Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 609047, 14 pages
doi:10.1155/2008/609047

Research Article

Boundary Asymptotic and Uniqueness of Solution for a Problem with $p(x)$-Laplacian

Ionel Rovenţa

Department of Mathematics, University of Craiova, 200585 Craiova, Romania

Correspondence should be addressed to Ionel Rovenţa, roventaionel@yahoo.com

Received 14 April 2008; Revised 10 June 2008; Accepted 30 August 2008

Recommended by Alexander Domoshnitsky

The goal of this paper is to prove the boundary asymptotic behavior of solutions for weighted $p(x)$-Laplacian equations that take infinite value on a bounded domain.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, and let f be a nonnegative, nondecreasing, and C^1 function on $\mathbb{R}^+$.

We study here the asymptotic boundary behaviour of solutions for the problem

$$\Delta_{p(x)}u = g(x)f(u), \quad x \in \Omega, \quad u(x) \longrightarrow \infty, \text{ as dist } (x, \partial\Omega) \longrightarrow 0, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with C^2 boundary. In additional conditions, the uniqueness of solutions is also discussed. For further details, see [1, 2].

Here, $\Delta_{p(x)}u := \text{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x))$ is the $p(x)$-Laplacian, a function defined on $\mathbb{R}^n$ with $1 < p(x) < \infty$.

First, Bieberbach in [3], considered the problem

$$\Delta u = g(x)f(u), \quad x \in \Omega, \quad u(x) \longrightarrow \infty, \text{ as dist } (x, \partial\Omega) \longrightarrow 0. \tag{1.2}$$

He shows that (1.2) admits a unique solution if Ω is a smooth planar domain, f is the exponential function, and g is constant 1. Rademacher [4] extended the results to the three-dimensional domains; Keller [5] and Osserman [6] gave a necessary and sufficient conditions to solve the problem (1.2) in the n-dimensional case if the domain satisfies inner and outer sphere conditions.

Bandle and Marcus [7] found the boundary asymptotic of solutions for (1.2), when g is continuous and positive, and f satisfies $f(mt) \le m^{1+\alpha} f(t)$ for all $0 < m < 1$ and all $t \ge t_0/m$. Cîrstea and Rădulescu [8–10] prove the uniqueness and asymptotic behavior of solutions for problem (1.2), when f is regularly varying, and $g \in C^{0,\alpha}(\overline{\Omega})$ is a nonnegative function which is allowed to vanish on the boundary.

In the study of p-Laplacian equations, the main difficulty arises from the lack of compactness, but here the $p(x)$-Laplacian possesses more complicated inhomogeneous nonlinearities, so we need some special techniques. Problems of this type arise in many areas of applied physics including nuclear physics, field theory, solid waves, and problems of false vacuum. Zhang in [11] gives the existence and singularity of blowup solutions for the problem

$$-\Delta_{p(x)} u + h(x,u) = 0, \quad x \in \Omega, \quad u(x) \longrightarrow \infty, \text{ as dist } (x, \partial\Omega) \longrightarrow 0, \tag{1.3}$$

where $\Omega = B(0,R) \subset \mathbb{R}^N$, $p(x)$ and $f(x,u)$ satisfy the following.

(H1) $p(x) \in C^1(\overline{\Omega})$ is a radial symmetric function and satisfies

$$1 < p^- \le p^+ < N, \quad \text{where } p^- = \inf_{\Omega} p(x), \quad p^+ = \sup_{\Omega} p(x). \tag{1.4}$$

(H2) $h(x,u)$ is radial with respect to x, $h(x,\cdot)$ is increasing, and $h(x,0) = 0$ for any $x \in \Omega$.

(H3) $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies

$$\begin{gathered} |f(x,t)| \le C_1 + C_2 |t|^{\alpha(x)-1}, \quad \text{for every } (x,t) \in \Omega \times \mathbb{R},\ C_1 \ge 0,\ C_2 \ge 0,\ \alpha \in C(\overline{\Omega}), \\ 1 \le \alpha(x) < p^*(x) := \frac{Np(x)}{N - p(x)}. \end{gathered} \tag{1.5}$$

2. Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. In that context, we refer to the book of Musielak [12] and the papers of Kováčik and Rákosník [13] and Fan et al. [14–16].

Set

$$L_+^\infty(\Omega) = \left\{ h;\ h \in L^\infty(\Omega),\ \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 1\ \forall x \in \overline{\Omega} \right\}. \tag{2.1}$$

For any $h \in L_+^\infty(\Omega)$, we define

$$h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x), \qquad h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x). \tag{2.2}$$

For any $p(x) \in L_+^\infty(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}. \tag{2.3}$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}. \tag{2.4}$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [13, Theorem 2.5], the Hölder inequality holds [13, Theorem 2.1], they are reflexive if and only if $1 < p^- \le p^+ < \infty$ [13, Corollary 2.7], and continuous functions are dense if $p^+ < \infty$ [13, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [13, Theorem 2.8]. If $0 < |\Omega| < \infty$ and $r_1,\ r_2$ are variable exponents so that $r_1(x) \le r_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{2.5}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx. \tag{2.6}$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations hold true:

$$\begin{gathered} |u|_{p(x)} > 1 \Longrightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \\ |u|_{p(x)} < 1 \Longrightarrow |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \\ |u_n - u|_{p(x)} \longrightarrow 0 \Longleftrightarrow \rho_{p(x)}(u_n - u) \longrightarrow 0. \end{gathered} \tag{2.7}$$

