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Research Article

Existence of Solutions for a Class of Elliptic Systems in $\mathbb{R}^N$ Involving the $(p(x), q(x))$-Laplacian

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In view of variational approach, we discuss a nonlinear elliptic system involving the $p(x)$-Laplacian. Establishing the suitable conditions on the nonlinearity, we proved the existence of nontrivial solutions.

1. Introduction

The paper concerns the existence of nontrivial solutions for the following nonlinear elliptic system:

$$\begin{aligned} -\Delta_{p(x)} u &= \frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \mathbb{R}^N, \\ -\Delta_{q(x)} v &= \frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{P,Q}$$

where $p(x)$ and $q(x)$ are two functions such that $1 < p(x)$, $q(x) < N$ $(N \geq 2)$, for every $x \in \mathbb{R}^N$. However, $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ and $\Delta_{p(x)}$ is the $p(x)$-Laplacian operator defined by $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$. Using a variational approach, the authors prove the existence of nontrivial solutions.

Over the last decades, the variable exponent Lebesgue space $L^{p(x)}$ and Sobolev space $W^{1,p(x)}$ [1–5] have been a subject of active research stimulated mainly by the development of the studies of problems in elasticity, electrorheological fluids, image processing, flow in porous media, calculus of variations, and differential equations with $p(x)$-growth conditions [6–13].

Among these problems, the study of $p(x)$-Laplacian problems via variational methods is an interesting topic. A lot of researchers have devoted their work to this area [14–22].

The operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, where p is a continuous nonconstant function. In particular, if $p(x) \equiv p$ (constant), it is the well-known p-Laplacian operator. However, the $p(x)$-Laplace operator possesses more complicated nonlinearity than p-Laplace operator due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties, as for example, we cannot use the Lagrange multiplier theorem and Morse theorem in a lot of problems involving this operator.

In literature, elliptic systems with standard and nonstandard growth conditions have been studied by many authors [23–28], where the nonlinear function F have different and mixed growth conditions and assumptions in each paper.

In [29], the authors show the existence of nontrivial solutions for the following p-Laplacian problem:

$$\begin{aligned} -\Delta_p u &= \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \mathbb{R}^N, \\ -\Delta_q v &= \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$ yields some mixed growth conditions and the primitive F being intimately connected with the first eigenvalue of an appropriate system. Using a weak version of the Palais-Smale condition, that is, Cerami condition, they apply the mountain pass theorem to get the nontrivial solutions of the the system.

In [30], the author obtains the existence and multiplicity of solutions for the following problem:

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \Omega, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) = \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \Omega, \\ u = 0, \quad v = 0 \text{ on } \Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $N \geq 2$, $(p,q) \in C(\overline{\Omega})^2$, $p(x) > 1$, $q(x) > 1$, for every $x \in \Omega$. The function F is assumed to be continuous in $x \in \overline{\Omega}$ and of class C^1 in $u, v \in \mathbb{R}$. Introducing some natural growth hypotheses on the right-hand side of the system which will ensure the mountain pass geometry and Palais-Smale condition for the corresponding Euler-Lagrange functional of the system, the author limits himself to the subcritical case for function F to obtain the existence and multiplicity results.

In the paper [31], Xu and An deal with the following problem:

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) + |v|^{q(x)-2}v = \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \mathbb{R}^N, \\ (u,v) \in W^{1,p(x)}(\mathbb{R}^N) \times W^{1,q(x)}(\mathbb{R}^N), \end{gathered} \tag{1.3}$$

where $N \geq 2$, $p(x)$, $q(x)$ are functions on $\mathbb{R}^N$. The function F is assumed to satisfy Caratheodory conditions and to be L^∞ in $x \in \mathbb{R}^N$ and C^1 in $u, v \in \mathbb{R}$. By the critical point theory, the authors use the two basic results on the existence of solutions of the system; these results correspond to the sublinear and superlinear cases for $p = 2$, respectively.

Inspired by the above-mentioned papers, we concern the existence of nontrivial solutions of problem (P,Q). We know that in the study of $p(x)$-Laplace equations in $\mathbb{R}^N$, the main difficulty arises from the lack of compactness. So, establishing some growth conditions on the right-hand side of the system which will ensure the mountain pass geometry and Cerami condition for the corresponding Euler-Lagrange functional J and applying a subcritical case for function F, we will overcome this difficulty.

2. Notations and preliminaries

We will investigate our problem (P,Q) in the variable exponent Sobolev space $W_0^{1,p(x)}(\mathbb{R}^N)$, so we need to recall some theories and basic properties on spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$.

Set

$$C_+(\mathbb{R}^N) = \left\{ h \in C(\mathbb{R}^N) : \inf_{x\in\mathbb{R}^N} h(x) > 1 \right\}. \tag{2.1}$$

For every $h \in C_+(\mathbb{R}^N)$, denote

$$h^- := \inf_{x\in\mathbb{R}^N} h(x), \qquad h^+ := \sup_{x\in\mathbb{R}^N} h(x). \tag{2.2}$$

Let us define by $\mathcal{U}(\mathbb{R}^N)$ the set of all measurable real-valued functions defined on $\mathbb{R}^N$. For $p \in C_+(\mathbb{R}^N)$, we denote the variable exponent Lebesgue space by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u \in \mathcal{U}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < \infty \right\}, \tag{2.3}$$

which is equipped with the norm, so-called Luxemburg norm [1, 3, 4]:

$$|u|_{p(x)} := |u|_{L^{p(x)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \tag{2.4}$$

and $(L^{p(x)}(\mathbb{R}^N), |\cdot|_{L^{p(x)}(\mathbb{R}^N)})$ becomes a Banach space, we call it generalized Lebesgue space.

Define the variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ by

$$W^{1,p(x)}(\mathbb{R}^N) = \{ u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \}, \tag{2.5}$$

and it can be equipped with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}} = |u|_{p(x)} + |\nabla u|_{p(x)} \quad \forall u \in W^{1,p(x)}(\mathbb{R}^N). \tag{2.6}$$

The space $W_0^{1,p(x)}(\mathbb{R}^N)$ is denoted by the closure of $C_0^\infty(\mathbb{R}^N)$ in $W^{1,p(x)}(\mathbb{R}^N)$ and it is equipped with the norm for all $u \in W_0^{1,p(x)}(\mathbb{R}^N)$:

$$\|u\|_{p(x)} = |\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\mathbb{R}^N). \tag{2.7}$$

If $p^- > 1$, then the spaces $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$, and $W_0^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Proposition 2.1 (see [1, 3, 4]). *The conjugate space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where $1/p'(x) + 1/p(x) = 1$. For any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p'(x)}(\mathbb{R}^N)$, we have*

$$\left|\int_{\mathbb{R}^N} uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.8}$$

