, the case of equality can occur only if $|Q_p v|^{p'} = |h|^p$, sign $Q_p v$ =sign h almost every, or, equivalently, if $v = h$. Comparing (3.10) with (2.7), we get $\mu = \lambda$.

The proof of Lemma 3.2 is complete. □

For convenience, we denote by (G,λ_n) all the function h_n, where h_n is sufficiently

(i)

$$\|G*h_n\|_q = \lambda_n := \lambda(p,q,G) = \lambda_n\|h_n\|_p, \tag{3.13}$$

(ii)

$$\int_0^{2\pi} G(x-y)(Q_q G*h_n)(x)dx = \lambda_n^q (Qh_n)(y)dy, \quad y\in\mathbb{T}. \tag{3.14}$$

In what follows, we need to convolute G with periodic kernel for

$$\phi_\sigma = \phi(\sigma,t) := \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma^2}(t-2n\pi)^2\right], \tag{3.15}$$

$\sigma > 0$. It is known that [8]

(i) $Z_c^\star(\phi_\sigma * f) \le S_c(f)$,

(ii) $\lim_{\sigma\to 0^+}\phi_\sigma * f = f$ uniformly holds for every continuous function f with 2π-period.

Let G be a B-kernel. $G_\sigma := \phi_\sigma * G$ is said to be the mollification of G by ϕ_σ. It is easily verified that G_σ is a B-kernel.

Lemma 3.3 (see [5]). *Suppose $h_{n,\sigma}\in(G_\sigma,\lambda_{n,\sigma})$, where $\lambda_{n,\sigma} := \lambda_n(p,q,G_\sigma)$. Then*

(i) $\lim_{\sigma\to 0^+}\lambda_{n,\sigma} = \lambda_n$,

(ii) *there exists a sequence of real number $\sigma_k > 0$ such that $\sigma_k\to 0^+$ and the corresponding sequence of continuous functions $\{h_{n,\sigma_k}\}_{k=1}^\infty$ is convergent uniformly on $\mathbb{T}$,*

(iii) *denote $h_n(x) = \lim_{k\to\infty} h_{n,\sigma_k}(x)$, then $h_n\in(G,\lambda_n)$.*

We recall an equivalent definition on the Bernstein n-width of a linear operator P from a linear normed space X to Y.

Definition 3.4 (see [2, page 149]). Let $P\in L(X,Y)$. Then, the Bernstein n-width is defined by

$$b_n(P(X),Y) = \sup_{X_{n+1}}\inf_{\substack{Px\in X_{n+1}\\ Px\ne 0}}\frac{\|Px\|_Y}{\|x\|_X}, \tag{3.16}$$

where X_{n+1} is any subspace of span $\{Px : x\in X\}$ of dimension $\ge n+1$.

Lemma 3.5. *Let G be a B-kernel. For each $p\in(1,\infty)$ and $n = 1,2,\ldots$, then*

$$b_{2n-1}(\widetilde{B}_p; L^p) \le \lambda_n := \lambda_n(p,p,G). \tag{3.17}$$

Proof. We first prove the theorem under the assumption that G is sufficiently smooth, and $Z_c^\star(c+G*h)\le S_c(h)$ is true. An example of such function is G_σ, the mollification of G by ϕ_σ. Assume that $b_{2n-1}(\widetilde{B}_p;L^p)>\lambda_n$. From the definition of Bernstein n-width, there exists a $2n$-dimensional linear subspace $L_{2n}:=\operatorname{lin}\{g_1,g_2,\dots,g_{2n}\}$, and a number $\gamma>\lambda_n$, such that $L_{2n}\cap\gamma S(L^p)\subseteq\widetilde{B}_p$, where $S(L^p)$ is the unit ball of L^p, that is,

$$\min_{c+G*h\in L_{2n}}\frac{\|c+(G*h)\|_p}{\|h\|_p}=\min_{f\in L_{2n}}\frac{\|f\|_p}{\|h\|_p}\ge\gamma>\lambda_n. \tag{3.18}$$

