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Research Article

Multiple Solutions for a Class of $p(x)$-Laplacian Systems

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We study the multiplicity of solutions for a class of Hamiltonian systems with the $p(x)$-Laplacian. Under suitable assumptions, we obtain a sequence of solutions associated with a sequence of positive energies going toward infinity.

1. Introduction and Main Results

Since the space $L^{p(x)}$ and $W^{1,p(x)}$ were thoroughly studied by Kováčik and Rákosník [1], variable exponent Sobolev spaces have been used in the last decades to model various phenomena. In [2], Růžička presented the mathematical theory for the application of variable exponent spaces in electro-rheological fluids.

In recent years, the differential equations and variational problems with $p(x)$-growth conditions have been studied extensively; see for example [3–6]. In [7], De Figueiredo and Ding discussed the multiple solutions for a kind of elliptic systems on a smooth bounded domain. Motivated by their work, we will consider the following sort of $p(x)$-Laplacian systems with "concave and convex nonlinearity":

$$\begin{gathered}
-\operatorname{div}\left(\left|\nabla u\right|^{p(x)-2}\nabla u\right)+|u|^{p(x)-2}u=H_u(x,u,v), \quad x\in\Omega,\\
-\operatorname{div}\left(\left|\nabla v\right|^{p(x)-2}\nabla v\right)+|v|^{p(x)-2}v=-H_v(x,u,v), \quad x\in\Omega,\\
u(x)=v(x)=0, \quad x\in\partial\Omega,
\end{gathered}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, p is continuous on $\overline{\Omega}$ and satisfies $1<p_-\le p(x)\le p_+<N$, and $H:\overline{\Omega}\times\mathbb{R}^2\to\mathbb{R}$ is a C^1 function. In this paper, we are mainly interested in the class

of Hamiltonians H such that

$$H(x,u,v)=\frac{|u|^{\alpha(x)}}{\alpha(x)}+\frac{|v|^{\beta(x)}}{\beta(x)}+F(x,u,v), \tag{1.2}$$

where $1<\alpha_-\leq\alpha(x)\leq p(x)$, $p(x)\ll\beta(x)\ll p^*(x)$. Here we denote

$$p_+=\sup_{x\in\Omega}p(x), \qquad p_-=\inf_{x\in\Omega}p(x), \tag{1.3}$$

and denote by $p(x)\ll\beta(x)$ the fact that $\inf_{x\in\Omega}(\beta(x)-p(x))>0$. Throughout this paper, $F(x,u,v)$ satisfies the following conditions:

(H1) $F\in C^1(\overline{\Omega}\times\mathbb{R}^2,\mathbb{R})$. Writing $z=(u,v)$, $F(x,0)\equiv 0$, $F_z(x,0)\equiv 0$;

(H2) there exist $p(x)<q_1(x)\ll p^*(x)$, $1<q_{2-}\leq q_2(x)<p(x)$ such that

$$\left|F_u(x,u,v)\right|,\left|F_v(x,u,v)\right|\leq a_0\left(1+|u|^{q_1(x)-1}+|v|^{q_2(x)-1}\right), \tag{1.4}$$

where a_0 is positive constant;

(H3) there exist $\mu(x),\nu(x)\in C^1(\overline{\Omega})$ with $p(x)\ll\mu(x)\ll p^*(x)$, $1<\nu_-\leq\nu(x)\leq p(x)$, and $R_0>0$ such that

$$\frac{1}{\mu(x)}F_u(x,u,v)u+\frac{1}{\nu(x)}F_v(x,u,v)v\geq F(x,u,v)>0, \tag{1.5}$$

when $|(u,v)|\geq R_0$.

As [8, Lemma 1.1], from assumption (H3), there exist $b_0,b_1>0$ such that

$$F(x,u,v)\geq b_0\left(|u|^{\mu(x)}+|v|^{\nu(x)}\right)-b_1, \tag{1.6}$$

for any $(x,u,v)\in\overline{\Omega}\times\mathbb{R}^2$. We can also get that there exists $b_2>0$ such that

$$\frac{1}{\mu(x)}F_u(x,u,v)u+\frac{1}{\nu(x)}F_v(x,u,v)v+b_2\geq F(x,u,v), \tag{1.7}$$

for any $(x,u,v)\in\overline{\Omega}\times\mathbb{R}^2$. In this paper, we will prove the following result.