Proposition 2.2 (see [1, 3, 4]). *Denote $\varrho_{p(x)}(u) = \int_{\mathbb{R}^N}|u(x)|^{p(x)}dx$ for all $u \in L^{p(x)}(\mathbb{R}^N)$, one has*

$$\min\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\} \leq \varrho_{p(x)}(u) \leq \max\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\}. \tag{2.9}$$

Proposition 2.3 (see [1]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\mathbb{R}^N)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{q(x)}(\mathbb{R}^N)$, $u \neq 0$. Then,*

$$\begin{aligned} |u|_{p(x)q(x)} \leq 1 &\Longrightarrow |u|_{p(x)q(x)}^{p^+} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-}, \\ |u|_{p(x)q(x)} \geq 1 &\Longrightarrow |u|_{p(x)q(x)}^{p^-} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+}. \end{aligned} \tag{2.10}$$

In particular, if $p(x) = p$ is constant, then

$$||u|^p|_{q(x)} = |u|_{pq(x)}^p. \tag{2.11}$$

Proposition 2.4 (see [3, 4]). *If $u, u_n \in L^{p(x)}(\mathbb{R}^N)$, $n = 1, 2, \ldots$, then the following statements are equivalent to each other:*

(1) $\lim_{n\to\infty}|u_n - u|_{p(x)} = 0$,

(2) $\lim_{n\to\infty}\varrho(u_n - u) = 0$,

(3) $u_n \to u$ *in measure in* $\mathbb{R}^N$ and $\lim_{n\to\infty}\varrho(u_n) = \varrho(u)$.

Definition 2.5. $1 < p(x) < N$ and for all $x \in \mathbb{R}^N$, let define

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

where $p^*(x)$ is the so-called critical Sobolev exponent of $p(x)$.

Proposition 2.6 (see [1, 32]). *Let $p(x) \in C_+^{0,1}(\mathbb{R}^N)$, that is, Lipschitz-continuous function defined on $\mathbb{R}^N$, then there exists a positive constant c such that*

$$|u|_{p^*(x)} \leq c\|u\|_{p(x)}, \tag{2.12}$$

for all $u \in W_0^{1,p(x)}(\mathbb{R}^N)$.

In the following discussions, we will use the product space

$$W_{p(x),q(x)} := W_0^{1,p(x)}(\mathbb{R}^N) \times W_0^{1,q(x)}(\mathbb{R}^N), \tag{2.13}$$

which is equipped with the norm

$$\|(u,v)\|_{p(x),q(x)} := \max\left\{\|u\|_{p(x)} + \|v\|_{q(x)}\right\} \quad \forall (u,v) \in W_{p(x),q(x)}, \tag{2.14}$$

where $\|u\|_{p(x)}$ (resp., $\|u\|_{q(x)}$) is the norm of $W_0^{1,p(x)}(\mathbb{R}^N)$ (resp., $W_0^{1,q(x)}(\mathbb{R}^N)$). The space $W^*_{p(x),q(x)}$ denotes the dual space of $W_{p(x),q(x)}$ and equipped with the norm $\|\cdot\|_{*,p(x),q(x)}$. Thus,

$$\|J'(u,v)\|_{*,p(x),q(x)} = \|D_1 J(u,v)\|_{*,p(x)} + \|D_2 J(u,v)\|_{*,q(x)}, \tag{2.15}$$

where $W^{-1,p'(x)}(\mathbb{R}^N)$ (resp., $W^{-1,q'(x)}(\mathbb{R}^N)$) is the dual space of $W_0^{1,p(x)}(\mathbb{R}^N)$ (resp., $W_0^{1,q(x)}(\mathbb{R}^N)$), and $\|\cdot\|_{*,p(x)}$ (resp., $\|\cdot\|_{*,q(x)}$) is its norm.

For (u,v) and (φ,ψ) in $W_{p(x),q(x)}$, let

$$\mathcal{F}(u,v) = \int_{\mathbb{R}^N} F(x,u(x),v(x))dx. \tag{2.16}$$

Then,

$$\mathcal{F}'(u,v)(\varphi,\psi) = D_1\mathcal{F}(u,v)(\varphi) + D_2\mathcal{F}(u,v)(\psi), \tag{2.17}$$

where

$$\begin{aligned} D_1\mathcal{F}(u,v)(\varphi) &= \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)\varphi\, dx, \\ D_2\mathcal{F}(u,v)(\psi) &= \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x,u,v)\psi\, dx. \end{aligned} \tag{2.18}$$

The Euler-Lagrange functional associated to (P,Q) is defined by

$$J(u,v) = \int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\mathbb{R}^N} \frac{1}{q(x)}|\nabla v|^{q(x)}dx - (u,v). \tag{2.19}$$

It is easy to verify that $J \in C^1(W_{p(x),q(x)},\mathbb{R})$ and that

$$J'(u,v)(\varphi,\psi) = D_1 J(u,v)(\varphi) + D_2 J(u,v)(\psi), \tag{2.20}$$

where

$$\begin{aligned} D_1 J(u,v)(\varphi) &= \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\, dx - D_1\mathcal{F}(u,v)(\varphi), \\ D_2 J(u,v)(\psi) &= \int_{\mathbb{R}^N} |\nabla v|^{q(x)-2}\nabla v \nabla\psi\, dx - D_2\mathcal{F}(u,v)(\psi). \end{aligned} \tag{2.21}$$

Definition 2.7. (u,v) is called a weak solution of the system (P,Q) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\, dx + \int_{\mathbb{R}^N} |\nabla v|^{q(x)-2}\nabla v\nabla\psi\, dx = \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)\varphi\, dx + \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x,u,v)\psi\, dx, \tag{2.22}$$

for all $(\varphi,\psi) \in W_{p(x),q(x)}$.

Definition 2.8. We say that J satisfies the Cerami condition (C) if every sequence $(\omega_n) \in W_{p(x),q(x)}$ such that

$$|J(\omega_n)| \le c, \qquad (1+\|\omega_n\|)J'(\omega_n) \longrightarrow 0 \tag{2.23}$$

contains a convergent subsequence in the norm of $W_{p(x),q(x)}$.