Next, we define the variable Sobolev space:

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\}. \tag{2.8}$$

On $W^{1,p(x)}(\Omega)$, we may consider one of the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \tag{2.9}$$

or

$$|||u||| = \inf\left\{\mu > 0;\ \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \le 1\right\}. \tag{2.10}$$

We also define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Assuming $p^- > 1$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

Set

$$I_{p(x)}(u) = \int_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx. \tag{2.11}$$

For all $u \in W_0^{1,p(x)}(\Omega)$, the following relations hold true:

$$\begin{aligned} \|u\| > 1 &\Longrightarrow \|u\|^{p^-} \le I_{p(x)}(u) \le \|u\|^{p^+}, \\ \|u\| < 1 &\Longrightarrow \|u\|^{p^+} \le I_{p(x)}(u) \le \|u\|^{p^-}. \end{aligned} \tag{2.12}$$

Finally, we remember some embedding results regarding variable exponent Lebesgue-Sobolev spaces. For the continuous embedding between variable exponent Lebesgue-Sobolev spaces we refer to [15, Theorem 1.1]. If $p : \Omega \to \mathbb{R}$ is Lipschitz continuous and $p^+ < N$, then for any $q \in L_+^\infty(\Omega)$ with $p(x) \le q(x) \le Np(x)/(N - p(x))$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. In what concerns the compact embedding, we refer to [15, Theorem 1.3]. If Ω is a bounded domain in $\mathbb{R}^n$, $p(x) \in C(\overline{\Omega})$, $p^+ > N$, then for any $q(x) \in L_+^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\overline{\Omega}}((Np(x)/(N-p(x))) - q(x)) > 0$ there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Our aim is to find the asymptotic boundary behaviour to the problem

$$\Delta_{p(x)} u = g(x) f(u), \quad x \in \Omega, \quad u(x) \longrightarrow \infty, \text{ as dist } (x, \partial\Omega) \longrightarrow 0. \tag{2.13}$$

Suppose $p \in L_+^\infty$ and $\lim_{\text{dist}(x,\partial\Omega)\to 0} p(x) = p^\#$. Consider $g \in C(\Omega)$ nonnegative, and f satisfies

$$f \in C^1([0,\infty)), \quad f(0) = 0, \; f(t) > 0 \text{ , and } f \text{ is nondecreasing on } (0,\infty), \tag{2.14}$$

$$\int_1^\infty \frac{1}{(F(t))^{1/p^\#}} dt < \infty, \quad \text{where } F(t) := \int_0^t f(s)ds. \tag{2.15}$$

We will refer to condition (2.15) as the generalized Keller-Osserman condition. Allow g to be unbounded on Ω or to vanish on $\partial\Omega$.

3. Boundary asymptotic and uniqueness

Let f satisfy the Keller-Osserman condition. Consider ψ as a function defined on $(0,\infty)$ such that

$$\psi(t) = \int_t^\infty \frac{1}{(q^\# F(s))^{1/p^\#}} ds, \tag{3.1}$$

where $q^\#$ is the Holder conjugate of $p^\#$, that is, $1/q^\# + 1/p^\# = 1$. The function ψ is decreasing so ψ has an inverse $\phi : (0, \psi(0+)) \to (0,\infty)$. Moreover, ϕ is a decreasing function with $\lim_{t\to 0_+} \phi(t) = \infty$, and some computations give us

$$\phi'(t) = (\psi^{-1}(t))' = -(q^\# F(\phi(t)))^{1/p^\#}, \qquad |\phi'(t)|^{p^\#-2}\phi''(t) = \frac{q^\#}{p^\#} f(\phi(t)). \tag{3.2}$$

Now we will see that

$$-\frac{\phi'(t)}{\phi''(t)} = \frac{p^{\#}}{q^{\#}} \frac{\left(q^{\#} F(\phi(t))\right)^{1/q^{\#}}}{f(\phi(t))}. \tag{3.3}$$

Definition 3.1. A positive measurable function defined on $[a, \infty)$, for some $a > 0$ is said to be regularly varying (at infinity) of index $q \in \mathbb{R}$, written $f \in \mathrm{RV}_q$, provided that

$$\lim_{z\to\infty} \frac{f(tz)}{f(z)} = t^q \quad \forall t > 0. \tag{3.4}$$

Remark 3.2. Let $f \in \mathrm{RV}_{\sigma+1}$ $(\sigma + 2 > p)$ be a nondecreasing and continuous function. Then, $F \in \mathrm{RV}_{\sigma+2}$ and therefore $F^{-1/p} \in \mathrm{RV}_{(-\sigma-2)/p}$. Since $(-\sigma - 2)/p < -1$, the observation made in [1] allows us to conclude that $F^{-1/p} \in L^1([1, \infty))$ and hence f satisfies the generalized Keller-Osserman condition.