Theorem 1.1. *Assume that hypotheses (H1)–(H3) are fulfilled. If $F(x,z)$ is even in z, then problem* (1.1) *has a sequence of solutions $\{z_n\}$ such that*

$$I(z_n)=\int_\Omega\left(\frac{\left|\nabla u_n\right|^{p(x)}+\left|u_n\right|^{p(x)}}{p(x)}-\frac{\left|\nabla v_n\right|^{p(x)}+\left|v_n\right|^{p(x)}}{p(x)}-H(x,z_n)\right)dx\longrightarrow\infty, \tag{1.8}$$

as $n\to\infty$.

2. Preliminaries

First we recall some basic properties of variable exponent spaces $L^{p(x)}(\Omega)$ and variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a domain. For a deeper treatment on these spaces, we refer to [1, 9–11].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1,\infty)$ and

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{2.1}$$

The variable exponent space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\int_\Omega |u(x)|^{p(x)} dx < \infty$. Under the assumption that $p_+ < \infty$, $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm (2.1).

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$ and it can be equipped with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}. \tag{2.2}$$

For $u \in W^{1,p(x)}(\Omega)$, if we define

$$|||u||| = \inf\left\{\lambda > 0 : \int_\Omega \frac{|u|^{p(x)} + |\nabla u|^{p(x)}}{\lambda^{p(x)}} dx \le 1\right\}, \tag{2.3}$$

then $|||u|||$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W^{1,p(x)}(\Omega)$.

By $W_0^{1,p(x)}(\Omega)$ we denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.2) and denote the dual space of $W_0^{1,p(x)}(\Omega)$ by $W^{-1,p'(x)}(\Omega)$. We know that if $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\|u\|_{1,p(x)}$ and $|\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Under the condition $1 < p_- \le p_+ < \infty, W_0^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space, then there exist $\{e_n\}_{n=1}^{+\infty} \subset W_0^{1,p(x)}(\Omega)$ and $\{f_m\}_{m=1}^{+\infty} \subset W^{-1,p'(x)}(\Omega)$ such that

$$\begin{gathered} f_m(e_n) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases} \\ W_0^{1,p(x)}(\Omega) = \overline{\operatorname{span}}\{e_i : i = 1,\ldots,n,\ldots\}, \\ W^{-1,p'(x)}(\Omega) = \overline{\operatorname{span}}\{f_j : j = 1,\ldots,m,\ldots\}. \end{gathered} \tag{2.4}$$

In the following, we will denote that $E = E^1 \oplus E^2$, where

$$E^1 = \{0\} \times W_0^{1,p(x)}(\Omega), \qquad E^2 = W_0^{1,p(x)}(\Omega) \times \{0\}. \tag{2.5}$$

For any $z \in E$, define the norm $\|z\| = \|(u,v)\| = |||u||| + |||v|||$. For any $n \in \mathbb{N}$, set $e_n^1 = (0, e_n)$, $e_n^2 = (e_n, 0)$ and

$$X_n = \operatorname{span}\{e_1^1, \dots, e_n^1\} \oplus E^2, \qquad X^n = E^1 \oplus \operatorname{span}\{e_1^2, \dots, e_n^2\}, \tag{2.6}$$

denote the complement of X^n in E by $(X^n)^\perp = \operatorname{span}\{e_{n+1}^2, e_{n+2}^2, \dots\}$.