In this paper, we will use the following assumptions:

($F1$) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $F(x,0,0) = 0$;

($F2$) for all $(u,v) \in \mathbb{R}^2$ and for a.e. $x \in \mathbb{R}^N$

$$\begin{gathered}\left|\frac{\partial F}{\partial u}(x,u,v)\right| \le a_1(x)|(u,v)|^{p^- -1} + a_2(x)|(u,v)|^{p^+ -1},\\ \left|\frac{\partial F}{\partial v}(x,u,v)\right| \le b_1(x)|(u,v)|^{q^- -1} + b_2(x)|(u,v)|^{q^+ -1},\end{gathered} \tag{2.24}$$

where

$$\begin{gathered}1 < p^-, \qquad q^- \le p^+, \qquad q^+ < (p^*)^-, (q^*)^-,\\ a_i \in L^{\delta(x)}(\mathbb{R}^N) \cap L^{\beta(x)}(\mathbb{R}^N), \quad b_i \in L^{\gamma(x)}(\mathbb{R}^N) \cap L^{\beta(x)}(\mathbb{R}^N), \quad i = 1,2,\\ \delta(x) = \frac{p(x)}{p(x)-1}, \qquad \gamma(x) = \frac{q(x)}{q(x)-1}, \qquad \widetilde{p}(x) = \frac{p^*(x)p(x)}{p^*(x)-p(x)},\\ \widetilde{q}(x) = \frac{q^*(x)q(x)}{q^*(x)-q(x)}, \qquad \beta(x) = \frac{p^*(x)q^*(x)}{p^*(x)q^*(x)-(p^*(x)+q^*(x))};\end{gathered} \tag{2.25}$$

($F3$) $(u,v)\cdot\nabla F(x,u,v) - F(x,u,v) \le 0$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2 \setminus \{(0,0)\}$, where $\nabla F = (\partial F/\partial u, \partial F/\partial v)$;

($F4$) suppose there exist two positive and bounded functions $a \in L^{N/p(x)}(\mathbb{R}^N)$ and $b \in L^{N/q(x)}(\mathbb{R}^N)$ such that

$$\begin{aligned}&\lim_{|(u,v)|\to 0}\sup \frac{p(x)q(x)|F(x,u,v)|}{q(x)a(x)|u|^{p(x)} + p(x)b(x)|v|^{q(x)}}\\ &\quad < \lambda_1 < \lim_{|(u,v)|\to+\infty}\inf \frac{p(x)q(x)|F(x,u,v)|}{q(x)a(x)|u|^{p(x)} + p(x)b(x)|v|^{q(x)}}.\end{aligned} \tag{2.26}$$

Let λ_1 denote the first eigenvalue of the nonlinear eigenvalue problem in $\mathbb{R}^N$:

$$\begin{gathered}-\Delta_{p(x)}u = \lambda a(x)|u|^{p(x)-2}u \quad \text{in } \mathbb{R}^N,\\ -\Delta_{q(x)}v = \lambda b(x)|v|^{q(x)-2}v \quad \text{in } \mathbb{R}^N.\end{gathered} \tag{2.27}$$

It is useful to recall the variational characterization:

$$\lambda_1 = \inf\left\{\frac{\int_{\mathbb{R}^N}((1/p(x))|\nabla u|^{p(x)} + (1/q(x))|\nabla v|^{q(x)})dx}{\int_{\mathbb{R}^N}((a(x)/p(x))|u|^{p(x)} + (b(x)/q(x))|v|^{q(x)})dx} : (u,v) \in W_{p(x),q(x)} \setminus \{(0,0)\}\right\}. \tag{2.28}$$

We will assume that λ_1 is a positive real number for all $(u,v) \in W_{p(x),q(x)} \setminus \{(0,0)\}$. For more details about the eigenvalue problems, we refer the reader to [17].

3. The main results

We will use the mountain pass theorem together with the following lemmas to get our main results.

Lemma 3.1. *Under the assumptions* $(F1)$ *and* $(F2)$*, the functional* $\mathcal{F}$ *is well defined, and it is of class* C^1 *on* $W_{p(x),q(x)}$*. Moreover, its derivative is*

$$\mathcal{F}'(u,v)(\omega,z)=\int_{\mathbb{R}^N}\frac{\partial F}{\partial u}(x,u,v)\omega\,dx+\int_{\mathbb{R}^N}\frac{\partial F}{\partial v}(x,u,v)z\,dx\quad\forall(u,v),(\omega,z)\in W_{p(x),q(x)}. \tag{3.1}$$

Proof. For all pair of real functions $(u,v)\in W_{p(x),q(x)}$, under the assumptions $(F1)$ and $(F2)$, we can write

$$\begin{aligned}F(x,u,v)&=\int_0^u\frac{\partial F}{\partial s}(x,s,v)ds+F(x,0,v)=\int_0^u\frac{\partial F}{\partial s}(x,s,v)ds+\int_0^v\frac{\partial F}{\partial s}(x,0,s)ds+F(x,0,0),\\ F(x,u,v)&\le c_1\left[a_1(x)\left(|u|^{p^-}+|v|^{p^--1}|u|\right)+a_2(x)\left(|u|^{p^+}+|v|^{p^+-1}|u|\right)+b_1(x)|v|^{q^-}+b_2(x)|v|^{q^+}\right].\end{aligned} \tag{3.2}$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^N}F(x,u,v)dx\le c_2\Bigg[\Bigg[\int_{\mathbb{R}^N}a_1(x)|u|^{p^-}dx+\int_{\mathbb{R}^N}a_1(x)|v|^{p^--1}|u|dx+\int_{\mathbb{R}^N}a_2(x)|u|^{p^+}dx\\ +\int_{\mathbb{R}^N}a_2(x)|v|^{p^+-1}|u|dx+\int_{\mathbb{R}^N}b_1(x)|v|^{q^-}dx+\int_{\mathbb{R}^N}b_2(x)|v|^{q^+}dx\Bigg],\end{aligned} \tag{3.3}$$

if we consider the fact that

$$W_0^{1,p(x)}(\mathbb{R}^N)\hookrightarrow L^{p^-p(x)}(\mathbb{R}^N)\Longrightarrow\left||u|^{p^-}\right|_{p(x)}=|u|^{p^-}_{p^-p(x)}\le c\|u\|^{p^-}_{p(x)}\quad\text{for } p^->1, \tag{3.4}$$

and if we apply Propositions 2.1, 2.3, and 2.6 and take $a_i\in L^{\delta(x)}(\mathbb{R}^N)\cap L^{\beta(x)}(\mathbb{R}^N)$, $b_i\in L^{\gamma(x)}(\mathbb{R}^N)$, then we have

