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Research Article

Existence and Analytic Approximation of Solutions of Duffing Type Nonlinear Integro-Differential Equation with Integral Boundary Conditions

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A generalized quasilinearization technique is developed to obtain a sequence of approximate solutions converging monotonically and quadratically to a unique solution of a boundary value problem involving Duffing type nonlinear integro-differential equation with integral boundary conditions. The convergence of order k ($k \geq 2$) for the sequence of iterates is also established. It is found that the work presented in this paper not only produces new results but also yields several old results in certain limits.

1. Introduction

Vascular diseases such as atherosclerosis and aneurysms are becoming frequent disorders in the industrialized world due to sedentary way of life and rich food. Causing more deaths than cancer, cardiovascular diseases are the leading cause of death in the world. In recent years, computational fluid dynamics (CFD) techniques have been used increasingly by researchers seeking to understand vascular hemodynamics. CFD methods possess the potential to augment the data obtained from in vitro methods by providing a complete characterization of hemodynamic conditions (blood velocity and pressure as a function of space and time) under precisely controlled conditions. However, specific difficulties in CFD studies of blood flows are related to the boundary conditions. It is now recognized that the blood flow in a given district may depend on the global dynamics of the whole circulation and the boundary conditions (e.g., the instantaneous velocity profile at the inlet section of the computed domain) for an in vitro blood flow computation need to be prescribed. Taylor et al. [1] assumed very long circular vessel geometry upstream the inlet section to obtain the analytic solution due to Womersley [2]. However, it is not always justified to assume a circular cross-section. In order to cope with this problem, an alternative approach prescribing integral boundary conditions is presented in [3]. The validity of this approach is verified by computing both steady and pulsated channel flows for Womersley number up to 15. In fact, the integral boundary conditions have various applications in other fields such as chemical engineering, thermoelasticity, underground water flow, population dynamics, and so forth, see for instance, [4–10] and references therein.

Integro-differential equations are encountered in many areas of science where it is necessary to take into account aftereffect or delay. Especially, models possessing hereditary properties are described by integro-differential equations in practice. Also, the governing equations in the problems of biological sciences such as spreading of disease by the dispersal of infectious individuals [11–13], the reaction-diffusion models in ecology to estimate the speed of invasion [14, 15], and so forth, are integro-differential equations. For the theoretical background of integro-differential equations, we refer the reader to the text [16].

In this paper, we study a boundary value problem for Duffing type nonlinear integro-differential equation (Duffing equation with both integral and nonintegral forcing terms of nonlinear type) with integral boundary conditions given by

$$
\begin{gathered}
-u''(t) - \gamma u'(t) = f(t, u(t)) + \int_0^t K(t, s, u(s))\,ds, \quad 0 < t < 1, \\
u(0) - \mu_1 u'(0) = \int_0^1 q_1(u(s))\,ds, \qquad u(1) + \mu_2 u'(1) = \int_0^1 q_2(u(s))\,ds,
\end{gathered}
\tag{1.1}
$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$, $q_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions, $\gamma \in \mathbb{R} - \{0\}$, and μ_i $(i = 1, 2)$ are nonnegative constants.

A generalized quasilinearization (QSL) technique due to Lakshmikantham [17, 18] is applied to obtain the analytic approximation of the solutions of the integral boundary value problem (1.1). In recent years, the QSL technique has been extensively developed and applied to a wide range of initial and boundary value problems for different types of differential equations, for instance, see [19–30] and the references therein. Section 2 contains some basic results. A monotone sequence of approximate solutions converging uniformly and quadratically to a unique solution of the problem (1.1) is obtained in Section 3. The convergence of order k $(k \geq 2)$ for the sequence of iterates is established in Section 4 .

2. Preliminary Results

For any $\rho, \theta_1, \theta_2 \in C[0,1]$, the nonhomogeneous linear problem

$$
\begin{gathered}
-u''(t) - \gamma u'(t) = \rho(t), \quad 0 < t < 1, \\
u(0) - \mu_1 u'(0) = \int_0^1 \theta_1(s)\,ds, \qquad u(1) + \mu_2 u'(1) = \int_0^1 \theta_2(s)\,ds,
\end{gathered}
\tag{2.1}
$$

has a unique solution

$$u(t) = G_1(t) + \int_0^1 G(t,s)\rho(s)ds, \tag{2.2}$$

where $G_1(t)$ is the unique solution of the problem

$$\begin{gathered} -u''(t) - \gamma u'(t) = 0, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \int_0^1 \theta_1(s)ds, \qquad u(1) + \mu_2 u'(1) = \int_0^1 \theta_2(s)ds, \end{gathered} \tag{2.3}$$

and is given by

$$\begin{aligned} G_1(t) &= \frac{1}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \\ &\quad \times \left[\left((-1+\gamma\mu_2)e^{-\gamma} + e^{-\gamma t}\right)\int_0^1 \theta_1(s)ds + \left(\left(1+\gamma\mu_1\right) - e^{-\gamma t}\right)\int_0^1 \theta_2(s)ds\right], \\ G(t,s) &= \Lambda \begin{cases} \left[(1-\gamma\mu_2) - e^{\gamma(1-s)}\right]\left[(1+\gamma\mu_1) - e^{-\gamma t}\right], & 0 \le t \le s, \\ \left[(1-\gamma\mu_2) - e^{\gamma(1-t)}\right]\left[(1+\gamma\mu_1) - e^{-\gamma s}\right], & s \le t \le 1, \end{cases} \\ \Lambda &= \frac{e^{\gamma s}}{\gamma\left[(1-\gamma\mu_2) - (1+\gamma\mu_1)e^{\gamma}\right]}. \end{aligned} \tag{2.4}$$

Here, we note that the associated homogeneous problem has only the trivial solution and $G(t,s) > 0$ on $(0,1)\times(0,1)$.

Definition 2.1. A function $\alpha \in C^2[0,1]$ is a lower solution of (1.1) if

$$\begin{gathered} -\alpha''(t) - \gamma\alpha'(t) \le f(t,\alpha(t)) + \int_0^t K(t,s,\alpha(s))ds, \quad 0 < t < 1, \\ \alpha(0) - \mu_1\alpha'(0) \le \int_0^1 q_1(\alpha(s))ds, \qquad \alpha(1) + \mu_2\alpha'(1) \le \int_0^1 q_2(\alpha(s))ds. \end{gathered} \tag{2.5}$$

Similarly, $\beta \in C^2[0,1]$ is an upper solution of (1.1) if the inequalities in the definition of lower solution are reversed.

Now, we present some basic results which are necessary to prove the main results.