3. The Proof of Theorem 1.1

Definition 3.1. We say that $z_0 = (u_0, v_0) \in E$ is a weak solution of problem (1.1), that is,

$$\begin{aligned}\int_\Omega \Big(&|\nabla u_0|^{p(x)-2} \nabla u_0 \nabla u + |u_0|^{p(x)-2} u_0 u - |\nabla v_0|^{p(x)-2} \nabla v_0 \nabla v \\ &- |v_0|^{p(x)-2} v_0 v - H_u(x, u_0, v_0) u - H_v(x, u_0, v_0) v \Big) dx = 0, \quad \forall z \in E.\end{aligned} \tag{3.1}$$

In this section, we denote that $V_m = \operatorname{span}\{e_i : i = 1, \dots, m\}$, for any $m \in \mathbb{N}$, and c_i is positive constant, for any $i = 0, 1, 2 \dots$.

Lemma 3.2. *Any (PS) sequence $\{z_n\} \subset E$, that is, $|I(z_n)| \le c$ and $I'(z_n) \to 0$, as $n \to \infty$, is bounded.*

Proof. Let $s > 0$ be sufficiently small such that $l_1 = \inf_{x\in\Omega}(1/p(x) - (1+s)/\mu(x)) > 0$, $l_2 = \inf_{x\in\Omega}((1+s)/\nu(x) - 1/p(x)) > 0$, $l_3 = \sup_{x\in\Omega}((1/\alpha(x) - (1+s))/\mu(x)) > 0$, $l_4 = \sup_{x\in\Omega}((1+s)/\nu(x) - 1/\beta(x)) > 0$.

Let $\{z_n\} \subset E$ be such that $|I(z_n)| \le c$ *and* $I'(z_n) \to 0$, as $n \to \infty$. We get

$$\begin{aligned} I(z_n) &- \left\langle I'(z_n), \left(\frac{1+s}{\mu(x)} u_n, \frac{1+s}{\nu(x)} v_n \right) \right\rangle \\ &= \int_\Omega \Bigg(\left(\frac{1}{p(x)} - \frac{1+s}{\mu(x)} \right) \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) + \frac{(1+s) u_n}{\mu(x)^2} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \mu \\ &\qquad + \left(\frac{1+s}{\nu(x)} - \frac{1}{p(x)} \right) \left(|\nabla v_n|^{p(x)} + |v_n|^{p(x)} \right) - \frac{(1+s) v_n}{\nu(x)^2} |\nabla v_n|^{p(x)-2} \nabla v_n \nabla \nu \\ &\qquad + \frac{1+s}{\mu(x)} F_u(x, u_n, v_n) u_n + \frac{1+s}{\nu(x)} F_v(x, u_n, v_n) v_n - F(x, u_n, v_n) \\ &\qquad + \left(\frac{1+s}{\mu(x)} - \frac{1}{\alpha(x)} \right) |u_n|^{\alpha(x)} + \left(\frac{1+s}{\nu(x)} - \frac{1}{\beta(x)} \right) |v_n|^{\beta(x)} \Bigg) dx \\ &\ge \int_\Omega \Bigg(l_1 |\nabla u_n|^{p(x)} + l_2 |\nabla v_n|^{p(x)} + sF(x, u_n, v_n) - l_3 |u_n|^{\alpha(x)} + l_4 |v_n|^{\beta(x)} \\ &\qquad + \frac{(1+s) u_n}{\mu(x)^2} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \mu - \frac{(1+s) v_n}{\nu(x)^2} |\nabla v_n|^{p(x)-2} \nabla v_n \nabla \nu - (1+s) b_2 \Bigg) dx. \end{aligned} \tag{3.2}$$

As $\mu(x), \nu(x) \in C^1(\overline{\Omega})$, by the Young inequality, we can get that for any $\varepsilon_1, \varepsilon_2 \in (0,1)$,

$$\begin{aligned}
\left|\frac{(1+s)u_n}{\mu(x)^2}|\nabla u_n|^{p(x)-2}\nabla u_n \nabla \mu\right| &\le c_0|\nabla u_n|^{p(x)-1}|u_n| \\
&\le c_0\left(\frac{\varepsilon_1(p(x)-1)}{p(x)}|\nabla u_n|^{p(x)} + \frac{\varepsilon_1^{1-p(x)}}{p(x)}|u_n|^{p(x)}\right) \\
&\le c_0\left(\varepsilon_1|\nabla u_n|^{p(x)} + \varepsilon_1^{1-p_+}|u_n|^{p(x)}\right), \\
\left|\frac{(1+s)v_n}{\nu(x)^2}|\nabla v_n|^{p(x)-2}\nabla v_n \nabla \nu\right| &\le c_1\left(\varepsilon_2|\nabla v_n|^{p(x)} + \varepsilon_2^{1-p_+}|v_n|^{p(x)}\right).
\end{aligned} \tag{3.3}$$