$$\begin{aligned}\int_{\mathbb{R}^N}F(x,u,v)dx&\le 2c_1\Big(|a_1|_{\delta(x)}\left||u|^{p^-}\right|_{p(x)}+|a_1|_{\beta(x)}\left||v|^{p^--1}\right|_{q^*(x)}|u|_{p^*(x)}+|a_2|_{\delta(x)}\left||u|^{p^+}\right|_{p(x)}\\ &\qquad+|a_2|_{\beta(x)}\left||v|^{p^+-1}\right|_{q^*(x)}|u|_{p^*(x)}+|b_1|_{\gamma(x)}\left||v|^{q^-}\right|_{q(x)}+|b_2|_{\gamma(x)}\left||v|^{q^+}\right|_{q(x)}\Big)\\ &=2c_1\Big(|a_1|_{\delta(x)}|u|^{p^-}_{p^-p(x)}+|a_1|_{\beta(x)}|v|^{p^--1}_{(p^--1)q^*(x)}|u|_{p^*(x)}+|a_2|_{\delta(x)}|u|^{p^+}_{p^+p(x)}\\ &\qquad+|a_2|_{\beta(x)}|v|^{p^+-1}_{(p^+-1)q^*(x)}|u|_{p^*(x)}+|b_1|_{\gamma(x)}|v|^{q^-}_{q^-q(x)}+|b_2|_{\gamma(x)}|v|^{q^+}_{q^+q(x)}\Big)\\ &\le c_3\Big(|a_1|_{\delta(x)}\|u\|^{p^-}_{p(x)}+|a_1|_{\beta(x)}\|v\|^{p^--1}_{q(x)}\|u\|_{p(x)}+|a_2|_{\delta(x)}\|u\|^{p^+}_{p(x)}\\ &\qquad+|a_2|_{\beta(x)}\|v\|^{p^+-1}_{q(x)}\|u\|_{p(x)}+|b_1|_{\gamma(x)}\|v\|^{q^-}_{q(x)}+|b_2|_{\gamma(x)}\|v\|^{q^+}_{q(x)}\Big)<\infty.\end{aligned} \tag{3.5}$$

Hence, $\mathcal{F}$ is well defined. Moreover, one can see easily that $\mathcal{F}'$ is also well defined on $W_{p(x),q(x)}$. Indeed, using $(F2)$ for all $(\omega, z) \in W_{p(x),q(x)}$, we have

$$\begin{aligned}
\mathcal{F}'(u,v)(\omega,z) &= \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)\omega\,dx + \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x,u,v)z\,dx, \\
\mathcal{F}'(u,v)(\omega,z) &\le \int_{\mathbb{R}^N} \left(a_1(x)|(u,v)|^{p^- -1} + a_2(x)|(u,v)|^{p^+ -1}\right)|\omega|dx \\
&\quad + \int_{\mathbb{R}^N} \left(b_1(x)|(u,v)|^{q^- -1} + b_2(x)|(u,v)|^{q^+ -1}\right)|z|dx \\
&\le \int_{\mathbb{R}^N} a_1(x)|u|^{p^- -1}|\omega|dx + \int_{\mathbb{R}^N} a_1(x)|v|^{p^- -1}|\omega|dx \\
&\quad + \int_{\mathbb{R}^N} a_2(x)|u|^{p^+ -1}|\omega|dx + \int_{\mathbb{R}^N} a_2(x)|v|^{p^+ -1}|\omega|dx \\
&\quad + \int_{\mathbb{R}^N} b_1(x)|u|^{q^- -1}|z|dx + \int_{\mathbb{R}^N} b_1(x)|v|^{q^- -1}|z|dx \\
&\quad + \int_{\mathbb{R}^N} b_2(x)|u|^{q^+ -1}|z|dx + \int_{\mathbb{R}^N} b_2(x)|v|^{q^+ -1}|z|dx,
\end{aligned} \tag{3.6}$$

and applying Propositions 2.1, 2.3, and 2.6 and considering the conditions $\tilde{p}(x) > p(x)$ and $\tilde{q}(x) > q(x)$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x,u,v)\omega\,dx &\le 2\Big(|a_1|_{\delta(x)}\left||u|^{p^- -1}\right|_{p^*(x)}|\omega|_{\tilde{p}(x)} + |a_1|_{\beta(x)}\left||v|^{p^- -1}\right|_{q^*(x)}|\omega|_{p^*(x)} \\
&\quad + |a_2|_{\delta(x)}\left||u|^{p^+ -1}\right|_{p^*(x)}|\omega|_{\tilde{p}(x)} + |a_2|_{\beta(x)}\left||v|^{p^+ -1}\right|_{q^*(x)}|\omega|_{p^*(x)}\Big) \\
&\le 2\Big(|a_1|_{\delta(x)}|u|^{p^- -1}_{(p^- -1)p^*(x)}|\omega|_{\tilde{p}(x)} + |a_1|_{\beta(x)}|v|^{p^- -1}_{(p^- -1)q^*(x)}|\omega|_{p^*(x)} \\
&\quad + |a_2|_{\delta(x)}|u|^{p^+ -1}_{(p^+ -1)p^*(x)}|\omega|_{\tilde{p}(x)} + |a_2|_{\beta(x)}|v|^{p^+ -1}_{(p^+ -1)q^*(x)}|\omega|_{p^*(x)}\Big) \\
&\le c_4\Big(|a_1|_{\delta(x)}\|u\|^{p^- -1}_{p(x)} + |a_1|_{\beta(x)}\|v\|^{p^- -1}_{q(x)} + |a_2|_{\delta(x)}\|u\|^{p^+ -1}_{p(x)} + |a_2|_{\beta}\|v\|^{p^+ -1}_{q(x)}\Big)\|\omega\|_{p(x)} \\
&< \infty,
\end{aligned} \tag{3.7}$$

and similarly

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x,u,v)z\,\mathrm{d}x \\
&\quad \le c_5\Big(|b_1|_{\beta(x)}\|u\|^{q^- -1}_{p(x)} + |b_1|_{\gamma(x)}\|v\|^{q^- -1}_{q(x)} + |b_2|_{\beta(x)}\|u\|^{q^+ -1}_{p(x)} + |b_2|_{\gamma(x)}\|v\|^{q^+ -1}_{q(x)}\Big)\|z\|_{q(x)} < \infty.
\end{aligned} \tag{3.8}$$

Now let us show that $\mathcal{F}$ is differentiable in sense of Fréchet, that is, for fixed $(u,v) \in W_{p(x),q(x)}$ and given $\varepsilon > 0$, there must be a $\delta = (\varepsilon, u, v) > 0$ such that

$$|\mathcal{F}(u+\omega, v+z) - \mathcal{F}(u,v) - \mathcal{F}'(u,v)(\omega,z)| \le \varepsilon(\|\omega\|_{p(x)} + \|z\|_{q(x)}), \tag{3.9}$$

for all $(\omega, z) \in W_{p(x),q(x)}$ with $(\|\omega\|_{p(x)} + \|z\|_{q(x)}) \le \delta$.

