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Research Article

Boundedness on Hardy-Sobolev Spaces for Hypersingular Marcinkiewicz Integrals with Variable Kernels

Xiangxing Tao,[1] Xiao Yu,[1,2] and Songyan Zhang[1]

[1] *Department of Mathematics, Faculty of Science, Ningbo University, Ningbo, Zhejiang Province 315211, China*
[2] *Department of Mathematics, Faculty of Science, Zhejiang University, Hangzhou, Zhejiang Province 315211, China*

Correspondence should be addressed to Songyan Zhang, zhangsongyan@nbu.edu.cn

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The existence and boundedness on Sobolev spaces and Hardy-Sobolev spaces for the hypersingular Marcinkiewicz integrals with variable kernels are derived.

1. Introduction

The function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$, if it satisfies the following two conditions:

(1) $\Omega(x,\lambda z) = \Omega(x,z)$, for any $x, z \in \mathbb{R}^n$ and any $\lambda > 0$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})} =: \sup_{r\geq 0,\, y\in\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |\Omega(rz'+y,z')|^q d\sigma(z')\right)^{1/q} < \infty$.

Let $\alpha \geq 0$ and $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, and let $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ satisfy the following cancellation property:

$$\int_{\mathbb{S}^{n-1}} \Omega(x,z') Y_m(z') d\sigma(z') = 0, \tag{1.1}$$

for all spherical harmonic polynomials $Y_m(z')$ with degree $\leq [\alpha]$ and for any $x \in \mathbb{R}^n$. We consider the hypersingular Marcinkiewicz integral $\mu_{\Omega,\alpha} f(x)$ defined by

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^{3+2\alpha}} \right)^{1/2}, \tag{1.2}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz class. In general, the way in which the integrals were given has to be interpreted. Let $\mathscr{H} = \{h(t) : \|h\|_{\mathscr{H}} = (\int_0^\infty |h(t)|^2(dt/t))^{1/2} < +\infty\}$, and let $h_f(t,x) = t^{-1-\alpha}\int_{|x-y|\le t}(\Omega(x,x-y)/|x-y|^{n-1})f(y)dy$. We have to show $h_f(\cdot,x) \in \mathscr{H}$ for every $x \in \mathbb{R}^n$ and $f \in \mathcal{S}$, and so that $\mu_{\Omega,\alpha}(f)(x) = \|h_f(\cdot,x)\|_{\mathscr{H}}$ can be regarded as a well-defined Hilbert-valued function. To see this, applying the Taylor's expansion, we can write

$$\begin{aligned} f(x-y) &= \sum_{|\kappa|\le[\alpha]} C_\kappa y^\kappa (D^\kappa f)(x) + \sum_{|\kappa|=[\alpha]+1} C_\kappa y^\kappa \int_0^1 (1-s)^{[\alpha]} (D^\kappa f)(x-sy)ds \\ &=: \sum_{|\kappa|\le[\alpha]} a_\kappa(x) y^\kappa + b(x,y)|y|^{[\alpha]+1} \end{aligned} \tag{1.3}$$

with some bounded functions $a_\kappa(x)$ and $b(x,y)$, where κ denotes the multi-indices. By cancellation property (1.1) of Ω, we have

$$\begin{aligned} h_f(t,x) &= t^{-1-\alpha} \int_0^t \int_{\mathbb{S}^{n-1}} \Omega(x,y') f(x-sy') d\sigma(y') ds \\ &= t^{-1-\alpha} \int_0^t s^{[\alpha]+1} \int_{\mathbb{S}^{n-1}} \Omega(x,y') b(x,sy') d\sigma(y') ds \\ &\le C t^{1+[\alpha]-\alpha}. \end{aligned} \tag{1.4}$$

Simultaneously, by choosing r so that $1 < r < q$ and r sufficiently closing to 1, the Hölder inequality for integrals implies

$$h_f(t,x) \le t^{-1-\alpha} \|f\|_{L^{r'}} \left(\int_{|y|\le t} \frac{|\Omega(x,y)|^r}{|y|^{(n-1)r}} dy \right)^{1/r} \le C t^{-\alpha-n+(n/r)}, \tag{1.5}$$

where $1/r + 1/r' = 1$. Hence, noting $1 + [\alpha] - \alpha > 0$ and $-\alpha - n + (n/r) < 0$, one can easily check that $\|h_f(\cdot,x)\|_{\mathscr{H}} \le C$ for any test function f and every $x \in \mathbb{R}^n$, and so $\mu_{\Omega,\alpha}(f)(x)$ is well defined for $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \ge 0$.

Historically, as high dimension corresponding to the Littlewood-Paley g-function, Stein [1] first introduced the classical Marcinkiewicz integral with convolution kernel $\Omega(x,z) \equiv \Omega(z)$ and in the special case $\alpha = 0$. The $L^p(H^p)$ boundedness, $1 < p \le 2$, or $p \le 1$ but is close to 1, for the classical Marcinkiewicz integrals was extended by many authors to various kernels, for example, [1–4]. On the other hand, because of connecting with the problems about the partial differential equations with variable coefficients, more attentions are focused on the variable kernels.

A fresh look at this problem is due to Ding et al. [5–7], where they showed the following theorem.

Theorem 1.1 (see [7]). *Let $\alpha = 0$, and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$, $q > 2(n-1)/n$, satisfy cancellation property* (1.1). *Then, the operator $\mu_{\Omega,0}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Moreover, if Ω satisfies the L^1-Dini condition, then $\mu_{\Omega,0}$ is bounded from Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

Our main results are as follows.

Theorem 1.2. *Let $\alpha > 0$, and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$, $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, satisfy cancellation property* (1.1). *Then,*

$$\left\|\mu_{\Omega,\alpha}(f)\right\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2_\alpha(\mathbb{R}^n)} \tag{1.6}$$

with the constant C independent of any $f \in L^2_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$.

Theorem 1.3. *Let $\alpha > 0$, and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$, $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, satisfy cancellation property* (1.1). *Then, for $n/(n+\alpha) < p \leq 1$,*

$$\left\|\mu_{\Omega,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p_\alpha(\mathbb{R}^n)} \tag{1.7}$$

with the constant C independent of any $f \in H^p_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$.