Remark 3.3. Let $f \in \mathrm{RV}_{\sigma+1}$. By the change of variable $s = tz$, we infer that

$$F(z) = \int_0^z f(s)ds = \int_0^1 zf(tz)dt. \tag{3.5}$$

Because $f \in \mathrm{RV}_{\sigma+1}$ is continuous, there exists an $\epsilon > 0$ such that for every $z > \epsilon$ we have $(f(tz)/f(z)) \le t^{\sigma+1} + 1$. Hence, using Lebesgue's dominated convergence theorem, the limit shifts with the integral and so

$$\lim_{z\to\infty} \frac{F(z)}{zf(z)} = \int_0^1 \lim_{z\to\infty} \frac{f(tz)}{f(z)} dt = \int_0^1 t^{\sigma+1} dt = \frac{1}{\sigma+2}. \tag{3.6}$$

Lemma 3.4. *Consider f satisfies the Keller-Osserman condition, then one has*

$$\lim_{s\to\infty} \frac{(F(s))^{1/q^{\#}}}{f(s)} = 0. \tag{3.7}$$

Proof. We sketch the proof only in the context of regularly varying functions. For further details regarding general case, see [17]. Let $f \in \mathrm{RV}_{\sigma+1}$ $(\sigma > p^{\#} - 2)$. From $f(z)/F(z) = (1/z)(zf(z)/F(z))$ and using $\lim_{z\to\infty}(F(z)/zf(z)) = 1/(\sigma + 2)$, then there exists a constant $C > 0$ such that $\lim_{z\to\infty}(F(z)/z^{\sigma+2}) = C$. Finally, $\lim_{z\to\infty}(f(z)/z^{\sigma+1}) = \lim_{z\to\infty}(F(z)/z^{\sigma+2})(zf(z)/F(z)) = C(\sigma + 2)$. It remains to consider the limit

$$\lim_{z\to\infty} \frac{F(z)^{1/q^{\#}}}{f(z)} = \lim_{z\to\infty} \frac{\left(F(z)/z^{\sigma+2}\right)^{1/q^{\#}}}{f(z)/z^{\sigma+1}} z^{-(1/p^{\#})(\sigma+2)+1} = 0, \tag{3.8}$$

because $-(1/p^{\#})(\sigma + 2) + 1 < 0$.

Another way to prove the lemma is to consider the derivative

$$\left((F(s))^{1/p^{\#}}\right)' = \frac{1}{p^{\#}} f(z)(F(z))^{-1/q^{\#}} > 0. \tag{3.9}$$

If $\left((F(s))^{1/p^{\#}}\right)'$ is bounded then there exists $M > 0$ such that $0 < \left((F(s))^{1/p^{\#}}\right)' < M$, $\forall t > 1$. By integrating from 1 to t, it follows that $0 < (F(s))^{1/p^{\#}} < Mt$ and also

$1/Mt < 1/(F(s))^{1/p^\#} < \infty,\ \forall t > 1$. If we integrate again from 1 to ∞, we obtain a contradiction with the Keller-Osserman condition.

In conclusion, the derivative of $(F(s))^{1/q^\#}$ is unbounded.

Using L'Hospitals rule, we should obtain much more

$$\lim_{z\to\infty} \frac{F(z)^{1/q^\#}}{z} = \infty. \tag{3.10}$$

□

The following lemma will be useful later; see [1].

Lemma 3.5. *Let $q^\# \in \mathbb{R}$ be the Holder conjugate of $p^\# > 1$. If $f \in \mathrm{RV}_{\sigma+1}$ $(\sigma > p^\# - 2)$, then $\lim_{z\to\infty}((F(z))^{1/q^\#}/f(z)\int_z^\infty (F(s))^{-1/p^\#}ds) = ((\sigma+2-p^\#)/p^\#(2+\sigma))$.*

Proof. By applying L'Hospitals rule, we obtain

$$\lim_{z\to\infty} \frac{z(F(z))^{-1/p^\#}}{\int_z^\infty (F(s))^{-1/p^\#}ds} = \lim_{z\to\infty}\left(\frac{1}{p^\#}\frac{zf(z)}{F(z)} - 1\right) = \frac{\sigma+2-p^\#}{p^\#}. \tag{3.11}$$

In conclusion,

$$\lim_{z\to\infty} \frac{(F(z))^{1/q^\#}}{f(z)\int_z^\infty (F(s))^{-1/p^\#}ds} = \lim_{z\to\infty} \frac{z(F(z))^{-1/p^\#}}{\int_z^\infty (F(s))^{-1/p^\#}ds}\frac{F(z)}{zf(z)} = \frac{\sigma+2-p^\#}{p^\#(2+\sigma)}. \tag{3.12}$$

□

Corollary 3.6. *Let $f \in \mathrm{RV}_{\sigma+1}$ $(\sigma > p^\#)$ be a continuous function. Then,*

$$\lim_{t\to 0} \frac{|\phi'(t)|^{p^\#-2}\phi'(t)}{tf(\phi(t))} = -\frac{q^\#}{p^\#}\frac{\sigma+p^\#-2}{\sigma+2}. \tag{3.13}$$

Proof. See [1, Corollary 3.2]. □

We say that $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ is a continuous local weak solution to the equation $\Delta_{p(x)}u = g(x)f(u)$ on the domain Ω if and only if

$$\int_D |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx = -\int_D g(x)f(u(x))\varphi dx, \quad \varphi \in W_0^{1,p(x)}(D), \tag{3.14}$$

for every subdomain $D \Subset \Omega$.