Theorem 2.2. *Let α and β be lower and upper solutions of the boundary value problem* (1.1)*, respectively, such that $\alpha(t) \le \beta(t)$, $t \in [0,1]$. If $f(t,u(t))$ and $K(t,s,u(s))$ are strictly decreasing in u for each $t \in [0,1]$ and for each $(t,s) \in [0,1]\times[0,1]$, respectively, and q_i satisfy a one-sided Lipschitz condition: $q_i(u) - q_i(v) \le L_i(u-v)$, $0 \le L_i < 1$, $i = 1,2$, then there exists a solution u of* (1.1) *such that $\alpha(t) \le u(t) \le \beta(t)$, $t \in [0,1]$.*

Proof. We define $P : J \times \mathbb{R} \to \mathbb{R}$ by $P(t,u(t)) = \max\{\alpha(t), \min\{u(t), \beta(t)\}\}$ and consider the modified problem

$$\begin{gathered} -u''(t) - \gamma u'(t) = \overline{f}(t,u(t)) + \int_0^t \overline{K}(t,s,u(s))ds, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s))ds, \qquad u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s)ds, \end{gathered} \tag{2.6}$$

where

$$\begin{gathered} \overline{f}(t,u(t)) = f(t,P(t,u(t))) + P(t,u(t)) - u(t), \qquad \overline{K}(t,s,u(s)) = K(t,s,P(t,u(s))), \\ \overline{q}_i(u) = \begin{cases} q_i(\beta), & \text{if } u > \beta, \\ q_i(u), & \text{if } \alpha \le u \le \beta, \\ q_i(\alpha), & \text{if } u < \alpha. \end{cases} \end{gathered} \tag{2.7}$$

As $\overline{f}, \overline{K}$, and $\overline{q}_i$ are continuous and bounded, an application of Schauder's fixed point theorem [16] ensures the existence of a solution of the problem (2.6). We note that any solution u of the problem (2.6) such that $\alpha(t) \le u(t) \le \beta(t),\ t \in [0,1]$, is also a solution of (1.1). Thus we need to show that $\alpha(t) \le u(t) \le \beta(t),\ t \in [0,1]$. As $P(t,\alpha(t)) = \max\{\alpha(t), \min\{\alpha(t), \beta(t)\}\} = \alpha(t)$ and $P(t,\beta(t)) = \max\{\alpha(t), \min\{\beta(t), \beta(t)\}\} = \beta(t)$, we have

$$\begin{gathered} \overline{f}(t,\alpha(t)) + \int_0^t \overline{K}(t,s,\alpha(s))ds = f(t,\alpha(t)) + \int_0^t K(t,s,\alpha(s))ds \ge -\alpha''(t) - \gamma\alpha'(t), \\ \alpha(0) - \mu_1\alpha'(0) \le \int_0^1 q_1(\alpha(s))ds = \int_0^1 \overline{q}_1(\alpha(s))ds, \\ \alpha(1) + \mu_2\alpha'(1) \le \int_0^1 q_2(\alpha(s))ds = \int_0^1 \overline{q}_2(\alpha(s))ds, \\ \overline{f}(t,\beta(t)) + \int_0^t \overline{K}(t,s,\beta(s))ds = f(t,\beta(t)) + \int_0^t K(t,s,\beta(s))ds \le -\beta''(t) - \gamma\beta'(t), \\ \beta(0) - \mu_1\beta'(0) \ge \int_0^1 q_1(\beta(s))ds = \int_0^1 \overline{q}_1(\beta(s))ds, \\ \beta(1) + \mu_2\beta'(1) \le \int_0^1 q_2(\beta(s))ds = \int_0^1 \overline{q}_2(\beta(s))ds, \end{gathered} \tag{2.8}$$

which imply that α, β are lower and upper solutions of (2.6), respectively, on $[0,1]$.

Now, we show that $\alpha \le u$ on $[0,1]$. If the inequality $\alpha \le u$ is not true on $[0,1]$, then $m(t) = \alpha(t) - u(t)$ will have a positive maximum at some $t_0 \in [0,1]$. We first suppose that $t_0 \in (0,1)$. Then $m(t_0) > 0$, $m'(t_0) = 0$, $m''(t_0) \le 0$. On the other hand, we have

$$\begin{aligned}
&\overline{f}(t_0,u(t_0))+\int_0^{t_0}\overline{K}(t_0,s,u(s))ds\\
&\quad=-u''(t_0)-\gamma u'(t_0)\\
&\quad\le-\alpha''(t_0)-\gamma\alpha'(t_0)\\
&\quad\le\overline{f}(t_0,\alpha(t_0))+\int_0^{t_0}\overline{K}(t_0,s,\alpha(s))ds,
\end{aligned}\tag{2.9}$$

which leads to a contradiction. Thus, there exists $t_1\in(0,1)$ such that $m(t_1)\le 0$. Assuming $t_0<t_1$ (the case $t_0>t_1$ is similar), we can find $t_2\in(t_0,t_1)$ such that $m(t_2)=0$ and $m(t)>0$ for every $t\in[t_0,t_2)$. Then

$$\begin{aligned}
m''(t)+\gamma m'(t)\ge&-f(t,\alpha(t))-\int_0^t K(t,s,\alpha(s))ds\\
&+f(t,P(t,u(t)))+P(t,u(t))-u(t)+\int_0^t K(t,s,P(s,u(s)))ds\\
=&\,m(t)>0,
\end{aligned}\tag{2.10}$$

which can alternatively be written as $(m'(t)e^{\gamma t})'>0$. Integrating from t_0 to t, and using $m'(t_0)=0$, we obtain $m'(t)>0$ for every $t\in[t_0,t_2)$ which together with $m'(t_0)=0$ implies that $m'(t)\ge 0$ for every $t\in[t_0,t_2)$. Thus, $m(t)$ is nondecreasing on $[t_0,t_2)$ which is a contradiction as $m(t)$ has a positive maximum value at $t=t_0$. Hence t_0 is an isolated maximum point. If $t_0=0$, then $m(0)>0, m'(0)=0, m''(0)\le 0$. On the other hand, we find that

$$\begin{aligned}
m(0)=\alpha(0)-u(0)&\le\mu_1 m'(0)+\int_0^1\left[q_1(\alpha(s))-\overline{q}_1(u(s))\right]ds\\
&=\int_0^1\left[q_1(\alpha(s))-\overline{q}_1(u(s))\right]ds.
\end{aligned}\tag{2.11}$$