By Theorems 1.2 and 1.3 and applying the interpolation theorem of sublinear operator, we obtain the $L^p - L^p_\alpha$ boundedness of $\mu_{\Omega,\alpha}$.

Corollary 1.4. *Let $\alpha > 0$, and $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$, $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, satisfy cancellation property* (1.1). *Then,*

$$\left\|\mu_{\Omega,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p_\alpha(\mathbb{R}^n)} \quad \text{for } 1 < p \leq 2 \tag{1.8}$$

with the constant C independent of any $f \in L^p_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$.

Further, we can derive the boundedness of $\mu_{\Omega,\alpha}$ on H^p_α for some $p \leq n/(n+\alpha)$ under an additional assumption, the $L^{1,\beta}$-Dini condition, on Ω as follows:

$$\int_0^1 \frac{\varpi(\delta)}{\delta^{1+\beta}} d\delta < \infty, \tag{1.9}$$

where $\varpi(\delta)$ is the integral modulus of continuity of Ω defined by

$$\varpi(\delta) = \sup_{r>0, |O|<\delta} \int_{\mathbb{S}^{n-1}} \left|\Omega(rz', Oz') - \Omega(rz', z')\right| d\sigma(z'), \tag{1.10}$$

where O is a rotation in $\mathbb{R}^n$ with $|O| = \|O - I\|$, and I is an identical operator.

Theorem 1.5. *Let* $\Omega(x,z') \in L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})$, $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, *satisfy cancellation property* (1.1) *and the* $L^{1,\beta}$*-Dini condition* (1.9) *with* $\beta \geq 0$. *Then, for* $\max\{n/(n+\alpha+\beta), 2n/(2n+2\alpha+1)\} < p \leq n/(n+\alpha)$, $0<\alpha<n/2$, *one has*

$$\left\|\mu_{\Omega,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p_\alpha(\mathbb{R}^n)} \tag{1.11}$$

with the constant C independent of any $f \in H^p_\alpha(\mathbb{R}^n)\cap\mathcal{S}(\mathbb{R}^n)$.

We will give the definitions for Sobolev spaces and Hardy-Sobolev spaces, and prove the key lemma (Lemma 2.2) for Bessel functions in the next section. The proof of Theorem 1.2 will be given in Section 3, and Theorems 1.3 and 1.5 will be proved in Section 4. Finally, as an application of Theorem 1.2, in Section 5 we will get the $L^2_\alpha - L^2$ boundedness for a class of the Littlewood-Paley type operators $\mu_{\Omega,\alpha,S}$ and $\mu^*_{\Omega,\alpha,\lambda}$ with variable kernels and index $\alpha \geq 0$, which relate to the Lusin area integral and the Littlewood-Paley g^*_λ function, respectively.

Here we point out that, by the same arguments in the paper, our theorems are also hold for the following hypersingular parametric Marcinkiewicz integrals

$$\mu^\rho_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty\left|\frac{1}{t^\rho}\int_{|x-y|\leq t}\frac{\Omega(x,x-y)}{|x-y|^{n-\rho}}f(y)dy\right|^2\frac{dt}{t^{1+2\alpha}}\right)^{1/2} \tag{1.12}$$

with the complex parameter ρ with $\mathrm{Re}(\rho)>0$.

Throughout the paper, C always denotes a positive constant not necessarily the same at each occurrence. we use $a \cong b$ to mean the equivalence of a and b; that is, there exists a positive constant C independent of a, b such that $C^{-1}a \leq b \leq Ca$.

2. Some notations and lemmas

Let us first recall the Triebel-Lizorkin space. Fix a radial function $\varphi(x) \in C^\infty$ satisfying $\mathrm{supp}(\varphi) \subseteq \{x : (1/2) < |x| \leq 2\}$ and $0 \leq \varphi(x) \leq 1$, and $\varphi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$. Let $\varphi_j(x) = \varphi(2^j x)$. Define the function $\psi_j(x)$ by $\widehat{\psi}_j(\xi) = \varphi_j(\xi)$. For $0<p$, $q<\infty$ and $\alpha\in\mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha,q}_p$ is the set of all distributions f satisfying

$$\dot{F}^{\alpha,q}_p(\mathbb{R}^n) = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}^{\alpha,q}_p} = \left\|\left(\sum_k\left|2^{-\alpha k}\psi_k * f\right|^q\right)^{1/q}\right\|_p < \infty\right\}. \tag{2.1}$$

For $1<p<\infty$, the homogeneous Sobolev spaces $L^p_\alpha(\mathbb{R}^n)$ are defined by $L^p_\alpha(\mathbb{R}^n) = \dot{F}^{\alpha,2}_p(\mathbb{R}^n)$, namely, $\|f\|_{L^p_\alpha} = \|f\|_{\dot{F}^{\alpha,2}_p}$. From [8], we know that for any $f \in L^2_\alpha(\mathbb{R}^n)$,

$$\|f\|_{L^2_\alpha(\mathbb{R}^n)} \cong \left(\int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|^2|\xi|^{2\alpha}d\xi\right)^{1/2}, \tag{2.2}$$

and if α is a nonnegative integer, then for any $f \in L^p_\alpha(\mathbb{R}^n)$,

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} \cong \sum_{|\tau|=\alpha}\left\|D^\tau f\right\|_{L^p(\mathbb{R}^n)}. \tag{2.3}$$

For $0 < p \leq 1$, we define the homogeneous Hardy-Sobolev space $H_\alpha^p(\mathbb{R}^n)$ by $H_\alpha^p(\mathbb{R}^n) = \dot{F}_p^{\alpha,2}(\mathbb{R}^n)$. It is well known that $H^p(\mathbb{R}^n) = \dot{F}_p^{0,2}(\mathbb{R}^n)$ for $0 < p \leq 1$, one can refer to [8] for the details.

Next we give two lemmas, which play important roles in the paper and are valuable in other applications of the Bessel functions.

Lemma 2.1 (see [9]). *Suppose $n \geq 2$ and $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ has the form $f(x) = f_0(|x|)P(x)$, where $P(x)$ is a solid spherical harmonic polynomial of degree m. Then, the Fourier transform of f has the form $\widehat{f}(\xi) = F_0(|\xi|)P(\xi)$, where*

$$F_0(r) = 2\pi i^{-m} r^{-((n+2m-2)/2)} \int_0^\infty f_0(s) J_{(n+2m-2)/2}(2\pi r s) s^{(n+2m)/2} ds, \tag{2.4}$$

and $r = |\xi|$, $J_m(s)$ is the Bessel function.

