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Research Article

The Direct and Converse Inequalities for Jackson-Type Operators on Spherical Cap

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Approximation on the spherical cap is different from that on the sphere which requires us to construct new operators. This paper discusses the approximation on the spherical cap. That is, the so-called Jackson-type operator $\{J_{k,s}^m\}_{k=1}^{\infty}$ is constructed to approximate the function defined on the spherical cap $D(x_0,\gamma)$. We thus establish the direct and inverse inequalities and obtain saturation theorems for $\{J_{k,s}^m\}_{k=1}^{\infty}$ on the cap $D(x_0,\gamma)$. Using methods of K-functional and multiplier, we obtain the inequality $C_1\|J_{k,s}^m(f)-f\|_{D,p} \le \omega^2(f,k^{-1})_{D,p} \le C_2 \max_{v\ge k}\|J_{v,s}^m(f)-f\|_{D,p}$ and that the saturation order of these operators is $O(k^{-2})$, where $\omega^2(f,t)_{D,p}$ is the modulus of smoothness of degree 2, the constants C_1 and C_2 are independent of k and f.

1. Introduction

In the past decades, many mathematicians dedicated to establish the Jackson and Bernstein-type theorems on the sphere (see [1–9]). Early works, such as Butzer and Johnen [3], Nikol'skii and Lizorkin [8, 9], and Lizorkin and Nikol'skii [5] had successfully established the direct and inverse theorems on the sphere. In 1991, Li and Yang [4] constructed Jackson operators on the sphere and obtained the Jackson and Bernstein-type theorems for the Jackson operators.

Jackson operator on the sphere is defined by (see [4])

$$J_{k,s}(f)(x) := \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} f(y)\mathfrak{D}_{k,s}(\arccos x\cdot y)f(y)\,d\omega(y), \tag{1.1}$$

where k and s are positive integers,

$$\mathfrak{D}_{k,s}(\theta) = \left(\frac{\sin(k\theta/2)}{\sin(\theta/2)}\right)^{2s} \tag{1.2}$$

is the classical Jackson kernel, f is measurable function of degree p on the sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $d\omega(y)$ is the elementary surface piece, $|\mathbb{S}^{n-1}|$ is the measurement of $\mathbb{S}^{n-1}$. For $f \in L^p(\mathbb{S}^{n-1})$, $(1 \leq p \leq \infty)$ ($L^\infty(\mathbb{S}^{n-1})$ is the collection of continuous functions on $\mathbb{S}^{n-1}$), Li and Yang [4] proved that

$$C_1 \| J_{k,s}(f) - f \|_{\mathbb{S}^{n-1},p} \leq \omega^2\left(f, k^{-1}\right) \leq C_2\ k^{-2} \sum_{v=1}^{k} v \| J_{v,s}(f) - f \|_{\mathbb{S}^{n-1},p}, \tag{1.3}$$

and the saturation order for $J_{k,s}$ is k^{-2}, where C_1 and C_2 are independent of positive integer k and f, and $\omega^2(f,t)$ is the modulus of smoothness of degree 2 on the unit sphere $\mathbb{S}^{n-1}$.

Naturally, we desire to obtain the similar results on the spherical caps. To achieve the goal, a key issue is to establish the inverse inequality on the cap.

Recently, Belinsky et al. [2] constructed mth translation operator S_θ^m when discussing the averages of functions on the sphere. This inspires us to construct the mth Jackson-type operator $J_{k,s}^m$ on the spherical cap. We then prove a strong-type converse inequality for $J_{k,s}^m$, which helps us get the direct and inverse theorems of approximation on the spherical cap. Also, we obtain that the saturation order for the constructed Jackson-type operator is k^{-2}, the same to that of the Jackson operator on the sphere.

2. Definitions and Auxiliary Notations

Throughout this paper, we denote by the letters C and C_i (i is either positive integers or variables on which C depends only) positive constants depending only on the dimension n. Their value may be different at different occurrences, even within the same formula. We will denote the points in $\mathbb{S}^{n-1}$ by x and y, and the elementary surface piece on $\mathbb{S}^{n-1}$ by $d\omega$. If it is necessary, we will write $d\omega(x)$ referring to the variable of the integration. The notation $a \approx b$ means that there exists a positive constant C such that $C^{-1}b \leq a \leq Cb$ where C is independent of a and b.

Next, we introduce some concepts and properties of sphere as well as caps (see [7, 10]). The volume of $\mathbb{S}^{n-1}$ is

$$\Omega_{n-1} := \int_{\mathbb{S}^{n-1}} d\omega = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{2.1}$$

Corresponding to $d\omega$, the inner product on $\mathbb{S}^{n-1}$ is defined by

$$\langle f, g \rangle := \int_{\mathbb{S}^{n-1}} f(x)\overline{g(x)}d\omega(x). \tag{2.2}$$

Denote by $L^p(\mathbb{S}^{n-1})$ the space of p-integrable functions on $\mathbb{S}^{n-1}$ endowed with the norms

$$\begin{gathered} \|f\|_\infty := \|f\|_{L^\infty(\mathbb{S}^{n-1})} := \operatorname*{ess\,sup}_{x\in\mathbb{S}^{n-1}} |f(x)|, \\ \|f\|_p := \|f\|_{L^p(\mathbb{S}^{n-1})} := \left\{ \int_{\mathbb{S}^{n-1}} |f(x)|^p d\omega(x) \right\}^{1/p} < \infty, \quad 1 \leq p < \infty. \end{gathered} \tag{2.3}$$