Define $L : W^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ as

$$\langle Lu, \varphi\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx + \int_\Omega g(x)f(u(x))\varphi dx, \tag{3.15}$$

for every $u \in W^{1,p(x)}(\Omega),\ v \in W_0^{1,p(x)}(\Omega)$.

Lemma 3.7 (Comparison principle). *Let $u, v \in W^{1,p(x)}(\Omega)$ satisfy $Lu - Lv \geq 0$ in $(W_0^{1,p(x)}(\Omega))^*$ and consider $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ (i.e., $u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in Ω.*

Proof. See [18]. □

Remark 3.8. L defined below satisfy the condition from [18]. Here, we use $Lu \geq Lv$ in the form

$$\int_{\Omega} |\nabla v|^{p(x)-2}\nabla v \nabla \varphi dx + \int_{\Omega} g(x) f(v(x))\varphi dx \leq \int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla \varphi dx + \int_{\Omega} g(x) f(u(x))\varphi dx. \tag{3.16}$$

Given a real number b, let Λ_b be the class of all positive monotonic functions $k \in L^1(0,\nu) \cap C^1(0,\nu)$ that satisfy

$$\lim_{t\to 0}\left(\int_0^t \frac{k(s)}{k(t)} ds\right)' = b. \tag{3.17}$$

We will let Λ stand for the union of all Λ_b as b ranges over $[0,\infty)$. For $k \in \Lambda$, we use the notation $\lambda(t) = \int_0^t k(s)ds$.

Remark 3.9. Observe that $\lim_{t\to 0^+} \int_0^t (k(s)/k(t))ds = 0$ for any $k \in \Lambda$. Moreover, in (3.17), $0 \leq b \leq 1$ if k is nondecreasing and $b \geq 1$ if k is nonincreasing. This is true because $\int_0^t (k(s)/k(t))ds = (1/k(t)) \int_0^t k(s)ds = t(k(\xi)/k(t))$, for an $\xi \in (0,t)$.

Let z be a C^2 function on a domain Ω in $\mathbb{R}^n$, f satisfies the Keller-Osserman condition and $v = \phi(z)$.

A direct computation show that

$$\begin{aligned}\Delta_{p(x)} v = (p(x)-1)|\phi'(z)|^{p(x)-2}\phi''(z)|\nabla z|^{p(x)} + |\phi'(z)|^{p(x)-2}\phi'(z)\Delta_{p(x)} z \\ + |\phi'(z)|^{p(x)-1}\ln(|\phi'(z)|)|\nabla z|^{p(x)-2}\nabla p(x)\nabla z(x).\end{aligned} \tag{3.18}$$

Theorem 3.10. *Suppose $f \in \mathrm{RV}_{\sigma+1}$, $(\sigma \geq p^{\#} - 2)$ satisfies (2.14). Let $\Omega \subsetneq \mathbb{R}^n$ be a bounded C^2 domain and let $g \in C(\Omega)$ be a nonnegative function such that*

$$\lim_{d(x)\to 0} \frac{g(x)}{[k(d(x))]^{p(x)}} = A \tag{3.19}$$

for some positive constant A and some $k \in \Lambda_b$. Then, any local weak solution u of (2.13) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi(\lambda(d(x)))} = \left[\frac{p^{\#} + b(2+\sigma-p^{\#})}{A(2+\sigma)}\right]^{1/(2+\sigma-p^{\#})}. \tag{3.20}$$

Proof. Consider $\rho > 0$ and

$$\Omega_\rho := \{x \in \Omega : 0 < d(x) < \rho\}. \tag{3.21}$$

We know that Ω is a C^2 bounded domain, so there exists positive constant μ, depending only on Ω, such that

$$d \in C^2(\overline{\Omega_\mu}), \qquad |\nabla d| \equiv 1 \quad \text{on } \Omega_\mu, \tag{3.22}$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$. The existence of such a constant is given by the smoothness of the domain Ω. For $\rho \in \Gamma := (0, \mu/2)$, let

$$\Omega_\rho^- := \Omega_\mu \setminus \overline{\Omega_\rho}, \qquad \Omega_\rho^+ := \Omega_{\mu-\rho}. \tag{3.23}$$

The proof concerns two cases which discuss the monotonicity of the function k.

In the first case, we take $k \in \Lambda_b$ for some b and k nonincreasing on $(0, \nu)$ for some $\nu > 0$. Without loss of generality, we can consider $\nu > \mu$.