Lemma 2.2. *For $\lambda \geq 0$, there exists a constant $C > 0$ depending only on λ such that*

$$\left| \int_0^t \frac{J_{m+\lambda}(\rho)}{\rho^\lambda} d\rho \right| \leq \frac{C}{m^\lambda} \quad \text{for } 0 < t < \infty,\ m = 1, 2, \ldots. \tag{2.5}$$

Proof. Let us write $\nu = m + \lambda$. In case $0 < t \leq \nu$, since $J_\nu(\rho) > 0$ for $0 < \rho < \nu$, it follows from [10, inequality (6.1)] that

$$\left| \int_0^t \frac{J_\nu(\rho)}{\rho^\lambda} d\rho \right| \leq \nu \left| \int_0^t \frac{J_\nu(\rho)}{\rho^{1+\lambda}} d\rho \right| \leq \frac{C\nu}{m^{1+\lambda}} \leq \frac{C}{m^\lambda}. \tag{2.6}$$

In case $\nu < t \leq 2\nu$, the second mean value theorem and [10, inequality (6.2)] yield

$$\begin{aligned} \int_\nu^t \frac{J_\nu(\rho)}{\rho^\lambda} d\rho &= \nu^{-\lambda} \int_\nu^{h'} J_\nu(\rho) d\rho \quad (\nu < h' \leq t < 2\nu) \\ &= \nu^{-\lambda+1} \int_1^h J_\nu(\nu\rho) d\rho \quad (1 < h < 2) \\ &= O(\nu^{-\lambda}), \end{aligned} \tag{2.7}$$

where the big oh is an absolute one. Thus,

$$\left| \int_0^t \frac{J_\nu(\rho)}{\rho^\lambda} d\rho \right| \leq \left| \int_0^\nu \frac{J_\nu(\rho)}{\rho^\lambda} d\rho \right| + \left| \int_\nu^t \frac{J_\nu(\rho)}{\rho^\lambda} d\rho \right| \leq \frac{C}{m^\lambda}. \tag{2.8}$$

In case $t > 2\nu$, we use the differential equation of J_ν which may be written as (see [10, page 221])

$$J_\nu(\rho) = -\frac{\rho J_\nu'(\rho)}{\rho^2 - \nu^2} - \frac{\rho^2 J_\nu''(\rho)}{\rho^2 - \nu^2}, \tag{2.9}$$

which implies

$$\left|\int_{2\nu}^{t}\frac{J_\nu(\rho)}{\rho^\lambda}d\rho\right| \le \left|\int_{2\nu}^{t}\frac{J'_\nu(\rho)}{\rho^{\lambda-1}(\rho^2-\nu^2)}d\rho\right| + \left|\int_{2\nu}^{t}\frac{J''_\nu(\rho)}{\rho^{\lambda-2}(\rho^2-\nu^2)}d\rho\right| =: I_1 + I_2. \tag{2.10}$$

Since $|J'_\nu(\rho)| \le 1$ and that $[\rho^\lambda(1-(\nu/\rho)^2)]^{-1}$ is a decreasing function for $\lambda \ge 0$ and $\rho \ge 2\nu$, we obtain by the second mean value theorem

$$\begin{aligned} I_2 &\le \frac{4}{3(2\nu)^\lambda}\left|\int_{2\nu}^{t_1}J''_\nu(\rho)d\rho\right| \qquad (2\nu \le t_1 \le t) \\ &\le \frac{4}{3(2\nu)^\lambda}\left|J'_\nu(t_1)-J'_\nu(2\nu)\right| \le \frac{C}{\nu^\lambda}. \end{aligned} \tag{2.11}$$

To estimate I_1, if $\lambda > 0$, one can easily see that

$$I_1 \le \int_{2\nu}^{t}\frac{d\rho}{\rho^{\lambda-1}(\rho^2-\nu^2)} \le \frac{4}{3}\int_{2\nu}^{t}\frac{d\rho}{\rho^{1+\lambda}} \le \frac{C}{\nu^\lambda}. \tag{2.12}$$

If $\lambda = 0$, then $\nu = m$ is an integer. Since $|J_m(\rho)| \le 1$ (see [9, page 137]), and $[\rho(1-(\nu/\rho)^2)]^{-1}$ is decreasing for $\rho \ge 2\nu$, we apply the second mean value theorem again to obtain

$$\begin{aligned} I_1 &= \left|\int_{2\nu}^{t}\frac{J'_m(\rho)}{\rho^{-1}(\rho^2-\nu^2)}d\rho\right| \\ &\le \frac{2}{3\nu}\left|\int_{2\nu}^{t_2}J'_m(\rho)d\rho\right| \qquad (2\nu \le t_2 \le t) \\ &\le \frac{2}{3\nu}\left|J_\nu(t_2)-J_\nu(2\nu)\right| \le C. \end{aligned} \tag{2.13}$$

Now we have gotten

$$\left|\int_{0}^{t}\frac{J_\nu(\rho)}{\rho^\lambda}d\rho\right| \le \left|\int_{0}^{2\nu}\frac{J_\nu(\rho)}{\rho^\lambda}d\rho\right| + \left|\int_{2\nu}^{t}\frac{J_\nu(\rho)}{\rho^\lambda}d\rho\right| \le \frac{C}{m^\lambda}, \tag{2.14}$$

which implies the lemma. □

3. The boundedness on Sobolev spaces

In this section, we derive the boundedness for the Marcinkiewicz integral $\mu_{\Omega,\alpha}$ on Sobolev spaces, and give the proof of Theorem 1.2.