If $u(t)<\alpha(t)$, then $\overline{q}_1(u(s))=q_1(\alpha(s))$, which, on substituting in (2.11), yields $m(0)\le 0$. This contradicts that $m(0)>0$. If $u(t)>\beta(t)$, then $\overline{q}_1(u(s))=q_1(\beta(s))$. Hence, in view of the fact that q_1 satisfies a one-sided Lipschitz condition, we have $q_1(\alpha(s))-\overline{q}_1(u(s))=(q_1(\alpha(s))-q_1(\beta(s)))\le L_1(\alpha(s)-\beta(s))$ which leads to $m(0)\le L_1\max_{t\in[0,1]}(\alpha(t)-\beta(t))=L_1(\alpha(0)-\beta(0))\le 0$, a contradiction. For $\alpha(t)\le u(t)\le\beta(t)$, we also get a contradiction $m(0)\le 0$. As before, there exists $t_2\in[0,t_1)$ such that $m(t_2)=0$ and $m(t)>0$ for every $t\in[0,t_2)$ which provides a contradiction that $m(t)$ is nondecreasing on $[0,t_2)$. Thus, $t_0=0$ is an isolated maximum point. In a similar manner, $t_0=1$ yields a contradiction. Hence it follows that $\alpha(t)\le u(t), t\in[0,1]$. On the same pattern, it can be shown that $u(t)\le\beta(t)$, $t\in[0,1]$. Thus, we conclude that $\alpha(t)\le u(t)\le\beta(t)$, $t\in[0,1]$. This completes the proof. □

Theorem 2.3. *Let α and β be, respectively, lower and upper solutions of the boundary value problem* (1.1). *If $f(t,u(t))$ and $K(t,s,u(s))$ are strictly decreasing in u for each $t\in[0,1]$ and for each*

$(t,s) \in [0,1] \times [0,1]$ *respectively, and* q_i *satisfies a one-sided Lipschitz condition:* $q_i(u) - q_i(v) \leq L_i(u-v),\ 0 \leq L_i < 1,\ i = 1,2,$ *then* $\alpha(t) \leq \beta(t),\ t \in [0,1]$.

Proof. We omit the proof as it follows the procedure employed in the proof of Theorem 2.2. □

3. Analytic Approximation of the Solution with Quadratic Convergence

Theorem 3.1. *Assume that*

(A_1) α *and* $\beta \in C^2[0,1]$ *are, respectively, lower and upper solutions of* (1.1) *such that* $\alpha(t) \leq \beta(t),\ t \in [0,1]$;

(A_2) $f \in C^2([0,1]\times\mathbb{R})$ *is such that* $f_u(t,u) < 0$ *for each* $t \in [0,1]$ *and* $(f_{uu}(t,u)+\phi_{uu}(t,u)) \geq 0$, *where* $\phi_{uu}(t,u)) \geq 0$ *for some continuous function* $\phi(t,u)$ *on* $[0,1] \times \mathbb{R}$;

(A_3) $K \in C^2([0,1] \times [0,1] \times \mathbb{R})$ *is such that* $K_u(t,s,u) < 0$ *for each* $(t,s) \in [0,1] \times [0,1]$ *with* $K_{uu}(t,s,u) \geq 0$.

(A_4) $q_i \in C^2(\mathbb{R})$ *is such that* $0 \leq q_i'(u) < 1$, *and* $q_i''(u) \geq 0$, $i = 1,2$.

Then, there exists a sequence $\{\alpha_n\}$ *of approximate solutions converging monotonically and quadratically to a unique solution of the problem* (1.1).

Proof. Let $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ be defined by $F(t,u) = f(t,u)+\phi(t,u)$ so that $F_{uu}(t,u) \geq 0$. Using the generalized mean value theorem together with (A_2), (A_3), and (A_4), we obtain

$$\begin{aligned} f(t,u) &\geq f(t,v) + F_u(t,v)(u-v) + \phi(t,v) - \phi(t,u), \\ K(t,s,u) &\geq K(t,s,v) + k_u(t,s,v)(u-v), \\ q_i(u) &\geq q_i(v) + {q'}_i(v)(u-v), \quad u,v \in \mathbb{R}. \end{aligned} \tag{3.1}$$

We set

$$\begin{aligned} g(t,u,v) &= f(t,v) + F_u(t,v)(u-v) + \phi(t,v) - \phi(t,u), \\ \widehat{K}(t,s,u;v) &= K(t,s,v) + k_u(t,s,v)(u-v). \end{aligned} \tag{3.2}$$

Observe that $g_u(t,u,v) = [F_u(t,v) - \phi_u(t,u)] \leq [F_u(t,u) - \phi_u(t,u)] = f_u(t,u) < 0$,

$$f(t,u) \geq g(t,u,v), \qquad f(t,u) = g(t,u,u), \tag{3.3}$$

and $\widehat{K}_u(t,s,u,v) < 0$,

$$K(t,s,u) \geq \widehat{K}(t,s,u;v), \qquad K(t,s,u) = \widehat{K}(t,s,u;u). \tag{3.4}$$

Let us define

$$Q_i(u,v) = q_i(v) + q_i'(v)(u-v), \tag{3.5}$$

so that $0 \le (\partial/\partial u)Q_i(u,v) = q_i' < 1$ and

$$q_i(u) \ge Q_i(u,v), \qquad q_i(u) = Q_i(u,u). \tag{3.6}$$

Now, we fix $\alpha_0 = \alpha$ and consider the problem

$$\begin{gathered} u''(t) + \gamma u'(t) + g(t,u,\alpha_0) + \int_0^t \widehat{K}(t,s,u(s);\alpha_0(s))ds = 0, \quad 0<t<1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s),\alpha_0(s))ds, \\ u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s),\alpha_0(s))ds. \end{gathered} \tag{3.7}$$

Using (A_1), (3.3), (3.4), and (3.6), we obtain

$$\begin{aligned} &\alpha_0''(t) + \gamma\alpha_0'(t) + g(t,\alpha_0,\alpha_0) + \int_0^t \widehat{K}(t,s,\alpha_0(s);\alpha_0(s))ds \\ &\quad = \alpha_0''(t) + \gamma\alpha_0'(t) + f(t,\alpha_0) + \int_0^t K(t,s,\alpha_0(s))ds \ge 0, \quad 0<t<1, \\ &\alpha_0(0) + \mu_1\alpha_0'(0) \le \int_0^1 q_1(\alpha_0(s))ds = \int_0^1 Q_1(\alpha_0(s),\alpha_0(s))ds, \\ &\alpha_0(1) + \mu_2\alpha_1'(0) \le \int_0^1 q_2(\alpha_0(s))ds = \int_0^1 Q_2(\alpha_0(s),\alpha_0(s))ds, \\ &\beta''(t) + \gamma\beta'(t) + g(t,\beta,\alpha_0) + \int_0^t \widehat{K}(t,s,\beta(s);\alpha_0(s))ds \\ &\quad \le \beta''(t) + \gamma\beta'(t) + f(t,\beta) + \int_0^t K(t,s,\beta(s))ds \le 0, \quad 0<t<1, \\ &\beta(0) - \mu_1\beta'(0) \ge \int_0^1 q_1(\beta(s))ds \ge \int_0^1 Q_1(\beta(s),\alpha_0(s))ds, \\ &\beta(1) + \mu_2\beta'(1) \ge \int_0^1 q_2(\beta(s))ds \ge \int_0^1 Q_2(\beta(s),\alpha_0(s))ds, \end{aligned} \tag{3.8}$$