Let B_r be the ball of radius r which is centered at the origin of $\mathbb{R}^N$ and denote $B_r' = \mathbb{R}^N - B_r$. Moreover, let us define the functional $\mathcal{F}_r$ on $W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$ as follows:

$$\mathcal{F}_r(u,v) = \int_{B_r} F(x,u(x),v(x))dx. \tag{3.10}$$

If we consider $(F1)$ and $(F2)$, it is easy to see that $\mathcal{F}_r \in C^1(W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r))$, and in addition for all $(\omega,z) \in W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$, we have

$$\mathcal{F}_r'(u,v)(\omega,z) = \int_{B_r} \frac{\partial F}{\partial u}(x,u,v)\omega\, dx + \int_{B_r} \frac{\partial F}{\partial v}(x,u,v)z\, dx. \tag{3.11}$$

Also as we know, the operator $\mathcal{F}_r' : W_{p(x),q(x)} \to W_{p(x),q(x)}^*$ is compact [3]. Then, for all $(u,v), (\omega,z) \in W_{p(x),q(x)}$, we can write

$$\begin{aligned} &\left|\mathcal{F}(u+\omega,v+z) - \mathcal{F}(u,v) - \mathcal{F}'(u,v)(\omega,z)\right| \\ &\quad\le \left|\mathcal{F}_r(u+\omega,v+z) - \mathcal{F}_r(u,v) - \mathcal{F}_r'(u,v)(\omega,z)\right| \\ &\quad\quad + \left|\int_{B_r'} \left(F(x,u+\omega,v+z) - F(x,u,v) - \frac{\partial F}{\partial u}(x,u,v)\omega - \frac{\partial F}{\partial v}(x,u,v)z\right)dx\right|. \end{aligned} \tag{3.12}$$

By virtue of the mean-value theorem, there exist $\zeta_1, \zeta_2 \in (0,1)$ such that

$$F(x,u+\omega,v+z) - F(x,u,v) = \frac{\partial F}{\partial u}(x,u+\zeta_1\omega,v)\omega + \frac{\partial F}{\partial v}(x,u,v+\zeta_2 z)z. \tag{3.13}$$

Using the condition $(F2)$, we have

$$\begin{aligned} &\left|\int_{B_r'} \left(\frac{\partial F}{\partial u}(x,u+\zeta_1\omega,v)\omega + \frac{\partial F}{\partial v}(x,u,v+\zeta_2 z)z - \frac{\partial F}{\partial u}(x,u,v)\omega - \frac{\partial F}{\partial v}(x,u,v)z\right)dx\right| \\ &\quad\le \left|\int_{B_r'} \left(a_1(x)\left(|u+\zeta_1\omega|^{p^- -1} - |u|^{p^- -1}\right) + a_2(x)\left(|u+\zeta_1\omega|^{p^+ -1} - |u|^{p^+ -1}\right)\right)|\omega|dx \right. \\ &\quad\quad \left. + \int_{B_r'} \left(b_1(x)\left(|v+\zeta_2 z|^{q^- -1} - |v|^{q^- -1}\right) + b_2(x)\left(|v+\zeta_2 z|^{q^+ -1} - |v|^{q^+ -1}\right)\right)|z|dx\right|. \end{aligned} \tag{3.14}$$

By help of the elementary inequality $|a+b|^s \le 2^{s-1}(|a|^s + |b|^s)$ for $a, b \in \mathbb{R}^N$, we can write

$$\begin{aligned} &\le (2^{p^- -1} - 1)\int_{B_r'} a_1(x)|u|^{p^- -1}|\omega|dx + (\zeta_1 2)^{p^- -1}\int_{B_r'} a_1(x)|\omega|^{p^- -1}|\omega|dx \\ &\quad + (2^{p^+ -1} - 1)\int_{B_r'} a_2(x)|u|^{p^+ -1}|\omega|dx + (\zeta_1 2)^{p^+ -1}\int_{B_r'} a_2(x)|\omega|^{p^+ -1}|\omega|dx \\ &\quad + (2^{q^- -1} - 1)\int_{B_r'} b_1(x)|v|^{q^- -1}|z|dx + (\zeta_2 2)^{q^- -1}\int_{B_r'} b_1(x)|z|^{q^- -1}|z|dx \\ &\quad + (2^{q^+ -1} - 1)\int_{B_r'} b_2(x)|v|^{q^+ -1}|z|dx + (\zeta_2 2)^{q^+ -1}\int_{B_r'} b_2(x)|z|^{q^+ -1}|z|dx, \end{aligned} \tag{3.15}$$

applying Propositions 2.1, 2.3, and 2.6, then we have

$$\begin{aligned}
&\leq c_6\Big(|a_1|_{\delta(x)}||u|^{p^--1}|_{p^*(x)}|\omega|_{\tilde{p}(x)}+|a_1|_{\delta(x)}||\omega|^{p^--1}|_{p^*(x)}|\omega|_{\tilde{p}(x)}\\
&\quad+|a_2|_{\delta(x)}||u|^{p^+-1}|_{p^*(x)}|\omega|_{\tilde{p}(x)}+|a_2|_{\delta(x)}||\omega|^{p^+-1}|_{p^*(x)}|\omega|_{\tilde{p}(x)}\\
&\quad+|b_1|_{\gamma(x)}||v|^{q^--1}|_{q^*(x)}|z|_{\tilde{q}(x)}+|b_1|_{\gamma(x)}||z|^{q^--1}|_{q^*(x)}|z|_{\tilde{q}(x)}\\
&\quad+|b_2|_{\gamma(x)}||v|^{q^+-1}|_{q^*(x)}|z|_{\tilde{q}(x)}+|b_2|_{\gamma(x)}||z|^{q^+-1}|_{q^*(x)}|z|_{\tilde{q}(x)}\Big)\,,\\
&\leq c_7\Big(\Big(|a_1|_{\delta(x)}\|u\|^{p^--1}_{p(x)}+|a_1|_{\delta(x)}\|\omega\|^{p^--1}_{p(x)}\Big)+\Big(|a_2|_{\delta(x)}\|u\|^{p^+-1}_{p(x)}+|a_2|_{\delta(x)}\|\omega\|^{p^+-1}_{p(x)}\Big)\Big)\|\omega\|_{p(x)}\\
&\quad+\Big(\Big(|b_1|_{\gamma(x)}\|v\|^{q^--1}_{q(x)}+|b_1|_{\gamma(x)}\|z\|^{q^--1}_{q(x)}\Big)+\Big(|b_2|_{\gamma(x)}\|v\|^{q^+-1}_{q(x)}+|b_2|_{\gamma(x)}\|z\|^{q^+-1}_{q(x)}\Big)\Big)\|z\|_{q(x)},
\end{aligned}\tag{3.16}$$

and by the fact that

$$\begin{aligned}
|a_i|_{L^{\delta(x)}(B_r')}&\longrightarrow 0,\\
|b_i|_{L^{\gamma(x)}(B_r')}&\longrightarrow 0
\end{aligned}\tag{3.17}$$

for $i=1,2$, as $r\to\infty$, and for r sufficiently large, it follows that

$$\left|\int_{B_r'}\left(F(x,u+\omega,v+z)-F(x,u,v)-\frac{\partial F}{\partial u}(x,u,v)\omega-\frac{\partial F}{\partial v}(x,u,v)\right)z)dx\right|\leq\varepsilon(\|\omega\|_{p(x)}+\|z\|_{q(x)}).\tag{3.18}$$