As in [9], let $\{Y_{m,1}, Y_{m,2}, \ldots, Y_{m,N(m)}\}$ denote an orthonormal basis of the space of normalized spherical harmonics of degree m, then by using (1.1) and the spherical harmonic development, see [11] for instance, we can decompose $\Omega(x, z')$ as follows:

$$\Omega(x, z') = \sum_{m=[\alpha]+1}^{+\infty} \sum_{j=1}^{N(m)} a_{m,j}(x) Y_{m,j}(z'), \tag{3.1}$$

where $N(m) \cong m^{n-2}$ and

$$a_{m,j}(x) = \int_{\mathbb{S}^{n-1}} \Omega(x, z') \overline{Y_{m,j}(z')} d\sigma(z'). \tag{3.2}$$

Denote

$$a_m(x) = \left(\sum_{j=1}^{N(m)} \left| a_{m,j}(x) \right|^2 \right)^{1/2}, \qquad b_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}, \tag{3.3}$$

then $\sum_{j=1}^{N(m)} b_{m,j}^2(x) = 1$. We also let

$$\mu_{m,j,\alpha}(f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \le t} \frac{Y_{m,j}(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^{3+2\alpha}} \right)^{1/2}. \tag{3.4}$$

Then, applying the Hölder inequality twice, we have for any $0 < \varepsilon < 1$

$$\begin{aligned} \left| \mu_{\Omega,\alpha}(f)(x) \right|^2 &= \int_0^\infty \left| \int_{|x-y| \le t} \sum_{m=[\alpha]+1}^{+\infty} a_m(x) \sum_{j=1}^{N(m)} b_{m,j}(x) \frac{Y_{m,j}(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\ &\le \left(\sum_{m=[\alpha]+1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=[\alpha]+1}^{+\infty} m^{\varepsilon(1+2\alpha)} \\ &\quad \cdot \int_0^{+\infty} \left| \int_{|x-y| \le t} \sum_{j=1}^{N(m)} b_{m,j}(x) \frac{Y_{m,j}(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\ &\le \left(\sum_{m=[\alpha]+1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=[\alpha]+1}^{+\infty} m^{\varepsilon(1+2\alpha)} \\ &\quad \cdot \int_0^{+\infty} \sum_{j=1}^{N(m)} \left| \int_{|x-y| \le t} \frac{Y_{m,j}(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^{3+2\alpha}} \\ &= \left(\sum_{m=[\alpha]+1}^{+\infty} a_m^2(x) m^{-\varepsilon(1+2\alpha)} \right) \sum_{m=[\alpha]+1}^{+\infty} m^{\varepsilon(1+2\alpha)} \sum_{j=1}^{N(m)} \left| \mu_{m,j,\alpha}(f)(x) \right|^2. \end{aligned} \tag{3.5}$$

By [11], we can observe that the series in the first parenthesis on the right-hand side of the inequality above is equal to $\|\Omega(x,\cdot)\|^2_{L^2_{-\gamma}(\mathbb{S}^{n-1})}$, where $L^2_{-\gamma}(\mathbb{S}^{n-1})$ is the Sobolev space on $\mathbb{S}^{n-1}$ with $\gamma = \varepsilon(1/2+\alpha)$ for any $0<\varepsilon<1$. So if we take ε sufficiently close to 1, then by the Sobolev imbedding theorem $L^q \subset L^2_{-\gamma}$, we have

$$\sum_m a_m^2(x) m^{-\varepsilon(1+2\alpha)} \le C\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})} \tag{3.6}$$

with $q > \max\{1, 2(n-1)/(n+2\alpha)\}$. Therefore, to prove the theorem, it remains to show that for ε sufficiently close to 1, we have

$$\sum_{m=[\alpha]+1}^{+\infty} m^{\varepsilon(1+2\alpha)} \sum_{j=1}^{N(m)} \int_{\mathbb{R}^n} \left|\mu_{m,j,\alpha}(f)(x)\right|^2 dx \le C\|f\|^2_{L^2_\alpha}. \tag{3.7}$$

Put $P_{m,j}(x) = Y_{m,j}(x)|x|^m$ and $\varphi_{m,j,t}(x) = t^{-1-\alpha} P_{m,j}(x)|x|^{-n-m+1}\chi_{\{|x|\le t\}}$. One can deduce from Lemma 2.1 that

$$\widehat{\varphi_{m,j,t}}(\xi) = F_0(|\xi|)P_{m,j}(|\xi|) = Y_{m,j}(\xi')|\xi|^m F_0(|\xi|), \tag{3.8}$$

where, by the changing of variable,

$$F_0(r) = i^{-m}(2\pi)^{(n/2-1)} r^{-m-1} t^{-1-\alpha} \int_0^{2\pi r t} \frac{J_{m+(n-2)/2}(\rho)}{\rho^{(n-2)/2}} d\rho. \tag{3.9}$$

Hence by Plancherel theorem, we have

$$\begin{aligned}
\sum_{j=1}^{N(m)} \int_{\mathbb{R}^n} \left|\mu_{m,j,\alpha}(f)(x)\right|^2 dx &= \sum_{j=1}^{N(m)} \int_{\mathbb{R}^n} \int_0^{+\infty} \left| \int_{\mathbb{R}^n} \varphi_{m,j,t}(x-y) f(y) dy \right|^2 \frac{dt}{t} dx \\
&= \sum_{j=1}^{N(m)} \int_0^{+\infty} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi_{m,j,t}(x-y) f(y) dy \right|^2 dx \frac{dt}{t} \\
&= \sum_{j=1}^{N(m)} \int_0^{+\infty} \int_{\mathbb{R}^n} \left|Y_{m,j}(\xi')\right|^2 |\xi|^{2m} \left|\widehat{f}(\xi)\right|^2 \left|F_0(|\xi|)\right|^2 d\xi \frac{dt}{t}.
\end{aligned} \tag{3.10}$$

By [11], we know that $\sum_{j=1}^{N(m)}|Y_{m,j}(z')|^2 \cong m^{n-2}$, thus (3.10) is bounded by

$$\begin{aligned} &Cm^{n-2}\int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|^2\int_0^{+\infty}\left(\frac{1}{2\pi|\xi|t}\int_0^{2\pi|\xi|t}\frac{J_{m+(n-2)/2}(\rho)}{\rho^{(n-2)/2}}d\rho\right)^2\frac{dt}{t^{1+2\alpha}}d\xi \\ &\quad\le Cm^{n-2}\int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|^2|\xi|^{2\alpha}\int_0^{m/2}\left(\frac{1}{r}\int_0^r\frac{J_{m+(n-2)/2}(\rho)}{\rho^{(n-2)/2}}d\rho\right)^2\frac{dr}{r^{1+2\alpha}}d\xi \\ &\qquad+Cm^{n-2}\int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|^2|\xi|^{2\alpha}\int_{m/2}^{+\infty}\left(\frac{1}{r}\int_0^r\frac{J_{m+(n-2)/2}(\rho)}{\rho^{(n-2)/2}}d\rho\right)^2\frac{dr}{r^{1+2\alpha}}d\xi \\ &\quad=: I_1+I_2. \end{aligned} \tag{3.11}$$