which imply that α_0 and β are, respectively, lower and upper solutions of (3.7). It follows by Theorem 2.2 and 2.3 that there exists a unique solution α_1 of (3.7) such that

$$\alpha_0(t) \le \alpha_1(t) \le \beta(t), \quad t \in [0,1]. \tag{3.9}$$

Next, we consider

$$\begin{gathered} u''(t) + \gamma u'(t) + g(t,u,\alpha_1) + \int_0^t \widehat{K}(t,s,u(s);\alpha_1(s))ds = 0, \quad 0<t<1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s),\alpha_1(s))ds, \\ u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s),\alpha_1(s))ds. \end{gathered} \tag{3.10}$$

Using the earlier arguments, it can be shown that α_1 and β are lower and upper solutions of (3.10), respectively, and hence by Theorem 2.2 and 2.3, there exists a unique solution α_2 of (3.10) such that

$$\alpha_1(t) \le \alpha_2(t) \le \beta(t), \quad t \in [0,1]. \tag{3.11}$$

Continuing this process successively yields a sequence $\{\alpha_n\}$ of solutions satisfying

$$\alpha_0(t) \le \alpha_1(t) \le \alpha_2(t) \le \cdots \le \alpha_n \le \beta(t), \quad t \in [0,1], \tag{3.12}$$

where the element α_n of the sequence $\{\alpha_n\}$ is a solution of the problem

$$\begin{gathered}
u''(t) + \gamma u'(t) + g(t,u,\alpha_{n-1}) + \int_0^t \widehat{K}(t,s,u(s);\alpha_{n-1}(s))ds = 0, \quad 0 < t < 1, \\
u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s),\alpha_{n-1}(s))ds, \\
u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s),\alpha_{n-1}(s))ds,
\end{gathered} \tag{3.13}$$

and is given by

$$\begin{aligned}
\alpha_n(t) = {} & \frac{-(1-\gamma\mu_2)e^{-\gamma} + e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 Q_1(\alpha_n(s),\alpha_{n-1}(s))ds \\
& + \frac{(1+\gamma\mu_1) - e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 Q_2(\alpha_n(s),\alpha_{n-1}(s))ds \\
& + \int_0^1 G(t,s)\left[g(s,\alpha_n(s),\alpha_{n-1}(s)) + \int_0^t \widehat{K}(s,l,u(l);\alpha_{n-1}(l))dl\right]ds.
\end{aligned} \tag{3.14}$$

Using the fact that $[0,1]$ is compact and the monotone convergence of the sequence $\{\alpha_n\}$ is pointwise, it follows by the standard arguments (Arzela Ascoli convergence criterion, Dini's theorem [21]) that the convergence of the sequence is uniform. If u is the limit point of the sequence, taking the limit $n \to \infty$ in (3.14), we obtain

$$\begin{aligned}
u(t) = {} & \frac{-(1-\gamma\mu_2)e^{-\gamma} + e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 q_1(u(s))ds \\
& + \frac{(1+\gamma\mu_1) - e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 q_2(u(s))ds \\
& + \int_0^1 G(t,s)\left[f(s,u(s)) + \int_0^t K(s,l,u(l))dl\right]ds,
\end{aligned} \tag{3.15}$$

thus, u is a solution of (1.1). Now, we show that the convergence of the sequence is quadratic. For that we set $e_n(t) = (u(t) - \alpha_n(t)) \geq 0,\ t \in [0,1]$. In view of (A_2), (A_3), and (3.2), it follows by Taylor's theorem that

$$
\begin{aligned}
e_n''(t) + \gamma e_n'(t)
&= u'' + \gamma u' - (\alpha_n'' + \gamma \alpha_n') \\
&= -f(t,u) - \int_0^t K(t,s,u(s))ds + g(t,\alpha_n,\alpha_{n-1}) + \int_0^t \widehat{K}(t,s,\alpha_n(s);\alpha_{n-1}(s))ds \\
&= -f(t,u) + f(t,\alpha_{n-1}) + F_u(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1}) + \phi(t,\alpha_{n-1}) - \phi(t,\alpha_n) \\
&\quad + \int_0^t \left[-K(t,s,u(s)) + K(t,s,\alpha_{n-1}(s)) + K_u(t,s,\alpha_{n-1}(s))(\alpha_n(s) - \alpha_{n-1}(s)) \right] ds \\
&= -f_u(t,c_1)(u - \alpha_{n-1}) - F_u(t,\alpha_{n-1})(u - \alpha_n) + F_u(t,\alpha_{n-1})(u - \alpha_{n-1}) -_{\phi}(t,c_2)(\alpha_n - \alpha_{n-1}) \\
&\quad - \int_0^t \left[K_u(t,s,d_1))(u - \alpha_{n-1}) - K_u(t,s,\alpha_{n-1}(s))(\alpha_n(s) - \alpha_{n-1}(s)) \right] ds \\
&= \left[-f_u(t,c_1) + F_u(t,\alpha_{n-1}) - \phi_u(t,c_2) \right] e_{n-1} + \left[-F_u(t,\alpha_{n-1}) + \phi_u(t,c_2) \right] e_n \\
&\quad - \int_0^t \left[K_u(t,s,d_1) - K_u(t,s,\alpha_{n-1}(s)) \right] e_{n-1}(s) ds - \int_0^t K_u(t,s,\alpha_{n-1}(s)) e_n(s) ds \\
&= \left[-F_u(t,c_1) + F_u(t,\alpha_{n-1}) + \phi_u(t,c_1) - \phi_u(t,c_2) \right] e_{n-1} + \left[-F_u(t,\alpha_{n-1}) + \phi_u(t,c_2) \right] e_n \\
&\quad - \int_0^t \left[K_u(t,s,d_1) - K_u(t,s,\alpha_{n-1}(s)) \right] e_{n-1}(s) ds - \int_0^t K_u(t,s,\alpha_{n-1}(s)) e_n(s) ds \\
&\geq \left[-F_u(t,u) + F_u(t,\alpha_{n-1}) + \phi_u(t,\alpha_{n-1}) - \phi_u(t,\alpha_n) \right] e_{n-1} + \left[-F_u(t,\alpha_{n-1}) + \phi_u(t,\alpha_{n-1}) \right] e_n \\
&\quad - \int_0^t \left[K_u(t,s,u(s)) - K_u(t,s,\alpha_{n-1}(s)) \right] e_{n-1}(s) ds - \int_0^t K_u(t,s,\alpha_{n-1}(s)) e_n(s) ds \\
&= \left[-F_{uu}(t,c_3) - \phi_{uu}(t,c_4) \right] e_{n-1}^2 - f_u(t,\alpha_{n-1}) e_n \\
&\quad - \int_0^t K_{uu}(t,s,d_2) e_{n-1}^2(s) ds - \int_0^t K_u(t,s,\alpha_{n-1}(s)) e_n(s) ds \\
&\geq -M \left\| e_{n-1} \right\|^2,
\end{aligned}
\tag{3.16}
$$