Let $\varepsilon_1, \varepsilon_2$ be sufficiently small such that

$$c_0\varepsilon_1 \le \frac{l_1}{2}, \qquad c_1\varepsilon_2 \le \frac{l_2}{2}, \tag{3.4}$$

then

$$\begin{aligned}
I(z_n) &- \left\langle I'(z_n), \left(\frac{1+s}{\mu(x)}u_n, \frac{1+s}{\nu(x)}v_n\right)\right\rangle \\
&\ge \int_\Omega \Bigg(\frac{l_1}{2}|\nabla u_n|^{p(x)} + \frac{l_2}{2}|\nabla v_n|^{p(x)} + s\left(b_0|u_n|^{\mu(x)} + b_0|v_n|^{\nu(x)} - b_1\right) \\
&\quad - \left(l_3|u_n|^{\alpha(x)} + c_0\varepsilon_1^{1-p_+}|u_n|^{p(x)}\right) + \left(l_4|v_n|^{\beta(x)} - c_1\varepsilon_2^{1-p_+}|v_n|^{p(x)}\right) - (1+s)b_2\Bigg)dx.
\end{aligned} \tag{3.5}$$

Note that $\alpha(x) \le p(x) \ll \mu(x)$, $p(x) \ll \beta(x)$, by the Young inequality, for any $\varepsilon_3, \varepsilon_4, \varepsilon_5 \in (0,1)$, we get

$$\begin{aligned}
|u_n|^{\alpha(x)} &\le \frac{\varepsilon_3\alpha(x)|u_n|^{\mu(x)}}{\mu(x)} + \frac{\mu(x)-\alpha(x)}{\mu(x)}\varepsilon_3^{\alpha(x)/(\alpha(x)-\mu(x))} \\
&\le \varepsilon_3|u_n|^{\mu(x)} + \varepsilon_3^{-\alpha_+/(\mu-\alpha)_-}, \\
|u_n|^{p(x)} &\le \frac{\varepsilon_4 p(x)}{\mu(x)}|u_n|^{\mu(x)} + \frac{\mu(x)-p(x)}{\mu(x)}\varepsilon_4^{p(x)/(p(x)-\mu(x))} \\
&\le \varepsilon_4|u_n|^{\mu(x)} + \varepsilon_4^{-p+/(\mu-p)_-}, \\
|v_n|^{p(x)} &\le \frac{\varepsilon_5 p(x)}{\beta(x)}|v_n|^{\beta(x)} + \frac{\beta(x)-p(x)}{\beta(x)}\varepsilon_5^{p(x)/(p(x)-\beta(x))} \\
&\le \varepsilon_5|v_n|^{\beta(x)} + \varepsilon_5^{-p+/(\beta-p)_-}.
\end{aligned} \tag{3.6}$$

Let ε_3, ε_4, ε_5 be sufficiently small such that $l_3\varepsilon_3 + c_0\varepsilon_1^{1-p_+}\varepsilon_4 \leq sb_0$ and $c_1\varepsilon_2^{1-p_+}\varepsilon_5 \leq l_4$, then we get

$$I(z_n) - \left\langle I'(z_n), \left(\frac{1+s}{\mu(x)}u_n, \frac{1+s}{\nu(x)}v_n\right)\right\rangle \geq \int_\Omega \left(\frac{l_1}{2}|\nabla u_n|^{p(x)} + \frac{l_2}{2}|\nabla v_n|^{p(x)} - c_2\right)dx. \quad (3.7)$$