Let

$$z^{\pm}(x) := \lambda(d(x)) \pm \lambda(\rho), \quad \text{for } x \in \Omega_{\rho}^{\pm}. \tag{3.24}$$

Some computations give us

$$\begin{aligned} |\nabla z^{\pm}|^{p(x)} &= k^{p(x)}(d)|\nabla d|^{p(x)}, \\ \Delta_{p(x)} z^{\pm} &= (p(x)-1)k^{p(x)-2}(d)k'(d)|\nabla d|^{p(x)} + k^{p(x)-1}(d)\Delta_{p(x)} d \\ &\quad + k^{p(x)-1}(d)\ln(k(d))|\nabla d|^{p(x)-2}\nabla p(x)\nabla d(x). \end{aligned} \tag{3.25}$$

Using these computations in (3.18) with $v^{\pm} = \phi(z^{\pm})$, we find that

$$\begin{aligned} \Delta_{p(x)} v^{\pm} &= (p(x)-1)|\phi'(z^{\pm})|^{p(x)-2}\phi''(z^{\pm})|\nabla z^{\pm}|^{p(x)} + |\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})\Delta_{p(x)} z^{\pm} \\ &\quad + |\phi'(z^{\pm})|^{p(x)-1}\ln(|\phi'(z^{\pm})|)|\nabla z^{\pm}|^{p(x)-2}\nabla p(x)\nabla z^{\pm}(x) \\ &= (p(x)-1)k^{p(x)}(d)|\phi'(z^{\pm})|^{p(x)-2}\phi''(z^{\pm})|\nabla d|^{p(x)} \\ &\quad + (p(x)-1)k^{p(x)-2}(d)k'(d)|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})|\nabla d|^{p(x)} \\ &\quad + k^{p(x)-1}(d)|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})\Delta_{p(x)} d \\ &\quad + k^{p(x)-1}(d)\ln(k(d))|\nabla d|^{p(x)-2}\nabla p(x)\nabla d(x)|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm}) \\ &\quad + k^{p(x)-1}(d)|\nabla d|^{p(x)-2}\nabla p(x)\nabla d(x)|\phi'(z^{\pm})|^{p(x)-1}\ln(|\phi'(z^{\pm})|). \end{aligned} \tag{3.26}$$

Given $0 < \epsilon < A/2$, we define two numbers $\vartheta^{\pm}$ by

$$\vartheta^{\pm} := \left[\frac{p^{\#} + b(2+\sigma-p^{\#})}{(A \pm 2\epsilon)(2+\sigma)}\right]^{1/(2+\sigma-p^{\#})}, \tag{3.27}$$

where A is the limit in (3.19).

Let $w^{\pm}(x) = \vartheta^{\pm}\phi(z^{\pm}(x))$, $x \in \Omega_{\rho}^{\pm}$. For simplicity, let $Lz := -\Delta_{p(x)} z + g(x)f(z)$.

In order to apply the *comparison principle*, we have to prove that for $\rho \in \Gamma$ and $\mu \geq 0$ sufficiently small, we have

$$Lw^{-} \geq 0 \quad \text{on } \Omega_{\rho}^{-}, \qquad Lw^{+} \leq 0 \quad \text{on } \Omega_{\rho}^{+}. \tag{3.28}$$

From (3.26), recalling that $p(x)-1 = p(x)/q(x)$ and $|\nabla d| = 1$ on Ω_{μ}, we find

$$\begin{aligned} Lw^{\pm} &= g(x)f(w^{\pm}) - \Delta_{p(x)}(w^{\pm}) \\ Lw^{\pm} &= (\vartheta^{\pm})^{p(x)-1}k^{p(x)}(d)f(\phi(z^{\pm})) \\ &\quad \times \left(\frac{g(x)f(\vartheta^{\pm}\phi(z^{\pm})}{(\vartheta^{\pm})^{p(x)-1}k^{p(x)}(d)f(\phi(z^{\pm}))} + D_1^{\pm}(x) + D_2^{\pm}(x) + D_3^{\pm}(x) + D_4^{\pm}(x)\right), \end{aligned} \tag{3.29}$$

where

$$D_1^{\pm}(x) = -\frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})}{k(d)f(\phi(z^{\pm}))}\Delta_{p(x)}d,$$

$$D_2^{\pm}(x) = -\frac{p(x)}{q(x)}\frac{k'(d)}{k^2(d)}\frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})|\nabla d|^{p(x)}}{f(\phi(z^{\pm}))},$$

$$D_3^{\pm}(x) = -\frac{p(x)}{q(x)}\frac{|\phi'(z^{\pm})|^{p(x)-2}\phi''(z^{\pm})}{f(\phi(z^{\pm}))},$$

$$D_4^{\pm}(x) = -\frac{|\phi'(z^{\pm})|^{p(x)-2}\left(\phi'(z^{\pm})\ln\left(k(d)\vartheta^{\pm}\right) + \left|\phi'(z^{\pm})\right|\ln\left(\left|\phi'(z^{\pm})\right|\right)\right)}{k(d)f\left(\phi(z^{\pm})\right)}|\nabla d|^{p(x)-2}\nabla p(x)\nabla d(x). \tag{3.30}$$