where $\alpha_{n-1} \leq c_1, c_3 \leq u$, $\alpha_{n-1} \leq c_2, c_4 \leq \alpha_n$, $\alpha_{n-1} \leq d_1, d_2 \leq u$, $M = \chi_1 + \chi_2 + \chi_3$, χ_1 is a bound on $\|F_{uu}\|$, χ_2 is a bound on $\|\phi_{uu}\|$ for $t \in (0,1)$, χ_3 provides a bound on $\int_0^t K_{uu} ds$. Further, in view of (3.5), we have

$$\begin{aligned} e_n(0) - \mu_1 e'_n(0) &= \int_0^1 \left[q_1(u(s)) - Q_1(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ &= \int_0^1 \left[q_1(u(s)) - q_1(\alpha_{n-1}(s)) - q_1'(\alpha_{n-1}(s))(\alpha_n - \alpha_{n-1})\right] ds \\ &= \int_0^1 \left[q_1'(\alpha_{n-1}(s)) e_n(s) + \frac{1}{2} q_2''(\zeta_1) e_{n-1}^2(s)\right] ds, \\ e_n(1) + \mu_2 e'_n(1) &= \int_0^1 \left[q_2(u(s)) - Q_2(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ &= \int_0^1 \left[q_2'(\alpha_{n-1}(s)) e_n(s) + \frac{1}{2} q_2''(\zeta_2) e_{n-1}^2(s)\right] ds, \end{aligned} \tag{3.17}$$

where $\alpha_{n-1} \leq \zeta_1, \zeta_2 \leq u$. In view of ($A_4$), there exist $\lambda_i < 1$ and $M_i \geq 0$ such that $q_i'(\alpha_{n-1}(s)) \leq \lambda_i$ and $(1/2)q_i''(\zeta_i) \leq M_i$ $(i = 1, 2)$. Let $\lambda = \max\{\lambda_1, \lambda_2\}$ and $M_3 = \max\{M_1, M_2\}$, then

$$\begin{aligned} e_n(0) - \mu_1 e'_n(0) &\leq \lambda \int_0^1 e_n(s) ds + M_3 \int_0^1 e_{n-1}^2(s) ds, \\ e_n(1) + \mu_2 e'_n(1) &\leq \lambda \int_0^1 e_n(s) ds + M_3 \int_0^1 e_{n-1}^2(s) ds. \end{aligned} \tag{3.18}$$

Using the estimates (3.16) and (3.18), we obtain

$$\begin{aligned} e_n(t) = {} & \frac{-(1-\gamma\mu_2)e^{-\gamma} + e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 \left[q_1(u(s)) - Q_1(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ & + \frac{(1+\gamma\mu_1) - e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 \left[q_2(u(s)) - Q_2(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ & + \int_0^1 G(t,s) \left[f(s, u(s)) + \int_0^t K(s, l, u(l)) dl \right. \\ & \qquad \left. - \left(g(t, \alpha_n, \alpha_{n-1}) + \int_0^t \widehat{K}(s, l, \alpha_n(l); \alpha_{n-1}(l)) dl \right) \right] ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{-(1-\gamma\mu_2)e^{-\gamma}+e^{-\gamma t}}{(1+\gamma\mu_1)-(1-\gamma\mu_2)e^{-\gamma}}\left[\lambda\int_0^1 e_n(s)ds+M_3\int_0^1 e_{n-1}^2(s)ds\right]\\
&\quad+\frac{(1+\gamma\mu_1)-e^{-\gamma t}}{(1+\gamma\mu_1)-(1-\gamma\mu_2)e^{-\gamma}}\left[\lambda\int_0^1 e_n(s)ds+M_3\int_0^1 e_{n-1}^2(s)ds\right]\\
&\quad-\int_0^1 G(t,s)\left[e_n''(s)+\gamma e'_n(s)\right]ds\\
&\leq \lambda\int_0^1 e_n(s)ds+M_3\int_0^1 e_{n-1}^2(s)ds+M\left\|e_{n-1}\right\|^2\int_0^1 G(t,s)ds\\
&\leq \lambda\left\|e_n\right\|+M_3\left\|e_{n-1}\right\|^2+M_4\left\|e_{n-1}\right\|^2\\
&=\lambda\left\|e_n\right\|+M_5\left\|e_{n-1}\right\|^2,
\end{aligned}\tag{3.19}$$

where M_4 provides a bound on $M\int_0^1 G(t,s)ds$ and $M_5 = M_4 + M_3$. Taking the maximum over $[0,1]$, we get

$$\left\|e_n\right\| \leq \frac{M_5}{1-\lambda}\left\|e_{n-1}\right\|^2, \tag{3.20}$$

where $\|u\| = \{|u(t)| : t \in [0,1]\}$. This establishes the quadratic convergence of the sequence of iterates. □

4. Higher Order Convergence

Theorem 4.1. *Assume that*

(B_1) *α and $\beta \in C^2[0,1]$ are, respectively, lower and upper solutions of* (1.1) *such that* $\alpha(t) \leq \beta(t)$, $t \in [0,1]$;

(B_2) *$f \in C^k([0,1]\times\mathbb{R})$ is such that* $(\partial^p/\partial u^p)f(t,u) < 0$ $(p = 1,2,3,\ldots,k-1)$ *and* $(\partial^k/\partial u^k)(f(t,u)+\phi(t,u)) \geq 0$ *with* $(\partial^k/\partial u^k)\phi(t,u)) \geq 0$ *for some continuous function* $\phi(t,u)$ *on* $C^k[[0,1]\times\mathbb{R}]$;

(B_3) *$K \in C^2([0,1]\times[0,1]\times\mathbb{R})$ satisfies* $(\partial^p/\partial u^p)K(t,s,u) < 0$ $(p = 1,2,3,\ldots,k-1)$ *and* $(\partial^k/\partial u^k)K(t,s,u)) \geq 0$.