It remains only to show that $\mathcal{F}'$ is continuous on $W_{p(x),q(x)}$. Let $(u_n,v_n),(u,v)\in W_{p(x),q(x)}$ such that $(u_n,v_n)\to(u,v)$. Then, for $(\omega,z)\in W_{p(x),q(x)}$, we have

$$\begin{aligned}
|\mathcal{F}'(u_n,v_n)(\omega,z)-\mathcal{F}'(u,v)(\omega,z)|\leq{}&|\mathcal{F}_r'(u_n,v_n)(\omega,z)-\mathcal{F}_r'(u,v)(\omega,z)|\\
&+\left|\int_{B_r'}\left(\frac{\partial F}{\partial u}(x,u_n,v_n)+\frac{\partial F}{\partial u}(x,u,v)\right)\omega\,dx\right|\\
&+\left|\int_{B_r'}\left(\frac{\partial F}{\partial v}(x,u_n,v_n)+\frac{\partial F}{\partial v}(x,u,v)\right)z\,dx\right|,
\end{aligned}\tag{3.19}$$

then by $(F2)$, we can write

$$\int_{B_r'}a_1(x)(|u_n|^{p^--1}+|u|^{p^--1}+|v_n|^{p^--1}+|v|^{p^--1})|\omega|dx\tag{3.20}$$

$$+\int_{B_r'}a_2(x)(|u_n|^{p^+-1}+|u|^{p^+-1}+|v_n|^{p^+-1}+|v|^{p^+-1})|\omega|dx\tag{I_1}$$

$$+\int_{B_r'}b_1(x)(|u_n|^{q^--1}+|u|^{q^--1}+|v_n|^{q^--1}+|v|^{q^--1})|z|dx\tag{I_2}$$

$$+\int_{B_r'}b_2(x)(|u_n|^{q^+-1}+|u|^{q^+-1}+|v_n|^{q^+-1}+|v|^{q^+-1})|z|dx.\tag{3.21}$$

Thus,

$$
\begin{aligned}
I_1 \le & \int_{B_r'} a_1(x)|u_n|^{p^- -1}|\omega|dx + \int_{B_r'} a_1(x)|u|^{p^- -1}|\omega|dx + \int_{B_r'} a_1(x)|v_n|^{p^- -1}|\omega|dx \\
& + \int_{B_r'} a_1(x)|v|^{p^- -1}|\omega|dx + \int_{B_r'} a_2(x)|u_n|^{p^+ -1}|\omega|dx + \int_{B_r'} a_2(x)|u|^{p^+ -1}|\omega|dx \\
& + \int_{B_r'} a_2(x)|v_n|^{p^+ -1}|\omega|dx + \int_{B_r'} a_2(x)|v|^{p^+ -1}|\omega|dx \\
\le & \, c_9\Big(|a_1|_{\delta(x)}\|u_n\|_{p(x)}^{p^- -1} + |a_1|_{\delta(x)}\|u\|_{p(x)}^{p^- -1} + |a_1|_{\beta(x)}\|v_n\|_{q(x)}^{p^- -1} + |a_1|_{\beta(x)}\|v\|_{q(x)}^{p^- -1} \\
& + |a_2|_{\delta(x)}\|u_n\|_{p(x)}^{p^+ -1} + |a_2|_{\delta(x)}\|u\|_{p(x)}^{p^+ -1} + |a_2|_{\beta(x)}\|v_n\|_{q(x)}^{p^+ -1} + |a_2|_{\beta(x)}\|v\|_{q(x)}^{p^+ -1}\Big)\|\omega\|_{p(x).}
\end{aligned}
\tag{3.22}
$$

Similarly,

$$
\begin{aligned}
I_2 \le c_{10}\Big(& |b_1|_{\beta(x)}\|u_n\|_{p(x)}^{q^- -1} + |b_1|_{\beta(x)}\|u\|_{p(x)}^{q^- -1} + |b_1|_{\gamma(x)}\|v_n\|_{q(x)}^{q^- -1} + |b_1|_{\gamma(x)}\|v\|_{q(x)}^{q^- -1} \\
& + |b_2|_{\beta(x)}\|u_n\|_{p(x)}^{q^+ -1} + |b_2|_{\beta(x)}\|u\|_{p(x)}^{q^+ -1} + |b_2|_{\gamma(x)}\|v_n\|_{q(x)}^{q^+ -1} + |b_2|_{\gamma(x)}\|v\|_{q(x)}^{q^+ -1}\Big)\|z\|_{q(x)}.
\end{aligned}
\tag{3.23}
$$

Since $\mathcal{F}_r'$ is continuous on $W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$, then we have

$$
|\mathcal{F}_r'(u_n, v_n)(\omega, z) - \mathcal{F}_r'(u, v)(\omega, z)| \longrightarrow 0, \tag{3.24}
$$

as $n \to \infty$. Moreover, using (3.17), when r sufficiently large, I_1 and I_2 tend also to 0. Hence,

$$
|\mathcal{F}'(u_n, v_n)(\omega, z) - \mathcal{F}'(u, v)(\omega, z)| \longrightarrow 0, \tag{3.25}
$$

as $(u_n, v_n) \to (u, v)$, this implies $\mathcal{F}'$ is continuous on $W_{p(x),q(x)}$. □

Lemma 3.2. *Under the assumptions* (*F*1) *and* (*F*2), $\mathcal{F}'$ *is compact from* $W_{p(x),q(x)}$ *to* $W^*_{p(x),q(x)}$.

Proof. Let (u_n, v_n) be a bounded sequence in $W_{p(x),q(x)}$. Then, there exists a subsequence (we denote again as (u_n, v_n)) which converges weakly in $W_{p(x),q(x)}$ to a $(u, v) \in W_{p(x),q(x)}$. Then, if we use the same arguments as above, we have

$$
\begin{aligned}
|\mathcal{F}'(u_n, v_n)(\omega, z) - \mathcal{F}'(u, v)(\omega, z)| \le & \, |\mathcal{F}_r'(u_n, v_n)(\omega, z) - \mathcal{F}_r'(u, v)(\omega, z)| \\
& + \left|\int_{B_r'} \left(\frac{\partial F}{\partial u}(x, u_n, v_n) - \frac{\partial F}{\partial u}(x, u, v)\right)\omega \, dx\right| \\
& + \left|\int_{B_r'} \left(\frac{\partial F}{\partial v}(x, u_n, v_n) - \frac{\partial F}{\partial v}(x, u, v)\right)z \, dx\right|.
\end{aligned}
\tag{3.26}
$$