Note that

$$\begin{aligned}\left|\left\langle I'(z_n), \left(\frac{1+s}{\mu(x)}u_n, \frac{1+s}{\nu(x)}v_n\right)\right\rangle\right| &\leq \|I'(z_n)\| \cdot \left(\left\|\left|\frac{1+s}{\mu(x)}u_n\right\|\right| + \left\|\left|\frac{1+s}{\nu(x)}v_n\right\|\right|\right) \\ &\leq c_3\|I'(z_n)\| \cdot \left(\left|\nabla\left(\frac{1+s}{\mu(x)}u_n\right)\right|_{p(x)} + \left|\nabla\left(\frac{1+s}{\nu(x)}v_n\right)\right|_{p(x)}\right) \\ &\leq c_4\|I'(z_n)\| \cdot \left(|\nabla u_n|_{p(x)} + |\nabla v_n|_{p(x)}\right),\end{aligned} \quad (3.8)$$

and for $n \in \mathbb{N}$ being large enough, we have

$$c_4\|I'(z_n)\| \leq \min\left\{\frac{l_1}{4}, \frac{l_2}{4}\right\}. \quad (3.9)$$

It is easy to know that if $|\nabla u_n|_{p(x)} \geq 1$ and $|\nabla v_n|_{p(x)} \geq 1$,

$$|\nabla u_n|_{p(x)} \leq \int_\Omega |\nabla u_n|^{p(x)}dx, \qquad |\nabla v_n|_{p(x)} \leq \int_\Omega |\nabla v_n|^{p(x)}dx, \quad (3.10)$$

thus we get

$$I(z_n) \geq \int_\Omega \left(\frac{l_1}{4}|\nabla u_n|^{p(x)} + \frac{l_2}{4}|\nabla v_n|^{p(x)} - c_2\right)dx, \quad (3.11)$$

then $|\nabla u_n|_{p(x)}$, $|\nabla v_n|_{p(x)}$ are bounded. Similarly, if $|\nabla u_n|_{p(x)} < 1$ or $|\nabla v_n|_{p(x)} < 1$, we can also get that $|\nabla u_n|_{p(x)}$, $|\nabla v_n|_{p(x)}$ are bounded. It is immediate to get that $\{z_n\}$ is bounded in E. $\square$

Lemma 3.3. *Any* (PS) *sequence contains a convergent subsequence.*

Proof. Let $\{z_n\} \subset E$ be a (PS) sequence. By Lemma 3.2, we obtain that $\{z_n\}$ is bounded in E. As E is reflexive, passing to a subsequence, still denoted by $\{z_n\}$, we may assume that there exists $z \in E$ such that $z_n \to z$ weakly in E. Then we can get $u_n \to u$ weakly in $W_0^{1,p(x)}(\Omega)$. Note that

$$\begin{aligned}\left\langle I'(z_n) - I'(z), (u_n - u, 0)\right\rangle = \int_\Omega \Big(\Big(& |\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\Big)\nabla(u_n - u) \\ & + \Big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\Big)(u_n - u) \\ & - \Big(|u_n|^{\alpha(x)-2}u_n - |u|^{\alpha(x)-2}u\Big)(u_n - u) \\ & - \left(F_u(x, u_n, v_n) - F_u(x, u, v)\right)(u_n - u)\Big)dx.\end{aligned} \tag{3.12}$$

It is easy to get that

$$\begin{gathered}\left\langle I'(z_n) - I'(z), (u_n - u, 0)\right\rangle \longrightarrow 0, \\ \int_\Omega F_u(x, u, v)(u_n - u)dx \longrightarrow 0,\end{gathered} \tag{3.13}$$

and $u_n \to u$ in $L^{p(x)}(\Omega)$, $u_n \to u$ in $L^{\alpha(x)}(\Omega)$, as $n \to \infty$. Then

$$\begin{gathered}\int_\Omega \Big(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\Big)(u_n - u)dx \longrightarrow 0, \\ \int_\Omega \Big(|u_n|^{\alpha(x)-2}u_n - |u|^{\alpha(x)-2}u\Big)(u_n - u)dx \longrightarrow 0,\end{gathered} \tag{3.14}$$