(B_4) *$q_j \in C^k(\mathbb{R})$ is such that* $(d^i/du^i)q_j(u) \leq (M/(\beta-\alpha)^{i-1})$ $(i = 1,2,\ldots,k-1, j = 1,2)$ *and* $(d^k/du^k)q_j(u) \geq 0$, *where* $M < 1/3$.

Then, there exists a monotone sequence $\{\alpha_n\}$ of approximate solutions converging uniformly and rapidly to the unique solution of the problem (1.1) *with the order of convergence* k $(k \geq 2)$.

Proof. Using Taylor's theorem and the assumptions (B_2)–(B_4), we obtain

$$
\begin{aligned}
f(t,u) &\ge \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} f(t,v) \frac{(u-v)^i}{i!} - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{(u-v)^k}{k!}, \\
K(t,s,u) &\ge \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} K(t,s,v) \frac{(u-v)^i}{i!}, \\
q_j(u) &\ge \sum_{i=0}^{k-1} \frac{d^i}{du^i} q_j(v) \frac{(u-v)^i}{i!},
\end{aligned}
\tag{4.1}
$$

where $\alpha \le v \le \xi \le u \le \beta$. We set

$$
\begin{aligned}
h(t,u,v) &= \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} f(t,v) \frac{(u-v)^i}{i!} - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{(u-v)^k}{k!}, \\
K^*(t,s,u,v) &= \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} K(t,s,v) \frac{(u-v)^i}{i!}, \\
Q_j^*(u,v) &= \sum_{i=0}^{k-1} \frac{d^i}{du^i} q_j(v) \frac{(u-v)^i}{i!},
\end{aligned}
\tag{4.2}
$$

and note that

$$
f(t,u) \ge h(t,u,v), \qquad f(t,u) = h(t,u,u), \tag{4.3}
$$

$$
K(t,s,u) \ge K^*(t,s,u,v), \qquad K(t,s,u) = K^*(t,s,u,u), \tag{4.4}
$$

$$
q_j(u) \ge Q_j^*(u,v), \qquad q_j(u) = Q_j^*(u,u), \tag{4.5}
$$

$$
\begin{aligned}
h_u(t,u,v) &= \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t,v) \frac{(u-v)^{i-1}}{(i-1)!} - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{(u-v)^{k-1}}{(k-1)!} \le 0, \\
K_u^*(t,s,u,v) &= \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} K(t,s,v) \frac{(u-v)^{i-1}}{(i-1)!} \le 0, \\
\frac{\partial}{\partial u} Q_j^*(u,v) &= \sum_{i=1}^{k-1} \frac{d^i}{du^i} q_j(v) \frac{(u-v)^{i-1}}{(i-1)!} \le \sum\nolimits_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{(\beta-\alpha)^{i-1}}{(i-1)!} \\
&\le M\left(3 - \frac{1}{2^{k-2}}\right) < 1.
\end{aligned}
\tag{4.6}
$$

Letting $\alpha_0 = \alpha$, we consider the problem

$$\begin{aligned} u''(t) + \gamma u'(t) + h(t, u, \alpha_0) + \int_0^t K^*(t, s, u; \alpha_0) ds = 0, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1^*(u(s), \alpha_0(s)) ds, \\ u(1) + \mu_2 u'(1) = \int_0^1 Q_2^*(u(s), \alpha_0(s)) ds. \end{aligned} \tag{4.7}$$

Using (B_1), (4.3), (4.4), and (4.5), we obtain

$$\begin{aligned} &\alpha_0''(t) + \gamma\alpha_0'(t) + h(t, \alpha_0, \alpha_0) + \int_0^t K^*(t, s, \alpha_0; \alpha_0) ds \\ &\quad = \alpha_0''(t) + \gamma\alpha_0'(t) + f(t, \alpha_0) + \int_0^t K(t, s, \alpha_0) ds \geq 0, \quad 0 < t < 1, \\ &\alpha_0(0) - \mu_1\alpha_0'(0) \leq \int_0^1 q_1(\alpha_0(s)) ds = \int_0^1 Q_1^*(\alpha_0(s), \alpha_0(s)) ds, \\ &\alpha_0(1) + \mu_2\alpha_1'(0) \leq \int_0^1 q_2(\alpha_0(s)) ds = \int_0^1 Q_2^*(\alpha_0(s), \alpha_0(s)) ds, \\ &\beta''(t) + \gamma\beta'(t) + h(t, \beta, \alpha_0) + \int_0^t K^*(t, s, \beta; \alpha_0) ds \\ &\quad \leq \beta''(t) + \gamma\beta'(t) + f(t, \beta) + \int_0^t K(t, s, \beta) ds \leq 0, \quad 0 < t < 1, \\ &\beta(0) - \mu_1\beta'(0) \geq \int_0^1 q_1(\beta(s)) ds \geq \int_0^1 Q_1^*(\beta(s), \alpha_0(s)) ds, \\ &\beta(1) + \mu_2\beta'(1) \geq \int_0^1 q_2(\beta(s)) ds \geq \int_0^1 Q_2^*(\beta(s), \alpha_0(s)) ds. \end{aligned} \tag{4.8}$$

Thus, it follows by definition that α_0 and β are, respectively, lower and upper solutions of (4.7). As before, by Theorem 2.2 and 2.3, there exists a unique solution α_1 of (4.7) such that

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta(t), \quad t \in [0, 1]. \tag{4.9}$$

Continuing this process successively, we obtain a monotone sequence $\{\alpha_n\}$ of solutions satisfying

$$\alpha_0(t) \leq \alpha_1(t) \leq \alpha_2(t) \leq \cdots \leq \alpha_n \leq \beta(t), \quad t \in [0, 1], \tag{4.10}$$

where the element α_n of the sequence $\{\alpha_n\}$ is a solution of the problem

$$u''(t) + \gamma u'(t) + h(t, u, \alpha_{n-1}) + \int_0^t K^*(t, s, u(s); \alpha_{n-1}(s))\,ds = 0, \quad 0 < t < 1,$$
$$u(0) - \mu_1 u'(0) = \int_0^1 Q_1^*(u(s), \alpha_{n-1}(s))\,ds, \tag{4.11}$$
$$u(1) + \mu_2 u'(1) = \int_0^1 Q_2^*(u(s), \alpha_{n-1}(s))\,ds,$$

and is given by

$$\begin{aligned}\alpha_n(t) = {}& \frac{-(1-\gamma\mu_2)e^{-\gamma} + e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 Q_1^*(\alpha_n(s), \alpha_{n-1}(s))\,ds \\ &+ \frac{(1+\gamma\mu_1) - e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 Q_2^*(\alpha_n(s), \alpha_{n-1}(s))\,ds \\ &+ \int_0^1 G(t,s)\left[h(s, \alpha_n(s), \alpha_{n-1}(s)) + \int_0^t K^*(s, l, \alpha_n(l); \alpha_{n-1}(l))\,dl\right]ds.\end{aligned} \tag{4.12}$$

Employing the arguments used in the proof of Theorem 3.1, we conclude that the sequence $\{\alpha_n\}$ converges uniformly to a unique solution u of (1.1).