Recalling that $\phi' < 0$ and $z^{\pm}(x) \le 2\lambda(d(x))$, we infer

$$\begin{aligned}|D_1^{\pm}(x)| &= -\frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})}{k(d)f(\phi(z^{\pm}))}|\Delta_{p(x)}d| \\ &\le -2\frac{\lambda(d(x))}{k(d(x))}\frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})}{z^{\pm}(x))f(\phi(z^{\pm}))}|\Delta_{p(x)}d|.\end{aligned} \tag{3.31}$$

Since $d \in C^2(\overline{\Omega_{\mu}})$, $\lim_{\operatorname{dist}(x,\partial\Omega)\to 0}p(x) = p^{\#}$, by Lemma 3.5 and Remark 3.9 from below, we obtain

$$\lim_{\Omega_{\rho}^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} D_1^{\pm}(x) = 0. \tag{3.32}$$

Using $\lim_{\operatorname{dist}(x,\partial\Omega)\to 0}p(x) = p^{\#}$ and Corollary 3.2, we have

$$\lim_{\Omega_{\rho}^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} \frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})}{z^{\pm}(x))f(\phi(z^{\pm}))} = -\frac{q^{\#}}{p^{\#}}\frac{\sigma+2-p^{\#}}{\sigma+2}. \tag{3.33}$$

With the same argument, we obtain

$$|D_4^{\pm}(x)| \le 2\frac{\lambda(d(x))}{k(d(x))}\ln\frac{k(d)\vartheta^{\pm}}{\phi'(z^{\pm})}\frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})}{z^{\pm}(x))f(\phi(z^{\pm}))}. \tag{3.34}$$

Next, we prove

$$\lim_{\Omega_{\rho}^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} \frac{\lambda(d(x))}{k(d(x))}\ln\frac{k(d)\vartheta^{\pm}}{|\phi'(z^{\pm})|} = 0. \tag{3.35}$$

Recall that

$$\phi'(z) = -\left(q^{\#}F(\phi(z))\right)^{1/p^{\#}} = -\left(q^{\#}\right)^{1/p^{\#}}A(z)\phi(z)^{(\sigma+2)/p^{\#}}, \tag{3.36}$$

where $A(z) = (F(\phi(z))/\phi(z)^{\sigma+2})^{1/p^{\#}}$.

If we integrate from 0 to t, we obtain

$$\frac{\phi(t)^{-\alpha+1}}{-\alpha+1} = -\left(q^{\#}\right)^{1/p^{\#}} \int_0^t A(s)ds, \tag{3.37}$$

where $\alpha = (\sigma+2)/p^{\#} > 1$. Here, when $x \to 0$, we have $d(x) \to \rho$ and $z^{\pm}(x) \to 0$.

Dividing by t and taking to the limit, we have

$$\lim_{t\to 0} \frac{\phi(t)^{-\alpha+1}}{t} = C, \tag{3.38}$$

where C is a suitable positive constant.

From (3.36), it follows that $\lim_{t\to 0}(\phi(t)/t^{\beta}) = -C_1$, where $\beta = (1/(-\alpha+1))((\sigma+2)/p^{\#}) < 0$, and C_1 is a positive suitable constant.

Now, it will be easy to infer that

$$\begin{aligned} \lim_{\Omega_\rho^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} \frac{\lambda(d(x))}{k(d(x))} \ln \frac{k(d)\vartheta^{\pm}}{|\phi'(z^{\pm})|} &= \lim_{\Omega_\rho^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} \frac{\lambda(d(x))}{k(d(x))} \ln \frac{k(d)\vartheta^{\pm}}{|\lambda^{\beta}(z^{\pm})|} \\ &= \lim_{t\to 0} \frac{\ln(k(t)/\lambda^{\beta}(t))}{k(t)/\lambda(t)} = 0. \end{aligned} \tag{3.39}$$

Finally, $\lim_{\Omega_\rho^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} D_4^{\pm}(x) = 0$.

Since k is nonincreasing, we have $k'(d)z^- \geq k'(d)\lambda(d)$ and $k'(d)z^+ \leq k'(d)\lambda(d)$. Therefore,

$$\begin{aligned} D_2^- + D_3^- &= -\frac{p(x)}{q(x)} \frac{k'(d)}{k^2(d)} \frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})}{f(\phi(z^{\pm}))} - \frac{p(x)}{q(x)} \frac{|\phi'(z^-)|^{p(x)-2}\phi''(z^-)}{f(\phi(z^-))} \\ &\geq -\frac{p(x)}{q(x)} \frac{k'(d)\lambda(d)}{k^2(d)} \frac{|\phi'(z^{\pm})|^{p(x)-2}\phi'(z^{\pm})}{z^- f(\phi(z^{\pm}))} - \frac{p(x)}{q(x)} \frac{|\phi'(z^-)|^{p(x)-2}\phi''(z^-)}{f(\phi(z^-))} = \widetilde{D_2^-} + D_3^-, \end{aligned} \tag{3.40}$$

and similarly $D_2^+ + D_3^+ \leq \widetilde{D_2^+} + D_3^+$.