as $n \to \infty$. By condition (H2), we obtain

$$\begin{aligned}&\int_\Omega \left|F_u(x, u_n, v_n)(u_n - u)\right|dx \\ &\quad \le \int_\Omega a_0\Big(1 + |u_n|^{q_1(x)-1} + |v_n|^{q_2(x)-1}\Big)|u_n - u|dx \\ &\quad \le a_1\Big(|u_n - u|_1 + \Big||u_n|^{q_1(x)-1}\Big|_{q_1'(x)} \cdot |u_n - u|_{q_1(x)} + \Big||v_n|^{q_2(x)-1}\Big|_{q_2'(x)} \cdot |u_n - u|_{q_2(x)}\Big).\end{aligned} \tag{3.15}$$

It is immediate to get that $|u_n - u|_1 \to 0$, $||u_n|^{q_1(x)-1}|_{q_1'(x)}$, $||v_n|^{q_2(x)-1}|_{q_2'(x)}$ are bounded and $|u_n - u|_{q_1(x)} \to 0$, $|u_n - u|_{q_2(x)} \to 0$, then we get

$$\begin{gathered}\int_\Omega F_u(x, u_n, v_n)(u_n - u)dx \longrightarrow 0, \\ \int_\Omega \Big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\Big)\nabla(u_n - u)dx \longrightarrow 0,\end{gathered} \tag{3.16}$$

as $n \to \infty$. Similar to [3, 4, Theorem 3.1], we divide Ω into two parts:

$$\Omega_1 = \{x \in \Omega : p(x) < 2\}, \qquad \Omega_2 = \{x \in \Omega : p(x) \ge 2\}. \tag{3.17}$$

On Ω_1, we have

$$\begin{aligned}
&\int_{\Omega_1} \left|\nabla u_n - \nabla u\right|^{p(x)} dx \\
&\quad\le c_5 \int_{\Omega_1} \left(\left(\left|\nabla u_n\right|^{p(x)-2}\nabla u_n - \left|\nabla u\right|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\right)^{p(x)/2} \\
&\qquad\qquad \times \left(\left|\nabla u_n\right|^{p(x)} + \left|\nabla u\right|^{p(x)}\right)^{(2-p(x))/2} dx \\
&\quad\le c_6 \left|\left(\left(\left|\nabla u_n\right|^{p(x)-2}\nabla u_n - \left|\nabla u\right|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\right)^{p(x)/2}\right|_{2/p(x),\Omega_1} \\
&\qquad \times \left|\left(\left|\nabla u_n\right|^{p(x)} + \left|\nabla u\right|^{p(x)}\right)^{(2-p(x))/2}\right|_{2/(2-p(x)),\Omega_1},
\end{aligned} \tag{3.18}$$

then $\int_{\Omega_1} |\nabla u_n - \nabla u|^{p(x)} dx \to 0$. On Ω_2, we have

$$\int_{\Omega_2} |\nabla u_n - \nabla u|^{p(x)} dx \le c_7 \int_{\Omega_2} (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u) dx \longrightarrow 0. \tag{3.19}$$

Thus we get $\int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx \to 0$. Then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to \infty$. Similarly, $v_n \to v$ in $W_0^{1,p(x)}(\Omega)$. □

Lemma 3.4. *There exists $R_m > 0$ such that $I(z) \le 0$ for all $z \in X^m$ with $\|z\| \ge R_m$.*

Proof. For any $z = (u, v) \in X^m$, $u \in V_m$, we have

$$\begin{aligned}
I(z) &\le \int_\Omega \left(\frac{\left|\nabla u\right|^{p(x)} + |u|^{p(x)}}{p(x)} - \frac{\left|\nabla v\right|^{p(x)} + |v|^{p(x)}}{p(x)} - F(x,u,v) \right) dx \\
&\le \int_\Omega \left(\frac{\left|\nabla u\right|^{p(x)} + |u|^{p(x)}}{p_-} - \frac{\left|\nabla v\right|^{p(x)} + |v|^{p(x)}}{p_+} - b_0 |u|^{\mu(x)} + b_1 \right) dx.
\end{aligned} \tag{3.20}$$