In order to prove that the convergence of the sequence is of order $k\,(k \geq 2)$, we set $e_n(t) = u(t) - \alpha_n(t)$ and $a_n(t) = \alpha_{n+1}(t) - \alpha_n(t),\ t \in [0,1]$ and note that

$$e_n(t) \geq 0, \qquad a_n(t) \geq 0, \qquad e_{n+1}(t) = e_n(t) - a_n(t), \qquad e_n^k \geq a_n^k. \tag{4.13}$$

Using Taylor's theorem, we find that

$$\begin{aligned}&e_n''(t) + \gamma e_n'(t) \\ &\quad = u''(t) + \gamma u'(t) - (\alpha_n''(t) + \gamma\alpha_n'(t)) \\ &\quad = -f(t, u(t)) - \int_0^t K(t, s, u(s))\,ds + h(t, \alpha_n, \alpha_{n-1}) + \int_0^t K^*(t, s, \alpha_n(s); \alpha_{n-1}(s))\,ds \\ &\quad = -f(t, u(t)) + \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{a_{n-1}^i}{i!} - \frac{\partial^k}{\partial u^k}\phi(t, \xi)\frac{a_{n-1}^k}{k!} \\ &\qquad + \int_0^t \left[-K(t, s, u(s)) + \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} K(t, s, \alpha_{n-1}(s))\frac{a_{n-1}^i}{i!}\right]ds\end{aligned}$$

$$
\begin{aligned}
&= -f\left(t,u(t)\right) + f\left(t,\alpha_{n-1}(t)\right) + \sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}f\left(t,\alpha_{n-1}\right)\frac{e_{n-1}^i}{i!}\\
&\quad - \sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}f\left(t,\alpha_{n-1}\right)\frac{\left(e_{n-1}^i - a_{n-1}^i\right)}{i!} - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{a_{n-1}^k}{k!}\\
&\quad - \int_0^t\left[\sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}K\left(t,s,\alpha_{n-1}(s)\right)\frac{\left(e_{n-1}^i - a_{n-1}^i\right)}{i!} + \frac{\partial^k}{\partial u^k}K(t,s,\eta)\frac{e_{n-1}^k}{k!}\right]ds\\
&= -\frac{\partial^k}{\partial u^k}f(t,\xi)\frac{e_{n-1}^k}{k!} - \sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}f\left(t,\alpha_{n-1}\right)\frac{\left(e_{n-1}-a_{n-1}\right)}{i!}\sum_{i=0}^{k-1}e_{n-1}^{i-1-l}a_{n-1}^l - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{a_{n-1}^k}{k!}\\
&\quad -\int_0^t\left[\sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}K\left(t,s,\alpha_{n-1}(s)\right)\frac{\left(e_{n-1}-a_{n-1}\right)}{i!}\sum_{i=0}^{k-1}e_{n-1}^{i-1-l}a_{n-1}^l + \frac{\partial^k}{\partial u^k}K(t,s,\eta)\frac{e_{n-1}^k}{k!}\right]ds\\
&\geq -\left(\frac{\partial^k}{\partial u^k}f(t,\xi) + \frac{\partial^k}{\partial u^k}\phi(t,\xi)\right)\frac{e_{n-1}^k}{k!} - \sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}f\left(t,\alpha_{n-1}\right)\frac{e_n}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l\\
&\quad - \int_0^t\left[\sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}K\left(t,s,\alpha_{n-1}(s)\right)\frac{e_n}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l + \frac{\partial^k}{\partial u^k}K(t,s,\eta)\frac{e_{n-1}^k}{k!}\right]ds\\
&= -\frac{\partial^k}{\partial u^k}F(t,\xi))\frac{e_{n-1}^k}{k!} - \sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}f\left(t,\alpha_{n-1}\right)\frac{e_n}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l\\
&\quad - \int_0^t\left[\sum_{i=1}^{k-1}\frac{\partial^i}{\partial u^i}K\left(t,s,\alpha_{n-1}(s)\right)\frac{e_n}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l + \frac{\partial^k}{\partial u^k}K(t,s,\eta)\frac{e_{n-1}^k}{k!}\right]ds\\
&\geq -N\frac{\left\|e_{n-1}\right\|^k}{k!},
\end{aligned}
\tag{4.14}
$$

where $N = N_1 + N_2$, N_1 is a bound for $\partial^k/\partial u^k F(t,\xi))$ and N_2 is a bound on $\int_0^t(\partial^k/\partial u^k)K(t,s,\eta)ds, \alpha_{n-1} \leq \eta \leq u$. Again, by Taylor's theorem and using (4.2), we obtain

$$
\begin{aligned}
q_j\left(u(s)\right) - Q_j^*\left(\alpha_n(s),\alpha_{n-1}(s)\right) &= \sum_{i=0}^{k-1}\frac{d^i}{du^i}q_j\left(\alpha_{n-1}\right)\frac{\left(u-\alpha_{n-1}\right)^i}{i!} + \frac{d^k}{du^k}q_j(c)\frac{\left(u-\alpha_{n-1}\right)^k}{k!}\\
&\quad - \sum_{i=0}^{k-1}\frac{d^i}{du^i}q_j\left(\alpha_{n-1}\right)\frac{\left(\alpha_n-\alpha_{n-1}\right)^i}{i!}\\
&= \left(\sum_{i=1}^{k-1}\frac{d^i}{du^i}q_j\left(\alpha_{n-1}\right)\frac{1}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l\right)e_n + \frac{d^k}{du^k}q_j(c)\frac{\left(e_{n-1}\right)^k}{k!}\\
&\leq \chi_j(t)e_n(t) + \frac{M}{\gamma^{k-1}}\frac{e_{n-1}^k}{k!}\\
&\leq \chi_j(t)e_n(t) + \frac{M}{\gamma^{k-1}}\frac{\left\|e_{n-1}\right\|^k}{k!},
\end{aligned}
\tag{4.15}
$$

where

$$\chi_j(t) = \sum_{i=1}^{k-1} \frac{d^i}{du^i} q_j(\alpha_{n-1}) \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l, \qquad \gamma = \max_{t\in[0,1]} \beta(t) - \min_{t\in[0,1]} \alpha(t). \tag{4.16}$$