From (3.19), we see that corresponding to ϵ, there is $0 < \delta_\epsilon < A/2$ such that

$$(A-\epsilon)k^{p(x)}(d(x)) \leq g(x) \leq (A+\epsilon)k^{p}(d(x)), \quad 0 < d(x) < \delta_\epsilon. \tag{3.41}$$

Suppose that the constant μ in (3.22) is chosen such that $\mu < \delta_\epsilon$. Using (3.40) in the left side of the inequality (3.29), we get

$$Lw^- \geq (\vartheta^-)^{p(x)-1} k^{p(x)}(d) f(\phi(z^-)) \left(\frac{(A-\epsilon) f(\vartheta^- \phi(z^-)}{(\vartheta^-)^{p(x)-1} f(\phi(z^-))} + D_1^-(x) + \widetilde{D_2^-(x)} + D_3^-(x) + D_4^-(x) \right). \tag{3.42}$$

Using the same method, we obtain

$$Lw^+ \leq (\vartheta^+)^{p(x)-1} k^{p(x)}(d) f(\phi(z^+)) \left(\frac{(A+\epsilon) f(\vartheta^+ \phi(z^+)}{(\vartheta^+)^{p(x)-1} f(\phi(z^+))} + D_1^+(x) + \widetilde{D_2^+(x)} + D_3^+(x) + D_4^+(x) \right). \tag{3.43}$$

From (3.2), we infer

$$\lim_{\Omega_\rho^\pm\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} -\frac{p(x)}{q(x)}\frac{|\phi'(z^\pm)|^{p(x)-2}\phi''(z^\pm)}{f(\phi(z^\pm))} = -1. \tag{3.44}$$

By Lemma 3.5, we have

$$\lim_{\Omega_\rho^\pm\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} \widetilde{D_2^\pm(x)} + D_3^\pm(x) = -\frac{p^\# + b(2+\sigma-p^\#)}{2+\sigma}. \tag{3.45}$$

Thus, as $\Omega_\rho^\pm \times \Gamma \ni (d(x),\rho) \to (0^+,0^+)$, the expression in the bracket of (3.42) converges to

$$(A-\epsilon)(\vartheta^-)^{2+\sigma-p^\#} - \frac{p^\# + b(2+\sigma-p^\#)}{2+\sigma} = \left[\frac{A-\epsilon}{A-2\epsilon} - 1\right]\frac{p^\# + b(2+\sigma-p^\#)}{2+\sigma} > 0, \tag{3.46}$$

and the expression in the bracket of (3.43) converges to

$$(A+\epsilon)(\vartheta^+)^{2+\sigma-p^\#} - \frac{p^\# + b(2+\sigma-p^\#)}{2+\sigma} = \left[\frac{A+\epsilon}{A+2\epsilon} - 1\right]\frac{p^\# + b(2+\sigma-p^\#)}{2+\sigma} < 0. \tag{3.47}$$

Therefore, for $\rho \in \Gamma$ and $\mu \geq 0$ sufficiently small, we conclude that

$$Lw^- \geq 0 \quad \text{on } \Omega_\rho^-, \qquad Lw^+ \leq 0 \quad \text{on } \Omega_\rho^+. \tag{3.48}$$

Thus,

$$\begin{aligned} -\Delta_p w^- &\geq -g(x)f(w^-) \quad \text{on } \Omega_\rho^-, \\ -\Delta_p w^+ &\leq -g(x)f(w^+) \quad \text{on } \Omega_\rho^+. \end{aligned} \tag{3.49}$$

Suppose now that k is nondecreasing. Given $0 < \epsilon < A/2$, from (3.19) we deduce

$$\begin{aligned} (A-\epsilon)k^{p(x)}(d(x)-\rho) &\leq (A-\epsilon)k^{p(x)}(d(x)) \leq g(x) \leq (A+\epsilon)k^{p(x)}(d(x)) \\ &\leq (A+\epsilon)k^{p(x)}(d(x)+\rho). \end{aligned} \tag{3.50}$$

Consider

$$w^\pm(x) = \vartheta^\pm\phi(\lambda(d(x)\pm\rho)) = \vartheta^\pm\phi(y^\pm) \quad \text{for } x \in \Omega_\rho^\pm. \tag{3.51}$$