Let $\eta(r)=(1/r)\int_0^r(J_{m+\lambda}(\rho)/\rho^\lambda)d\rho$. Using [12, Lemma 2.1] (the proof of Lemma 2.1(b) and the estimation in [12, page 565]), we have $\eta(r)\le J_{m+\lambda}(r)/r^\lambda$ and $|J_\nu(r)|\le(r/2)^\nu/\Gamma(\nu+1)$. Thus, by Stirling's formula, we have for $m\ge[\alpha]+1$ that

$$\begin{aligned} \int_0^{m/2}\eta(r)^2\frac{dr}{r^{1+2\alpha}} &\le \int_0^{m/2}\frac{r^{2m-1-2\alpha}}{\left(2^{m+\lambda}\Gamma(m+\lambda+1)\right)^2}dr \\ &\le C\left(\frac{(m/2)^{m+\alpha}}{2^m\Gamma(m+\lambda+1)}\right)^2 \\ &\le \frac{C}{m^{2\lambda+2-2\alpha}}. \end{aligned} \tag{3.12}$$

Now we let $\lambda=(n-2)/2$ and use the inequality above to see that

$$I_1\le Cm^{-2-2\alpha}\int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|^2|\xi|^{2\alpha}d\xi\le Cm^{-2-2\alpha}\|f\|_{L_\alpha^2}^2. \tag{3.13}$$

To estimate I_2, using Lemma 2.2 that $|\int_0^r(J_{m+\lambda}(t)/t^\lambda)dt|\le C/m^\lambda$, we obtain

$$\begin{aligned} I_2 &\le Cm^{n-2}\int_{R^n}\left|\widehat{f}(\xi)\right|^2|\xi|^{2\alpha}\int_{m/2}^{+\infty}\left(\frac{C}{m^{(n-2)/2}}\right)^2\frac{dr}{r^{3+2\alpha}}d\xi \\ &\le C\int_{R^n}\left|\widehat{f}(\xi)\right|^2|\xi|^{2\alpha}m^{-2-2\alpha}d\xi \\ &\le Cm^{-2-2\alpha}\|f\|_{L_\alpha^2}^2. \end{aligned} \tag{3.14}$$

Combining the estimates of I_1 and I_2, we get from (3.10) and (3.11) that

$$\sum_{m=[\alpha]+1}^{+\infty}m^{\varepsilon(1+2\alpha)}\sum_{j=1}^{N(m)}\int_{\mathbb{R}^n}\left|\mu_{m,j,\alpha}(f)(x)\right|^2dx\le C\sum_{m=[\alpha]+1}^{+\infty}m^{\varepsilon(1+2\alpha)-2-2\alpha}\|f\|_{L_\alpha^2}^2, \tag{3.15}$$

which implies (3.7). Then, we complete the proof of the theorem.

4. The boundedness on Hardy-Sobolev spaces

In order to show the boundedness for the operator $\mu_{\Omega,\alpha}$ on Hardy-Sobolev spaces, and prove Theorems 1.3 and 1.5, we first recall the atomic decomposition for Hardy-Sobolev spaces.

Definition 4.1 (see [13]). For $\alpha \geq 0$, the function $a(x)$ is called a $(p, 2, \alpha)$ atom if it satisfies the following three conditions:

(1) $\text{supp}(a) \subset B$ with a ball $B \subset \mathbb{R}^n$;

(2) $\|a\|_{L^2_\alpha} \leq |B|^{1/2-1/p}$;

(3) $\int_{\mathbb{R}^n} a(x)P(x) = 0$, for any polynomial $P(x)$ of degree $\leq N = [n(1/p-1)\alpha]$.

By [13], we know that every $f \in H^p_\alpha(\mathbb{R}^n)$ can be written as a sum of $(p, 2, \alpha)$ atoms $a_j(x)$, that is, $f = \sum_j \lambda_j a_j$ which converges in $H^p_\alpha(\mathbb{R}^n)$ and in the sense of distributions. Moreover, $\|f\|^p_{H^p_\alpha} \cong \sum_j |\lambda_j|^p$.

To consider the hypersingular Marcinkiewicz integral $\mu_{\Omega,\alpha}(f)$ on Hardy-Sobolev spaces $H^p_\alpha(\mathbb{R}^n)$, we can start with f in a nice dense class of function, say $f \in H^p_\alpha(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$. Then, by the Lebesgue dominated convergence theorem, we have

$$\|h_f(\cdot, x)\|_{\mathscr{H}} \leq \sum_j |\lambda_j| \cdot \|h_{a_j}(\cdot, x)\|_{\mathscr{H}}. \tag{4.1}$$

Recalling $\mu_{\Omega,\alpha}(f)(x) = \|h_f(\cdot, x)\|_{\mathscr{H}}$, so we obtain for $0 < p \leq 1$

$$\|\mu_{\Omega,\alpha}(f)\|^p_{L^p(\mathbb{R}^n)} \leq \sum_j |\lambda_j|^p \|\mu_{\Omega,\alpha}(a_j)\|^p_{L^p(\mathbb{R}^n)}. \tag{4.2}$$

Now we are in the position to prove the following theorem.

Proof of Theorem 1.3. Because of (4.2), it is sufficient to show that

$$\|\mu_{\Omega,\alpha}(a)\|^p_{L^p} \leq C \tag{4.3}$$

with the constant C independent of any $(p, 2, \alpha)$ atom a. By translation argument, we can assume that $\text{supp}(a) \subset B(0, \rho)$. We first note that

$$\|\mu_{\Omega,\alpha}(a)\|^p_{L^p} \leq \int_{|x|\leq 8\rho} |\mu_{\Omega,\alpha}(a)|^p dx + \int_{|x|>8\rho} |\mu_{\Omega,\alpha}(a)|^p dx =: J_1 + J_2. \tag{4.4}$$

By Theorem 1.2 and size condition (2) of a, it is not difficult to deduce that

$$J_1 \leq C\|\mu_{\Omega,\alpha}(a)\|^p_{L^2}\rho^{n(1-p/2)} \leq C\|a\|^p_{L^2_\alpha}\rho^{n(1-p/2)} \leq C. \tag{4.5}$$

Next we estimate J_2 according to three cases.