In the following, we will consider $\int_\Omega ((|\nabla u|^{p(x)} + |u|^{p(x)})/p_- - b_0|u|^{\mu(x)}) dx$.
(i) If $|||u||| \le 1$. We have

$$\int_\Omega \left(\frac{\left|\nabla u\right|^{p(x)} + |u|^{p(x)}}{p_-} - b_0 |u|^{\mu(x)} \right) dx \le \frac{1}{p_-}. \tag{3.21}$$

(ii) If $|||u||| > 1$. Note that $\mu, p \in C(\overline{\Omega})$, $p(x) \ll \mu(x)$. For any $x \in \overline{\Omega}$, there exists $Q(x)$ which is an open subset of $\overline{\Omega}$ such that

$$p_x = \sup_{y \in Q(x)} p(y) < \mu_x = \inf_{y \in Q(x)} \mu(y), \tag{3.22}$$

then $\{Q(x)\}_{x\in\overline{\Omega}}$ is an open covering of $\overline{\Omega}$. As $\overline{\Omega}$ is compact, we can pick a finite subcovering $\{Q(x)\}_{i=1}^{n}$ for $\overline{\Omega}$. Thus there exists a sequence of open set $\{\Omega_i\}_{i=1}^{n}$ such that $\Omega = \bigcup_{i=1}^{n}\Omega_i$ and

$$p_{i+} = \sup_{x\in\Omega_i} p(x) < \mu_{i-} = \inf_{x\in\Omega_i} \mu(x), \tag{3.23}$$

for $i = 1,\dots,n$. Denote that $r_i = |||u|||_{\Omega_i}$, then we have

$$\begin{aligned}
&\int_\Omega \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p_-} - b_0|u|^{\mu(x)} \right) dx \\
&\quad = \sum_{i=1}^{n} \int_{\Omega_i} \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p_-} - b_0|u|^{\mu(x)} \right) dx \\
&\quad = \sum_{r_i>1} \int_{\Omega_i} \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p_-} - b_0|u|^{\mu(x)} \right) dx \\
&\qquad + \sum_{r_i\le 1} \int_{\Omega_i} \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p_-} - b_0|u|^{\mu(x)} \right) dx \\
&\quad \le \sum_{r_i>1} \left(\frac{|||u|||_{\Omega_i}^{p_{i+}}}{p_-} - b_0 k_{m_i} |||u|||_{\Omega_i}^{\mu_{i-}} \right) + \frac{n}{p_-},
\end{aligned} \tag{3.24}$$

where $k_{m_i} = \inf_{u\in V_m|_{\Omega_i},\, |||u|||_{\Omega_i}=1} \int_{\Omega_i} |u|^{\mu(x)} dx$. As $V_m|_{\Omega_i}$ is a finite dimensional space, we have $k_{m_i} > 0$, for $i = 1,\dots,n$.

We denote by s_i the maximum of polynomial $t^{p_{i+}}/p_- - b_0 k_{m_i} t^{\mu_{i-}}$ on $[0,\infty)$, for $i = 1,\dots,n$. Then there exists $t_0 > 1$ such that

$$\frac{t^{p_{i+}}}{p_-} - b_0 k_{m_i} t^{\mu_{i-}} + c_8 \le 0, \tag{3.25}$$

for $t > t_0$ and $i = 1,\dots,n$, where $c_8 = \sum_{i=1}^{n} s_i + n/p_- + b_1 \operatorname{meas}\Omega$.

Let $R_m = \max\{2, 2(p_+(c_8 + 1/p_-))^{1/p_-}, 2nt_0\}$. If $\|z\| \ge R_m$, we get $|||u||| \ge R_m/2$ or $|||v||| \ge R_m/2$.