Since the restriction operator is continuous, then $(u_n, v_n) \rightharpoonup (u, v)$ in $W_0^{1,p(x)}(B_r) \times W_0^{1,q(x)}(B_r)$. Because of the compactness of $\mathcal{F}_r'$, the first expression on the right-hand side of the equation tends to 0, as $n \to \infty$, and as we did above, when r sufficiently large, I_1 and I_2 tend also to 0. This implies $\mathcal{F}'$ is compact from $W_{p(x),q(x)}$ to $W^*_{p(x),q(x)}$. □

Lemma 3.3. *If* $(F1)$, $(F2)$, *and* $(F3)$ *hold, then* J *satisfies the condition* (C), *that is, there exists a sequence* $(u_n, v_n) \in W_{p(x),q(x)}$ *such that*

(i) $|J(u_n, v_n)| \leq c$,

(ii) $(1 + \|u_n\|_{p(x)} + \|v_n\|_{q(x)})\|J'(u_n, v_n)\|_{*,p(x),q(x)} \rightarrow 0$ *as* $n \rightarrow +\infty$

contains a convergent subsequence.

Proof. By the assumption (ii), it is clear that $J'(u_n, v_n)(\omega, z) \leq \xi_n \rightarrow 0$ as $n \rightarrow \infty$ for all $(\omega, z) \in W_{p(x),q(x)}$. Let us choose $(\omega, z) = (u_n, v_n)$, then we have

$$\begin{aligned} \xi_n &\geq J'(u_n, v_n)(u_n, v_n) \\ &\geq \|u_n\|_{p(x)}^{p^-} + \|v_n\|_{q(x)}^{q^-} - \int_{\mathbb{R}^N} \left(\frac{\partial F}{\partial u}(x, u_n, v_n) u_n + \frac{\partial F}{\partial v}(x, u_n, v_n) v_n \right) dx. \end{aligned} \tag{3.27}$$

Moreover, by the assumption (i), we can write

$$c \geq -J(u_n, v_n) \geq -\frac{1}{p^+}\|u_n\|_{p(x)}^{p^-} - \frac{1}{q^+}\|v_n\|_{q(x)}^{q^-} + \int_{\mathbb{R}^N} F(x, u_n, v_n) dx. \tag{3.28}$$

Using the assumption $(F3)$, it follows that

$$\begin{aligned} \xi_n + c &\geq J'(u_n, v_n)(u_n, v_n) - J(u_n, v_n) \\ &\geq \left(1 - \frac{1}{p^+}\right)\|u_n\|_{p(x)}^{p^-} + \left(1 - \frac{1}{q^+}\right)\|v_n\|_{q(x)}^{q^-} + \int_{\mathbb{R}^N} F(x, u_n, v_n) dx \\ &\quad - \int_{\mathbb{R}^N} \left(\frac{\partial F}{\partial u}(x, u_n, v_n) u_n + \frac{\partial F}{\partial v}(x, u_n, v_n) v_n \right) dx \\ &\geq \left(1 - \frac{1}{p^+}\right)\|u_n\|_{p(x)}^{p^-} + \left(1 - \frac{1}{q^+}\right)\|v_n\|_{q(x)}^{q^-}. \end{aligned} \tag{3.29}$$

Thus, the sequence (u_n, v_n) is bounded in $W_{p(x),q(x)}$. Then, there exists a subsequence (we denote again as (u_n, v_n)) which converges weakly in $W_{p(x),q(x)}$.

We recall the elementary inequalities:

$$2^{2-p}|a-b|^p \leq (a|a|^{p-2} - b|b|^{p-2})\cdot(a-b) \quad \text{if } p \geq 2, \tag{3.30}$$

$$(p-1)|a-b|^2(|a|+|b|)^{p-2} \leq (a|a|^{p-2} - b|b|^{p-2})\cdot(a-b) \quad \text{if } 1 < p < 2, \tag{3.31}$$

for all $a, b \in \mathbb{R}^N$, where $\cdot$ denotes the standard inner product in $\mathbb{R}^N$. We will show that (u_n, v_n) contains a Cauchy subsequence. Let us define the sets

$$\begin{aligned} \mathsf{U}_p &= \{x \in \mathbb{R}^N : p(x) \geq 2\}, \qquad \mathsf{V}_p = \{x \in \mathbb{R}^N : 1 < p(x) < 2\}, \\ \mathsf{U}_q &= \{x \in \mathbb{R}^N : q(x) \geq 2\}, \qquad \mathsf{V}_q = \{x \in \mathbb{R}^N : 1 < q(x) < 2\}. \end{aligned} \tag{3.32}$$

For all $x \in \mathbb{R}^N$, we put

$$\begin{aligned} \Phi_{n,k} &= (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_k|^{p(x)-2}\nabla u_k)\cdot(\nabla u_n - \nabla u_k), \\ \Psi_{n,k} &= (|\nabla u_n| + |\nabla u_k|)^{2-p(x)}. \end{aligned} \tag{3.33}$$

Therefore for $p(x) \geq 2$, using (3.30), we have

$$
\begin{aligned}
2^{2-p^+}\int_{U_p}|\nabla u_n-\nabla u_k|^{p(x)}dx &\leq \int_{U_p}[(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u_k|^{p(x)-2}\nabla u_k)\cdot(\nabla u_n-\nabla u_k)]dx \\
&\leq \int_{\mathbb{R}^N}\Phi_{n,k}\,dx := T_{n,k} \\
&= (D_1J(u_n,v_n)-D_1J(u_k,v_k)+D_1\digamma(u_n,v_n)-D_1\digamma(u_k,v_k))(u_n-u_k) \\
&\leq \left(\|D_1J(u_n,v_n)\|_{*,p(x),q(x)}+\|D_1J(u_k,v_k)\|_{*,p(x),q(x)}\right)\|u_n-u_k\|_{p(x)} \\
&\quad+\|D_1\digamma(u_n,v_n)-D_1\digamma(u_k,v_k)\|_{*,p(x),q(x)}\|u_n-u_k\|_{p(x)}.
\end{aligned}
\tag{3.34}
$$