Case 1. $0 < \alpha < n/2$. In this case, one may refer to [14, Lemma 5.5] to get

$$\int_B |a(x)|dx \le C\rho^{n-n/p+\alpha}. \tag{4.6}$$

Since $0 < p \le 1$, the Minkowski inequality and the Hölder inequality give that

$$\begin{aligned}
J_2 &= \int_{|x|>8\rho}\left(\int_0^{+\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-1}}a(y)dy\right|^2\frac{dt}{t^{3+2\alpha}}\right)^{p/2}dx\\
&\le \int_{|x|>8\rho}\left|\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}}|a(y)|dy\right|^p dx\\
&= \sum_{j=3}^{+\infty}\int_{2^j\rho<|x|<2^{j+1}\rho}\left|\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}}|a(y)|dy\right|^p dx\\
&\le \sum_{j=3}^{+\infty}\left(2^j\rho\right)^{n(1-p)}\left(\int_{2^j\rho<|x|<2^{j+1}\rho}\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}}|a(y)|dy\,dx\right)^p\\
&\le \sum_{j=3}^{+\infty}\left(2^j\rho\right)^{n(1-p)}\left(\int_B|a(y)|dy\int_{2^j\rho<|x|<2^{j+1}\rho}\frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}}dx\right)^p\\
&\le C\|\Omega\|^p_{L^\infty\times L^1}\left(\int_B|a(y)|dy\right)^p\cdot\sum_{j=3}^{+\infty}\left(2^j\rho\right)^{n(1-p)-\alpha p}.
\end{aligned} \tag{4.7}$$

Thus, by (4.6) and the condition $p > n/(n+\alpha)$, we get

$$J_2 \le C\|\Omega\|^p_{L^\infty\times L^1}\sum_{j=3}^{+\infty}2^{j(n-np-\alpha p)} \le C. \tag{4.8}$$

Case 2. $\alpha > n/2$ and $\alpha - n/2$ is not an integer. Thus there is an integer $k \ge 1$ such that $n/2 - 1 < \alpha - k < n/2$. By Pitt's theorem, one has the following.

Lemma 4.2 (see [15]). *Let $a(x)$ be a $(p,2,\alpha)$ atom with support in $B = B(0,\rho)$, β denotes a multi-index with $|\beta| = k$, then $\int_0^1\int_B|D^\beta a(sy)|dy\,ds \le C\rho^{n-n/p+\alpha-k}$.*

Now using Taylor's expansion on $a(y)$ at $y = 0$, and applying the cancellation property of Ω, we see that

$$h_a(t,x) = t^{-1-\alpha}\sum_{|\beta|=k}C_\beta\int_{\substack{|x-y|\le t\\|y|<\rho}}\frac{\Omega(x,x-y)}{|x-y|^{n-1}}\int_0^1(1-s)^{k-1}y^\beta D^\beta a(sy)ds\,dy, \tag{4.9}$$

and so, by the Minkowski inequality,

$$\mu_{\Omega,\alpha}(a)(x) \le C\rho^k \sum_{|\beta|=k} \int_{|y|<\rho} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\alpha}} \int_0^1 |D^\beta a(sy)| ds\, dy. \tag{4.10}$$

Hence, by the same argument as in (4.7) and using Lemma 4.2, we have

$$\begin{aligned} J_2 &\le C\rho^{kp} \|\Omega\|^p_{L^\infty \times L^1} \left(\int_B \sum_{|\beta|=k} \int_0^1 |D^\beta a(sy)| ds\, dy \right)^p \cdot \sum_{j=3}^{+\infty} (2^j \rho)^{n(1-p)-\alpha p} \\ &\le C\|\Omega\|^p_{L^\infty \times L^1} \sum_{j=3}^{+\infty} 2^{j(n-np-\alpha p)} \le C. \end{aligned} \tag{4.11}$$

Case 3. $\alpha \ge n/2$ and $\alpha - n/2$ is a nonnegative integer. In this case, we can take a small positive real $0 < \varepsilon < 1$ satisfying $n/(n+\alpha-\varepsilon) < p$ and $[\alpha+\varepsilon] = [\alpha]$, then by Cases 1 and 2, we have

$$\|\mu_{\Omega,\alpha-\varepsilon}(f)\|_{L^p} \le C\|f\|_{H^p_{\alpha-\varepsilon}}, \qquad \|\mu_{\Omega,\alpha+\varepsilon}(f)\|_{L^p} \le C\|f\|_{H^p_{\alpha+\varepsilon}}. \tag{4.12}$$

By interpolation, see [15], we get $\|\mu_{\Omega,\alpha}(f)\|_{L^p} \le C\|f\|_{H^p_\alpha}$. □

Next we are going to proof the following.

Proof of Theorem 1.5. By similar arguments, it is enough to show the uniform boundedness of $\|\mu_{\Omega,\alpha}(a)\|_{L^p}$ for any $(p,2,\alpha)$-atom a with the support $B(0,\rho)$. Write

$$\|\mu_{\Omega,\alpha}(a)\|^p_p = \int_{|x|\le 8\rho} |\mu_{\Omega,\alpha}(a)(x)|^p dx + \int_{|x|>8\rho} |\mu_{\Omega,\alpha}(a)(x)|^p dx =: W + V. \tag{4.13}$$

It is easy to see that $W \le C$. For V, we decompose it further into

$$\begin{aligned} V &\le \int_{|x|>8\rho} \left| \int_0^{+\infty} \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} a(y) dy \right|^2 \frac{dt}{t^{3+2\alpha}} \right|^{p/2} dx \\ &\le \int_{|x|>8\rho} \left| \int_0^{|x|+2\rho} \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} a(y) dy \right|^2 \frac{dt}{t^{3+2\alpha}} \right|^{p/2} dx \\ &\quad + \int_{|x|>8\rho} \left| \int_{|x|+2\rho}^{+\infty} \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} a(y) dy \right|^2 \frac{dt}{t^{3+2\alpha}} \right|^{p/2} dx \\ &=: V_1 + V_2. \end{aligned} \tag{4.14}$$