For every $f\in L_{2n}$, $f=\sum_{j=1}^{2n}\xi_jg_j$, define a mapping $f\to\xi=(\xi_1,\xi_2,\dots,\xi_{2n})\in\mathbb{R}^{2n}$. Using the similar method as that in [9, pages 214–216], we get $\|h\|_p=\left(\sum_{j=1}^{2n}c_j|\xi_j|^p\right)^{1/p}$, where $c_j=\int_{(j-1)\pi/n}^{j\pi/n}|h(x)|^p dx$, $j=1,\dots,2n$, and $c_j=\int_0^{\pi/n}|h(x)|^p dx=c_1$, $j=1,\dots,2n$, if $h\in D_n$. By (3.18), we have

$$\min_{\xi\in\mathbb{R}^{2n}\setminus\{0\}}\frac{\|\sum_{j=1}^{2n}\xi_jg_j\|_p}{\left(\sum_{j=1}^{2n}c_j|\xi_j|^p\right)^{1/p}}>\lambda_n. \tag{3.19}$$

Let

$$S^{2n-1}:=\left\{\xi:\xi=(\xi_1,\dots,\xi_{2n})\in\mathbb{R}^{2n},\sum_{i=1}^{2n}\xi_i=0,\sum_{i=1}^{2n}|\xi_i|=2\pi\right\}. \tag{3.20}$$

For every vector $\xi\in S^{2n-1}$, we take

$$h_0^\xi(t)=\begin{cases}(2\pi)^{-1/p}\operatorname{sign}\xi_k, & \text{for } t\in(t_{k-1},t_k),\ k=1,\dots,2n,\\ 0, & \text{for } t=t_k,\ k=1,\dots,2n-1,\end{cases} \tag{3.21}$$

where $t_0=0$, $t_k=\sum_{i=1}^k|\xi_i|$, $k=1,\dots,2n$, and let

$$f_0^\xi(x)=(G*h_0^\xi)(x)+\beta_0,\quad 1<p<\infty, \tag{3.22}$$

where β_0 satisfies the condition

$$\inf_{c\in\mathbb{R}}\|(G*h_0)+c\|_p=\|(G*h_0)+\beta_0\|_p. \tag{3.23}$$

Next, for $p=q$, we consider the iterative procedure (3.3)-(3.4) beginning with h_0^ξ and f_0^ξ instead of h_0 and f_0, respectively. The analogues of Lemmas 3.1 and 3.2 hold. Moreover, for the limit element f^ξ, there exists $\widehat{\xi}\in S^{2n-1}$ such that $f^{\widehat{\xi}}$ has at least $2n$ simple zeros in $[0,2\pi)$ (i.e., $S_c(f^{\widehat{\xi}})\ge 2n$). Indeed, let $O_k^{2n-1}=\{\xi:\xi\in S^{2n-1},Z_c^\star(f_k^\xi)\le 2n-2\}$, where the function f_k^ξ defined by (3.3). Clearly, the set O_k^{2n-1} is open in S^{2n-1}. Let $H_k^{2n-1}=S^{2n-1}\setminus O_k^{2n-1}$. Then, H_k^{2n-1} is a nonempty closed set, and that $H_{k+1}^{2n-1}\subset H_k^{2n-1}$, $k\in\mathbb{N}$. First, we prove that H_k^{2n-1} is nonempty. For fixed $0<x_1<x_2<\cdots<x_{2n-1}<2\pi$, let $\eta(\xi)=(\eta_1(\xi),\quad\eta_2(\xi),\dots,\eta_{2n}(\xi))$, where

$$\eta_i(\xi)=\begin{cases}\int_{\mathbb{T}}h_0^\xi(t)dt, & \text{for } i=1,\\ f_k^\xi(x_{i-1}), & \text{for } i=2,\dots,2n.\end{cases} \tag{3.24}$$

It is easily seen that $\eta(\xi)$ is a continuous and odd mapping. By Borsuk's theorem [10], there exists a $\overline{\xi} \in S^{2n-1}$ such that $\eta(\overline{\xi}) = 0$. Then, $Z_c^\star(f_k^{\overline{\xi}}) = 2n-1$, that is, $\overline{\xi} \in H_k^{2n-1}$. Thus, H_k^{2n-1} is a nonempty. Next, we prove $H_{k+1}^{2n-1} \subset H_k^{2n-1}$, $k \in \mathbb{N}$. Assume, on the contrary, there exists a $\widetilde{\xi} \in H_{k+1}^{2n-1}$, but $\widetilde{\xi} \notin H_k^{2n-1}$. Thus, $S_c(f_k^{\widetilde{\xi}}) \leq Z_c^\star(f_k^{\widetilde{\xi}}) \leq 2n-2$ results in $S_c(Q_q f_k^{\widetilde{\xi}}) \leq 2n-2$. By (3.4), we get

$$S_c\left(Q_p h_{k+1}^{\widetilde{\xi}}\right) \leq 2n-2, \qquad S_c\left(h_k^{\widetilde{\xi}}\right) \leq 2n-2. \tag{3.25}$$