When $1 < p(x) < 2$, employing (3.31) and Proposition 2.2, it follows

$$
\begin{aligned}
\int_{V_p}|\nabla u_n-\nabla u_k|^{p(x)}dx &\leq \int_{V_p}|\nabla u_n-\nabla u_k|^{p(x)}(|\nabla u_n|+|\nabla u_k|)^{p(x)(p(x)-2)/2}(|\nabla u_n|+|\nabla u_k|)^{p(x)(2-p(x))/2}dx \\
&\leq 2\left||\nabla u_n-\nabla u_k|^{p(x)}.\Psi_{n,k}^{-p(x)/2}\right|_{2/p(x)}\times\left|\Psi_{n,k}^{p(x)/2}\right|_{2/(2-p(x))} \\
&\leq 2\max\left\{\left(\int_{\mathbb{R}^N}|\nabla u_n-\nabla u_k|^2\Psi_{n,k}^{-1}dx\right)^{p^-/2},\left(\int_{\mathbb{R}^N}|\nabla u_n-\nabla u_k|^2\Psi_{n,k}^{-1}\,dx\right)^{p^+/2}\right\} \\
&\quad\times\max\left\{\left(\int_{\mathbb{R}^N}\Psi_{n,k}^{p(x)/(2-p(x))}dx\right)^{(2-p^-)/2},\left(\int_{\mathbb{R}^N}\Psi_{n,k}^{p(x)/(2-p(x))}dx\right)^{(2-p^+)/2}\right\} \\
&\leq 2\max\left\{(p^- -1)^{-p^-/2}\cdot T_{n,k}^{p^-/2},(p^- -1)^{-p^+/2}\cdot T_{n,k}^{p^+/2}\right\} \\
&\quad\times\max\left\{\left(\int_{\mathbb{R}^N}\Psi_{n,k}^{p(x)/(2-p(x))}dx\right)^{(2-p^-)/2},\left(\int_{\mathbb{R}^N}\Psi_{n,k}^{p(x)/(2-p(x))}dx\right)^{(2-p^+)/2}\right\}.
\end{aligned}
\tag{3.35}
$$

Since $T_{n,k}$ is uniformly bounded in $W_0^{1,p(x)}(\mathbb{R}^N)$ in accordance with n, k, and by the fact that $\|J'(u_m,v_m)\|_{*,p(x),q(x)} \to 0$ as $m \to +\infty$, $\digamma'$ is compact and by Proposition 2.4, we have

$$
\lim_{n,k\to+\infty}\int_{\mathbb{R}^N}|\nabla u_n-\nabla u_k|^{p(x)}dx = 0. \tag{3.36}
$$

Applying the same arguments, we can find a subsequence of (u_n, v_n) such that

$$
\lim_{n,k\to+\infty}\int_{\mathbb{R}^N}|\nabla v_n-\nabla v_k|^{q(x)}dx = 0. \tag{3.37}
$$

Therefore by Proposition 2.2, for a convenient subsequence, we have

$$
\lim_{n,k\to+\infty}\|(u_n,v_n)-(u_k,v_k)\|_{*,p(x),q(x)} = 0. \tag{3.38}
$$

Hence, (u_n, v_n) contains a Cauchy subsequence and so contains a strongly convergent subsequence. The proof is complete. □

Lemma 3.4. *Under the assumptions* $(F1)$–$(F4)$, *the functional* J *satisfies the following.*

(i) *There exists* $\rho, \sigma > 0$ *such that* $\|u\|_{p(x)} + \|v\|_{q(x)} = \rho$ *implies* $J(u,v) \geq \sigma > 0$.

(ii) *There exists* $E \in W_{p(x),q(x)}$ *such that* $\|E\|_{p(x),q(x)} > \rho$ *and* $J(E) \leq 0$.

Proof. By $(F4)$, we can find $\rho > 0$ such that $\|u\|_{p(x)} + \|v\|_{q(x)} = \rho$, so we have

$$\begin{gathered} F(x,u,v) < \lambda_1\left(\frac{a(x)}{p(x)}|u|^{p(x)} + \frac{b(x)}{q(x)}|v|^{q(x)}\right), \\ \int_{\mathbb{R}^N} F(x,u,v) < \lambda_1 \int_{\mathbb{R}^N}\left(\frac{a(x)}{p(x)}|u|^{p(x)} + \frac{b(x)}{q(x)}|v|^{q(x)}\right)dx, \end{gathered} \tag{3.39}$$

since $\lambda_1 > 0$, then we have

$$\begin{gathered} \int_{\mathbb{R}^N} F(x,u,v) < \int_{\mathbb{R}^N}\left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{q(x)}|\nabla v|^{q(x)}\right)dx, \\ 0 < \int_{\mathbb{R}^N}\left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{q(x)}|\nabla v|^{q(x)}\right)dx - \int_{\mathbb{R}^N} F(x,u,v) = J(u,v). \end{gathered} \tag{3.40}$$

Hence, there exists $\sigma > 0$ such that $J \geq \sigma > 0$.

Let (τ, θ) be an eigenfunction relative to λ_1. Then, using the assumption $(F4)$, we can obtain for $\epsilon > 0$ and t sufficiently large,

$$F\left(x, t^{1/p(x)}\tau, t^{1/q(x)}\theta\right) \geq t(\lambda_1 + \epsilon)\left(\frac{a(x)}{p(x)}|\tau|^{p(x)} + \frac{b(x)}{q(x)}|\theta|^{q(x)}\right). \tag{3.41}$$

Thus,

$$\begin{aligned} J\left(t^{1/p(x)}\tau, t^{1/q(x)}\theta\right) &= t\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}|\nabla \tau|^{p(x)} + \frac{1}{q(x)}|\nabla \theta|^{q(x)}\right)dx \\ &\quad - \int_{\mathbb{R}^N} F\left(x, t^{1/p(x)}\tau, t^{1/q(x)}\theta\right)dx \\ &\leq \int_{\mathbb{R}^N}\left(\frac{1}{p(x)}|\nabla \tau|^{p(x)} + \frac{1}{q(x)}|\nabla \theta|^{q(x)}\right)dx \\ &\quad - t(\lambda_1 + \epsilon)\left(\int_{\mathbb{R}^N}\frac{a(x)}{p(x)}|\tau|^{p(x)}dx + \int_{\mathbb{R}^N}\frac{b(x)}{q(x)}|\theta|^{q(x)}dx\right), \end{aligned} \tag{3.42}$$

then it follows

$$J\left(t^{1/p(x)}\tau, t^{1/q(x)}\theta\right) \leq -\epsilon t\left(\frac{1}{p^+}\int_{\mathbb{R}^N} a(x)|\tau|^{p(x)}dx + \frac{1}{q^+}\int_{\mathbb{R}^N} b(x)|\theta|^{q(x)}dx\right). \tag{3.43}$$

So, we can conclude that $\lim_{t\to+\infty} J(t^{1/p(x)}\tau, t^{1/q(x)}\theta) = -\infty$. Hence, for t sufficiently large, $J(t^{1/p(x)}\tau, t^{1/q(x)}\theta) \leq 0$. As a consequence, we can say that the functional $J(u,v)$ has a critical point; and as we know, the critical points of $J(u,v)$ are the weak solutions of the system (P,Q). □

Theorem 3.5. *The system* (P,Q) *has at least one nontrivial solution* (u,v).

Proof. By Lemmas 3.3 and 3.4, we can apply the mountain pass theorem to obtain that the system (P,Q) has a nontrivial weak solution. □

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