Note that for $|x| > 8\rho$, $|y| \le \rho$, we have $|x-y| \cong |x| \cong |x|+2\rho$, so the mean value theorem gives $|(|x|+2\rho)^{-2-2\alpha} - |x-y|^{-2-2\alpha}| \le C\rho|x-y|^{-3-2\alpha}$. Thus, by the Minkowski inequality for integrals,

$$V_1 \le C\rho^{p/2}\sum_{j=3}^{+\infty}\int_{2^j\rho\le|x|<2^{j+1}\rho}\left|\int_B \frac{|\Omega(x,x-y)|}{|x-y|^{n+1/2+\alpha}}|a(y)|dy\right|^p dx. \tag{4.15}$$

Now, like (4.7), we write V_1 as the following sum, and apply the Hölder inequality for integrals to obtain

$$\begin{aligned}
V_1 &\le C\rho^{p/2}\sum_{j=3}^{+\infty}(2^j\rho)^{n(1-p)}\left(\int_{2^j\rho\le|x|<2^{j+1}\rho}\left|\int_B \frac{|\Omega(x,x-y)|}{|x-y|^{n+1/2+\alpha}}|a(y)|dy\right|dx\right)^p \\
&\le C\rho^{p/2}\|\Omega\|^p_{L^\infty\times L^1}\sum_{j=3}^{+\infty}(2^j\rho)^{-(1/2+\alpha)p}(2^j\rho)^{n(1-p)}\left(\int_B|a(y)|dy\right)^p \\
&\le C\|\Omega\|^p_{L^\infty\times L^1}\sum_{j=3}^{+\infty}(2^j)^{-(1/2+\alpha)p+n(1-p)} \le C,
\end{aligned} \tag{4.16}$$

because of $p > n/(n+\alpha+1/2)$ and the inequality (4.6).

To deal with V_2, we note that $B(0,\rho) \subset \{y : |x-y| \le t\}$ for $t \ge |x|+2\rho$ and $|x| > 8\rho$. So by the cancelation of the atom $a(x)$, we have

$$\begin{aligned}
V_2 &= \int_{|x|>8\rho}\left(\int_{|x|+2\rho}^{+\infty}\left|\int_{|x-y|\le t}\left(\frac{\Omega(x,x-y)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x-y|^{n-1}}\right.\right.\right. \\
&\qquad\left.\left.\left. + \frac{\Omega(x,x)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x|^{n-1}}\right)\cdot a(y)dy\right|^2\frac{dt}{t^{3+2\alpha}}\right)^{p/2}dx \\
&\le \int_{|x|>8\rho}\left|\int_{|x|+2\rho}^{+\infty}\left|\int_{|x-y|\le t}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x-y|^{n-1}}\right||a(y)|dy\right|^2\frac{dt}{t^{3+2\alpha}}\right|^{p/2}dx \\
&\quad + \int_{|x|>8\rho}\left|\int_{|x|+2\rho}^{+\infty}\left|\int_{|x-y|\le t}\left|\frac{\Omega(x,x)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x|^{n-1}}\right||a(y)|dy\right|^2\frac{dt}{t^{3+2\alpha}}\right|^{p/2}dx \\
&=: V_{21} + V_{22}.
\end{aligned} \tag{4.17}$$

Applying the Minkowski inequality, decomposition of integral domain, the Hölder inequality, and the Fubini theorem successively, we can obtain

$$\begin{aligned} V_{21} &\leq \sum_{j=3}^{+\infty} \int_{2^j\rho\leq|x|<2^{j+1}\rho} \left| \int_B \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^{n+\alpha}} |a(y)| dy \right|^p dx \\ &\leq \sum_{j=3}^{+\infty} (2^j\rho)^{n(1-p)} \left(\int_{2^j\rho\leq|x|<2^{j+1}\rho} \int_B \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^{n+\alpha}} |a(y)| dy\, dx \right)^p \\ &= \sum_{j=3}^{+\infty} (2^j\rho)^{n(1-p)} \left(\int_B |a(y)| \left(\int_{2^j\rho\leq|x|<2^{j+1}\rho} \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^{n+\alpha}} dx \right) dy \right)^p. \end{aligned} \tag{4.18}$$

Since $|y| < \rho$ and $|x| > 2\rho$, we have $|x-y| \geq (1/2)|x|$ and so $|(x-y)/|x-y|-x/|x|| \leq 4(|y|/|x|)$. Thus by using the $L^{1,\beta}$-Dini condition of Ω, the inner integral on the right-hand side of the integral inequality above is controlled by

$$\begin{aligned} &\int_{2^j\rho}^{2^{j+1}\rho} \int_{\mathbb{S}^{n-1}} \left| \Omega\left(rx', \frac{rx'-y}{|rx'-y|} \right) - \Omega(rx',x') \right| d\sigma(x') r^{-1-\alpha} dr \leq C \int_{2^j\rho}^{2^{j+1}\rho} r^{-1-\alpha} \varpi_1\left(\frac{4|y|}{r}\right) dr \\ &\quad\leq C \int_{4|y|/2^{j+1}\rho}^{4|y|/2^j\rho} |y|^{-\alpha} \frac{\varpi_1(\delta)}{\delta^{1+\beta}} \delta^{\alpha+\beta} d\delta \\ &\quad\leq C 2^{-j(\alpha+\beta)} \rho^{-\alpha} \int_0^1 \frac{\varpi_1(\delta)}{\delta^{1+\beta}} d\delta \\ &\quad\leq C 2^{-j(\alpha+\beta)} \rho^{-\alpha}. \end{aligned} \tag{4.19}$$

From this, and using (4.6) and the condition $p > n/(n+\alpha+\beta)$, we can thus deduce that

$$\begin{aligned} V_{21} &\leq C \sum_{j=3}^{+\infty} (2^j\rho)^{n(1-p)} \left(\int_B |a(y)| 2^{-j(\alpha+\beta)} \rho^{-\alpha} dy \right)^p \\ &\leq C \sum_{j=3}^{+\infty} 2^{j(n(1-p)-(\alpha+\beta)p)} \leq C. \end{aligned} \tag{4.20}$$