Here, $y^\pm(x) := \lambda(d(x)\pm\rho)$. For simplicity, we will use $d^\pm$ for $d(x)\pm\rho$. Similarly, we obtain

$$\begin{aligned} Lw^\pm &= (\vartheta^\pm)^{p(x)-1}k^{p(x)}(d\pm)f(\phi(y^\pm)) \\ &\quad\times\left(\frac{g(x)f(\vartheta^\pm\phi(y^\pm))}{(\vartheta^\pm)^{p(x)-1}k^{p(x)}(d\pm)f(\phi(y^\pm))} + T_1^\pm(x) + T_2^\pm(x) + T_3^\pm(x) + T_4^\pm(x)\right), \end{aligned} \tag{3.52}$$

where

$$T_1^{\pm}(x) = -\frac{|\phi'(y^{\pm})|^{p(x)-2}\phi'(y^{\pm})}{k(d\pm)f(\phi(y^{\pm}))}\Delta_{p(x)}d,$$
$$T_2^{\pm}(x) = -\frac{p(x)}{q(x)}\frac{k'(d\pm)}{k^2(d\pm)}\frac{|\phi'(y^{\pm})|^{p(x)-2}\phi'(y^{\pm})|\nabla d|^{p(x)}}{f(\phi(y^{\pm}))},$$
$$T_3^{\pm}(x) = -\frac{p(x)}{q(x)}\frac{|\phi'(y^{\pm})|^{p(x)-2}\phi''(y^{\pm})}{f(\phi(y^{\pm}))},$$
$$T_4^{\pm}(x) = -\frac{|\phi'(y^{\pm})|^{p(x)-2}(\phi'(y^{\pm})\ln(k(d\pm)\vartheta^{\pm}) + |\phi'(y^{\pm})|\ln(|\phi'(y^{\pm})|))}{k(d\pm)f(\phi(y^{\pm}))}|\nabla d|^{p(x)-2}\nabla p(x)\nabla d(x). \tag{3.53}$$

By Lemma 3.5 and Remark 3.9, we obtain a similar inequality by replacing $z^{\pm}$ with $y^{\pm}$:

$$\lim_{\Omega_\rho^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} T_1^{\pm}(x) = 0,$$
$$\lim_{\Omega_\rho^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} T_4^{\pm}(x) = 0, \tag{3.54}$$
$$\lim_{\Omega_\rho^{\pm}\times\Gamma\ni(d(x),\rho)\to(0^+,0^+)} T_2^{\pm}(x) + T_3^{\pm}(x) = -\frac{p^{\#} + b(2+\sigma-p^{\#})}{2+\sigma}.$$

Now, suppose u is a nonnegative solution of (2.13). We note that

$$u \le w^- + C_u(\mu) \quad \text{on } \partial\Omega_\rho^-, \tag{3.55}$$

where $C_u(\mu) := \max\{u(x) \mid d(x) \ge \mu\}$. Since ϕ and k are nonincreasing follows $w^+(x) \le \vartheta^+\phi(\lambda(\mu))$ for $d(x) = \mu - \rho$. Therefore,

$$w^+ \le u + C_w(\mu) \quad \text{on } \partial\Omega_\rho^+, \tag{3.56}$$

where $C_w(\mu) := \vartheta^+\phi(\lambda(\mu))$. Furthermore, u and w^+ satisfy the hypothesis from comparison principle on Ω_ρ^+ for $G(x,t) = -g(x)f(t)$. Moreover, since f is nondecreasing, $w^- + C_u(\mu)$ and $u + C_w(\mu)$ satisfy the hypothesis from comparison principle on Ω_ρ^- and Ω_ρ^+, respectively, for $G(x,t) = -g(x)f(t)$. Therefore by comparison principle, we infer

$$u(x) \le w^-(x) + C_u(\mu)\ x \in \partial\Omega_\rho^-, \qquad w^+(x) \le u + C_w(x)(\mu)\ x \in \partial\Omega_\rho^+. \tag{3.57}$$

Hence for $x \in \Omega_\rho^+ \cap \Omega_\rho^-$, we have

$$\vartheta^+ - \frac{C_w(\mu)}{\phi(\lambda(d(x)) \pm \lambda(\rho))} \le \frac{u(x)}{\phi(\lambda(d(x)) \pm \lambda(\rho))} \le \vartheta^- + \frac{C_u(\mu)}{\phi(\lambda(d(x)) \pm \lambda(\rho))},$$
$$\vartheta^+ - \frac{C_w(\mu)}{\phi(\lambda(d(x) \pm \rho))} \le \frac{u(x)}{\phi(\lambda(d(x) \pm \rho))} \le \vartheta^- + \frac{C_u(\mu)}{\phi(\lambda(d(x) \pm \rho))}. \tag{3.58}$$

Letting $\rho \to 0$, we see that

$$\vartheta^+ - \frac{C_w(\mu)}{\phi(\lambda(d(x))} \le \frac{u(x)}{\phi(\lambda(d(x)))} \le \vartheta^- + \frac{C_u(\mu)}{\phi(\lambda(d(x)))}, \tag{3.59}$$

for all $x \in \Omega_\mu$. On recalling that $\phi(t) \to \infty$ as $t \to 0$, we obtain

$$\vartheta^+ \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi(\lambda(d(x)))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\phi(\lambda(d(x)))} \le \vartheta^-. \tag{3.60}$$

The claimed result follows if we take $\epsilon \to 0^+$. If we want to prove the uniqueness of solutions for (2.13), we need additional condition on f; see [1]. The proof is similar to the proof from [1]. □

Acknowledgment

The author would like to express his gratitude to the referee and to the editor for their detailed and helpful comments. The author has been partially supported by CNCSIS Grant no. 589/2007.

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