According to (3.3), we have $Z_c^\star(f_{k+1}^{\widetilde{\xi}}) \leq 2n-2$, namely, $\widetilde{\xi} \notin H_{k+1}^{2n-1}$. A contradiction follows from the above. We have constructed a system of nonempty closed nested sets. Their intersection is nonempty. Let $\widehat{\xi} \in \bigcap_{k=1}^{\infty}(H_k^{2n-1})$. According to Lemma 3.2, there exists $(f^{\widehat{\xi}}(x), \lambda^p) \in \Gamma(p,p,G)$ such that $\lim_{k\to\infty} f_k^{\widehat{\xi}}(x) = f^{\widehat{\xi}}(x)$, $x \in [0,2\pi)$. Thus, $Z_c^\star(f^{\widehat{\xi}}) \geq 2n-1$. In view of Lemma 2.3, zeros of $f^{\widehat{\xi}}(x)$ are simple. Therefore, $S_c(f^{\widehat{\xi}}) \geq 2n-1$. But since the function $f^{\widehat{\xi}}(x)$ is periodic, we actually have $S_c(f^{\widehat{\xi}}) \geq 2n$. We write $S_c(f^{\widehat{\xi}}) = 2N$.

For the spectral function $f^{\widehat{\xi}}$ corresponding to spectral value $\lambda(\widehat{\xi})$, by Lemma 2.7, and the nonincreasing property of Kolmogorov n-widths in n, and $d_{2n}(\widetilde{B}_p; L^p) = \lambda_n(p,p,G)$ [7], we have

$$\lambda(\widehat{\xi}) \leq \lambda_N = d_{2N}\left(\widetilde{B}_p; L^p\right) \leq d_{2n}\left(\widetilde{B}_p; L^p\right) = \lambda_n. \tag{3.26}$$

Therefore, by Lemmas 3.1, 3.2, and (3.26), we have

$$\min_{\xi\in\mathbb{R}^{2n}\setminus\{0\}} \frac{\|\sum_{j=1}^{2n}\xi_j g_j\|_p}{\left(\sum_{j=1}^{2n} c_j|\xi_j|^p\right)^{1/p}} \leq \frac{\|\sum_{j=1}^{2n}\widehat{\xi}_j g_j\|_p}{(c_1)^{1/p}\left(\sum_{j=1}^{2n}|\widehat{\xi}_j|^p\right)^{1/p}} = \|f^{\widehat{\xi}}\|_p = \lambda(\widehat{\xi}) \leq \lambda_n, \tag{3.27}$$

which is contradicted with (3.19).

For a general B-kernel G, set $G_\sigma = \phi_\sigma * G$, and $h_\sigma = \phi_\sigma * h$, $\lambda_{n,\sigma} = \phi_\sigma * \lambda_n$. For $f = c + G*h \in \widetilde{B}_p$, we set $f_\sigma = c + G_\sigma * h$. From the results obtained in the pervious case, we have

$$\frac{\|G_\sigma * h + c\|_p}{\|h_\sigma\|_p} = \frac{\|f_\sigma\|_p}{\|h_\sigma\|_p} \leq \lambda_{n,\sigma}. \tag{3.28}$$

According to Lemma 3.3, we get $\|G*h + c\|_p / \|h\|_p \leq \lambda_n(p,p,G)$. Therefore, we obtain $b_{2n-1}(\widetilde{B}_p; L^p) \leq \lambda_n(p,p,G)$. The proof of Lemma 3.5 is complete. □

Proof of theorem

Now, we consider the proof of Theorem 1.3.

Proof. By Lemma 3.5, if G is B-kernel, for each $p \in (1,\infty)$ and $n = 1,2,\ldots$, we have $b_{2n-1}(\widetilde{B}_p; L^p) \leq \lambda_n(p,p,G)$. On the other hand, by [5], for each $1 < p \leq q < \infty$ and $n = 1,2,\ldots$, then $b_{2n-1}(\widetilde{B}_p; L^q) \geq \lambda_n(p,q,G)$. Thus, we have $b_{2n-1}(\widetilde{B}_p; L^p) = \lambda_n(p,p,G)$ for $p \in (1,\infty)$ and $n \in \mathbb{N}^+$. The result (1.8) is obvious since $s_{2n}(\widetilde{B}_p; L^p) = \lambda_n(p,p,G)$ [5]. Theorem 1.3 is proved completely. □

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