Finally, we give the estimate of V_{22}. Obviously, for $|y| < \rho$ and $|x| > 2\rho$, we have $|x| \cong |x-y| \cong |x| + 2\rho$, and by mean value theorem, $|1/|x-y|^{n-1} - 1/|x|^{n-1}| \leq C(|y|/|x|^n)$. Therefore, by Minkowski's inequality for integrals, the inequality (4.6) and the condition

$n/(n+1+\alpha) < p < 1$, we can obtain

$$\begin{aligned} V_{22} &\le C\int_{|x|>8\rho}\left|\int_B |a(y)|\left|\frac{\Omega(x,x)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x|^{n-1}}\right|\frac{1}{|x|^{1+\alpha}}dy\right|^p dx \\ &\le C\int_{|x|>8\rho}\left|\int_B |a(y)|\frac{|\Omega(x,x)|\cdot|y|}{|x|^{n+1+\alpha}}dy\right|^p dx \\ &\le C\rho^p\left(\int_B |a(y)|dy\right)^p\int_{|x|>8\rho}\frac{|\Omega(x,x)|^p}{|x|^{(n+1+\alpha)p}}dx \\ &\le \rho^p\rho^{p(n-n/p+\alpha)}\rho^{-(n+1+\alpha)p+n}\|\Omega\|_{L^\infty\times L^p} \le C. \end{aligned} \tag{4.21}$$

Combining (4.13), (4.14), (4.16), (4.20), and (4.21), we have proved that

$$\left\|\mu_{\Omega,\alpha}(a)\right\|_{L^p}^p \le W + V_1 + V_{21} + V_{22} \le C, \tag{4.22}$$

which is our desired inequality. The proof of the theorem is finished. □

5. Final remark

By similar argument as that for $\mu_{\Omega,\alpha}$, we can consider the Littlewood-Paley type operators $\mu_{\Omega,\alpha,S}$ and $\mu^*_{\Omega,\alpha,\lambda}$ with variable kernels and index $\alpha \ge 0$, which relate to the Lusin area integral and the Littlewood-Paley g^*_λ function, as follows:

$$\begin{aligned} \mu_{\Omega,\alpha,S}(f)(x) &= \left(\iint_{\Gamma(x)}\left|\int_{|y-z|\le t}\frac{\Omega(y,y-z)}{|y-z|^{n-1}}f(z)dz\right|^2\frac{dy\,dt}{t^{n+3+2\alpha}}\right)^{1/2}, \\ \mu^*_{\Omega,\alpha,\lambda}(f)(x) &= \left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|\le t}\frac{\Omega(y,y-z)}{|y-z|^{n-1}}f(z)dz\right|^2\frac{dy\,dt}{t^{n+3+2\alpha}}\right)^{1/2} \end{aligned} \tag{5.1}$$

with $\lambda > 1$, where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

As an application of Theorem 1.2, we have the following conclusion.

Theorem 5.1. *Let $\alpha \ge 0$, and assume that $\Omega(x,z') \in L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})$, $q > \max\{1, 2(n-1)/(n+2\alpha)\}$, satisfy* (1.1). *Then,*

$$\left\|\mu_{\Omega,\alpha,S}(f)\right\|_{L^2(\mathbb{R}^n)} \le 2^{\lambda n}\left\|\mu^*_{\Omega,\alpha,\lambda}(f)\right\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2_\alpha(\mathbb{R}^n)} \tag{5.2}$$

with the constant C independent of any $f \in L^2_\alpha(\mathbb{R}^n)\cap\mathcal{S}(\mathbb{R}^n)$.

The proof of the theorem is based on the following lemma.

Lemma 5.2. *Let $\lambda > 1$, there exists a positive constant $C_{\lambda,n}$ such that for any nonnegative and locally integrable function φ,*

$$\int_{\mathbb{R}^n} \left(\mu^*_{\Omega,\alpha,\lambda}(f)(x)\right)^2 \varphi(x)dx \le C_{\lambda,n} \int_{\mathbb{R}^n} \left(\mu_{\Omega,\alpha}(f)(x)\right)^2 M(\varphi)(x)dx, \tag{5.3}$$

where M denotes the Hardy-Littlewood maximal operator.

Proof. Directly from the definition, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \left(\mu^*_{\Omega,\alpha,\lambda}(f)(x)\right)^2 \varphi(x)dx &= \int_{\mathbb{R}^n} \iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|F_{\Omega,t}(y)\right|^2 \frac{dy\,dt}{t^{n+3+2\alpha}} \varphi(x)dx \\ &\le \int_{\mathbb{R}^n} \int_0^\infty \left|F_{\Omega,t}(y)\right|^2 \frac{dt}{t^{3+2\alpha}} \left(\sup_{t>0} \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \varphi(x) \frac{1}{t^n} dx\right) dy \\ &\le C_{\lambda,n} \int_{\mathbb{R}^n} \left(\mu_{\Omega,\alpha}(f)(y)\right)^2 M(\varphi)(y)dy,\end{aligned} \tag{5.4}$$

which implies the lemma. □

Now, if we take the supremum in inequality (5.3) over all φ satisfying $\|\varphi\|_{L^\infty} \le 1$, and apply Theorem 1.2, we can see

$$\begin{aligned}\left\|\mu^*_{\Omega,\alpha,\lambda}(f)\right\|^2_{L^2(\mathbb{R}^n)} &\le C_{\lambda,n} \left(\sup_{\|\varphi\|_{L^\infty} \le 1} \left\|M(\varphi)\right\|_{L^\infty}\right) \int_{\mathbb{R}^n} \left(\mu_{\Omega,\alpha}(f)(x)\right)^2 dx \\ &\le C_{\lambda,n} \|f\|^2_{L^2_\alpha(\mathbb{R}^n)}.\end{aligned} \tag{5.5}$$

Moreover, by the observation that $\mu_{\Omega,\alpha,S}(f)(x) \le 2^{\lambda n} \mu^*_{\Omega,\alpha,\lambda}(f)(x)$, we have obtained Theorem 5.1.

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