(i) If $|||u||| \ge R_m/2$, $|||u||| \ge nt_0 > 1$. It is easy to verify that there exists at least i_0 such that $|||u|||_{\Omega_{i_0}} \ge t_0 > 1$, thus

$$I(z) \le \frac{|||u|||_{\Omega_{i_0}}^{p_{i_0+}}}{p_-} - b_0 k_{m_{i_0}} |||u|||_{\Omega_{i_0}}^{\mu_{i_0-}} + c_8 \le 0. \tag{3.26}$$

(ii) If $|||v||| \geq R_m/2$, $|||v||| \geq (p_+(c_8 + 1/p_-))^{1/p_-}$. We obtain

$$I(z) \leq c_8 + \frac{1}{p_-} - \frac{|||v|||^{p_-}}{p_+} \leq 0. \tag{3.27}$$

Now we get the result. □

Lemma 3.5. *There exist $r_m > 0$ and $a_m \to \infty$ $(m \to \infty)$ such that $I(z) \geq a_m$, for any $z \in (X^{m-1})^{\perp}$ with $\|z\| = r_m$.*

Proof. For $z = (u, v) \in (X^{m-1})^{\perp}$, $v = 0$. By condition (H2), there exists $c_9 > 0$ such that

$$\left|F(x,u,0)\right| \leq c_9 |u|^{q_1(x)} + c_9. \tag{3.28}$$

Let $\|z\| \geq 1$, we get

$$\begin{aligned} I(z) &= \int_\Omega \left(\frac{\left|\nabla u\right|^{p(x)} + |u|^{p(x)}}{p(x)} - \frac{|u|^{\alpha(x)}}{\alpha(x)} - F(x,u,0) \right) dx \\ &\geq \int_\Omega \left(\frac{\left|\nabla u\right|^{p(x)} + |u|^{p(x)}}{p_+} - \frac{|u|^{\alpha(x)}}{\alpha_-} - c_9 |u|^{q_1(x)} - c_9 \right) dx \\ &\geq \int_\Omega \left(\frac{\left|\nabla u\right|^{p(x)} + |u|^{p(x)}}{p_+} - c_{10} |u|^{q_1(x)} \right) dx - c_{11}. \end{aligned} \tag{3.29}$$

Denote that

$$\theta_m = \sup_{\substack{u \in V_m^{\perp} \\ |||u||| \leq 1}} \int_\Omega |u|^{q_1(x)} dx, \tag{3.30}$$

thus

$$I(z) \geq \frac{|||u|||^{p_-}}{p_+} - c_{10} \theta_m \|u\|^{q_{1+}} - c_{11}. \tag{3.31}$$

Let

$$r_m = \max\left\{ 1, \left(\frac{p_-}{c_{10} p_+ q_{1+} \theta_m} \right)^{1/(q_{1+} - p_-)}, \left(\frac{2 c_{11} p_+ q_{1+}}{q_{1+} - p_-} \right)^{1/p_-} \right\}. \tag{3.32}$$

By [5, Lemma 3.3], we get that $\theta_m \to 0$, as $m \to \infty$, then

$$\begin{aligned} I(z) &\geq r_m^{p_-} \frac{(q_{1+} - p_-)}{p_+ q_{1+}} - c_{11} \\ &\triangleq a_m, \end{aligned} \tag{3.33}$$

when m is sufficiently large and $\|z\| = r_m$. It is easy to get that $a_m \to \infty$, as $m \to \infty$. □

Lemma 3.6. *I is bounded from above on any bounded set of X^m.*

Proof. For $z = (u, v) \in X^m$. We get

$$I(z) \leq \int_\Omega \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} - F(x, u, v) \right) dx. \tag{3.34}$$

By conditions (H2) and (H3), we know that if $|(u, v)| \geq R_0$, $F(x, u, v) \geq 0$ and if $|(u, v)| < R_0$, $|F(x, u, v)| \leq c_0$. Then

$$I(z) \leq \int_\Omega \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} + c_{12} \right) dx, \tag{3.35}$$

and it is easy to get the result. □

Proof of Theorem 1.1. By Lemmas 3.2–3.6 above, and [7, Proposition 2.1 and Remark 2.1], we know that the functional I has a sequence of critical values $c_k \to \infty$, as $k \to \infty$. Now we complete the proof. □

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