Making use of (B_4), we find that

$$\chi_j(t) \le \sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l \le \sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{1}{(i-1)!} (\beta-\alpha)^{i-1} < 3M < 1. \tag{4.17}$$

Thus, we can find $\lambda < 1$ such that $\chi_j(t) \le \lambda$, $t \in [0,1]$ and consequently, we have

$$\begin{aligned} e_n(0) - \mu_1 e_n'(0) &= \int_0^1 \left[q_1(u(s)) - Q_1^*(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ &\le \lambda \int_0^1 e_n(s)ds + \frac{M}{\gamma^{k-1}k!} \|e_{n-1}\|^k, \\ e_n(1) + \mu_2 e_n'(1) &= \int_0^1 \left[q_2(u(s)) - Q_2^*(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ &\le \lambda \int_0^1 e_n(s)ds + \frac{M}{\gamma^{k-1}k!} \|e_{n-1}\|^k. \end{aligned} \tag{4.18}$$

By virtue of (4.14) and (4.18), we have

$$\begin{aligned} e_n(t) = {} & \frac{-(1-\gamma\mu_2)e^{-\gamma} + e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 \left[q_1(u(s)) - Q_1^*(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ & + \frac{(1+\gamma\mu_1) - e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \int_0^1 \left[q_2(u(s)) - Q_2^*(\alpha_n(s), \alpha_{n-1}(s))\right] ds \\ & + \int_0^1 G(t,s) \Bigg[f(s,u(s)) + \int_0^t K(s,l,u(l))dl - h(t,\alpha_n(s),\alpha_{n-1}(s)) \\ & \qquad - \int_0^t K^*(s,l,\alpha_n(l);\alpha_{n-1}(l))dl \Bigg] ds \\ \le {} & \frac{-(1-\gamma\mu_2)e^{-\gamma} + e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \left[\lambda \int_0^1 e_n(s)ds + \frac{M}{\gamma^{k-1}k!} \int_0^1 e_{n-1}^k(s)ds \right] \\ & + \frac{(1+\gamma\mu_1) - e^{-\gamma t}}{(1+\gamma\mu_1) - (1-\gamma\mu_2)e^{-\gamma}} \left[\lambda \int_0^1 e_n(s)ds + \frac{M}{\gamma^{k-1}k!} \int_0^1 e_{n-1}^k(s)ds \right] \\ & - \int_0^1 G(t,s) \left[e_n''(s) + \gamma e_n'(s)\right] ds \\ \le {} & \lambda \int_0^1 e_n(s)ds + \frac{M}{\gamma^{k-1}k!} \int_0^1 e_{n-1}^k(s)ds + \frac{N}{k!} \|e_{n-1}\|^k \int_0^1 G(t,s)ds \\ \le {} & \lambda \|e_n\| + \frac{M}{\gamma^{k-1}k!} \|e_{n-1}\|^k + \frac{N_3}{k!} \|e_{n-1}\|^k = \lambda \|e_n\| + N_4 \|e_{n-1}\|^k, \end{aligned} \tag{4.19}$$

where $N_4 = M + \gamma^{k-1} N_3 / \gamma^{k-1} k!$ and N_3 is a bound on $N\int_0^1 G(t,s)ds$. Taking the maximum over $[0,1]$ and solving the above expression algebraically, we obtain

$$\|e_n\| \leq \frac{N_4}{1-\lambda} \|e_{n-1}\|^k. \tag{4.20}$$

This completes the proof. □

5. Discussion

An algorithm for approximating the analytic solution of Duffing type nonlinear integro-differential equation with integral boundary conditions is developed by applying the method of generalized quasilinearization. A monotone sequence of iterates converging uniformly and quadratically (rapidly) to a unique solution of an integral boundary value problem (1.1) is presented. For the illustration of the results, let us consider the following integral boundary value problem:

$$\begin{aligned} -u''(t) + 1.5u'(t) &= -te^{u(t)+1} - 2u(t) + \int_0^t (2-t)e^{-u(s)-1} ds, 0 < t < 1, \\ u(0) - \frac{1}{4}u'(0) &= \frac{1}{2}\int_0^1 \left(\frac{1}{2}u(s) - 1\right) ds, \\ u(1) + \frac{1}{2}u'(1) &= \int_0^1 \left(\frac{1}{2}u(s) + 1\right) ds. \end{aligned} \tag{5.1}$$

Let $\alpha(t) = -1$ and $\beta(t) = t$ be, respectively, lower and upper solutions of (5.1). Clearly $\alpha(t)$ and $\beta(t)$ are not the solutions of (5.1) and $\alpha(t) < \beta(t)$, $t \in [0,1]$. Moreover, the assumptions $(A_1), (A_2), (A_3)$, and (A_4) of Theorem 3.1 are satisfied by choosing $\phi(t,u) = Mu^2$, $M > 0$. Thus, the conclusion of Theorem 3.1 applies to the problem (5.1).

The results established in this paper provide a diagnostic tool to predict the possible onset of diseases such as cardiac disorder and chaos in speech by varying the nonlinear forcing functions $f(t,u), K(t,s,u)$, and $q_i(u)$ appropriately in (1.1). If the nonlinearity $f(t,u)$ in (1.1) is of convex type, then the assumption (A_2) in Theorem 3.1 reduces to $f_{uu}(t,u) \geq 0$ and (B_2) in Theorem 4.1 becomes $\partial^k/\partial u^k f(t,u) \geq 0$ (that is, $\phi(t,u) = 0$ in this case). The existence results for Duffing type nonlinear integro-differential equations with Dirichlet boundary conditions can be recorded by taking $q_1(\cdot) = 0 = q_2(\cdot)$ and $\mu_1 = 0 = \mu_2$ in (1.1). Further, for $q_1(\cdot) = a, q_2(\cdot) = b$ (a and b are constants) and $\mu_1 = 0 = \mu_2$ in (1.1), our results become the existence results for Duffing type nonlinear integro-differential equations with nonhomogeneous Dirichlet boundary conditions. If we take $\mu_1 = 0 = \mu_2$ in (1.1), our problem reduces to the Dirichlet boundary value problem involving Duffing type nonlinear integro-differential equations with integral boundary conditions. In case, we fix $q_1(\cdot) = a, q_2(\cdot) = b$ in (1.1), the existence results for Duffing type nonlinear integro-differential equations with separated boundary conditions appear as a special case of our results. By taking $K(t,s,u) \equiv 0$ in (1.1), the results of [30] appear as a special case of our work. The results for forced Duffing equation involving a purely integral type of nonlinearity subject to integral boundary conditions follow by taking $f(t,u) \equiv 0$ in (1.1).

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