We denote by $D(x_0,\gamma)$ the spherical cap with center x_0 and angle $0<\gamma\le\pi/2$, that is,

$$D(x_0,\gamma) := \left\{x \in \mathbb{S}^{n-1} : x\cdot x_0 \ge \cos\gamma\right\}, \tag{2.4}$$

and by $D(\gamma)$ the volume of $D(x_0,\gamma)$, that is,

$$D(\gamma) := \int_0^\gamma \left|\mathbb{S}^{n-2}\right| \sin^{n-2}\theta d\theta. \tag{2.5}$$

Then for fixed x_0 and γ, $L^p(D(x_0,\gamma))$ is a Banach space endowed with the norm $\|\cdot\|_{D,p}$ defined by

$$\begin{aligned} \|f\|_{D,\infty} &:= \operatorname{ess}\sup_{x\in D(x_0,\gamma)} |f(x)|, \\ \|f\|_{D,p} &:= \left\{\int_{D(x_0,\gamma)} |f(x)|^p d\omega(x)\right\}^{1/p}, \quad 1\le p<\infty. \end{aligned} \tag{2.6}$$

For any $f\in L^p(D(x_0,\gamma))$, we note

$$f^*(x) = \begin{cases} f(x), & x\in D(x_0,\gamma), \\ 0, & x\in \mathbb{S}^{n-1}\setminus D(x_0,\gamma), \end{cases} \tag{2.7}$$

and clearly, $f^*\in L^p(\mathbb{S}^{n-1})$ and $\|f^*\|_p = \|f\|_{D,p}$. This allows us to introduce some operators on spherical cap using existing operators on the sphere.

Definition 2.1. Suppose that $T : L^p(\mathbb{S}^{n-1}) \to L^p(\mathbb{S}^{n-1})$ is an operator on $\mathbb{S}^{n-1}$, then

$$\begin{aligned} &T_{x_0,\gamma} : L^p(D(x_0,\gamma)) \longrightarrow L^p(D(x_0,\gamma)), \\ &T_{x_0,\gamma}(f)(x) = T(f^*)(x), \quad x\in D(x_0,\gamma) \end{aligned} \tag{2.8}$$

is called the operator on $D(x_0,\gamma)$ introduced by T. We may use the notation T instead of $T_{x_0,\gamma}$ for convenience without mixing up.

We now make a brief introduction of projection operators $Y_j(\cdot)$ by ultraspherical (Gegenbauer) polynomials $\{G_j^\lambda\}_{j=1}^\infty$ $(\lambda=(n-2)/2)$ for discussion of saturation property of Jackson operators.

Ultraspherical polynomials $\{G_j^\lambda\}_{j=1}^\infty$ are defined in terms of the generating function (see [11]):

$$\frac{1}{(1-2tr+r^2)^\lambda} = \sum_{j=0}^\infty G_j^\lambda(t) r^j, \tag{2.9}$$

where $|r|<1$, $|t|\le 1$.

For any $\lambda > 0$, we have (see [11])

$$G_1^{\lambda}(t) = 2\lambda t,$$
$$\frac{d}{dt}G_j^{\lambda}(t) = 2\lambda G_{j-1}^{\lambda+1}(t). \tag{2.10}$$

When $\lambda = (n-2)/2$ (see [7]),

$$G_j^{\lambda}(t) = \frac{\Gamma(2\lambda + j)}{\Gamma(j+1)\Gamma(2\lambda)} P_j^n(t), \quad j = 0, 1, 2, \ldots, \tag{2.11}$$

where $P_j^n(t)$ is the Legendre polynomial of degree j. Particularly,

$$G_j^{\lambda}(1) = \frac{\Gamma(2\lambda + j)}{\Gamma(j+1)\Gamma(2\lambda)} P_j^n(1) = \frac{\Gamma(2\lambda + j)}{\Gamma(j+1)\Gamma(2\lambda)}, \quad j = 0, 1, 2, \ldots. \tag{2.12}$$

Therefore,

$$P_j^n(t) = \frac{G_j^{\lambda}(t)}{G_j^{\lambda}(1)}. \tag{2.13}$$

Besides, for any $j = 0, 1, 2, \ldots,$ and $|t| \leq 1$, $|P_j^n(t)| \leq 1$ (see [10]).

The projection operators is defined by

$$Y_j(f)(x) = \frac{\Gamma(\lambda)(n+\lambda)}{2\pi^{n/2}} \int_{\mathbb{S}^{n-1}} G_j^{\lambda}(x \cdot y) f^*(y) d\omega(y). \tag{2.14}$$

It follows from (2.10) and (2.13) that

$$\lim_{t \to 1-} \frac{1 - \left(P_j^n(t)\right)^m}{1 - \left(P_1^n(t)\right)^m} = \frac{j(j+2\lambda)}{2\lambda + 1}. \tag{2.15}$$

In the same way, we define the inner product on $D(x_0, \gamma)$ as follows:

$$\langle f, g \rangle_D := \int_{D(x_0,\gamma)} f(x) \overline{g(x)} d\omega(x). \tag{2.16}$$

We denote by $\widetilde{\Delta}$ the Laplace-Beltrami operator

$$\widetilde{\Delta} f := \sum_{i=1}^{n} \frac{\partial^2 g(x)}{\partial x_i^2} \bigg|_{|x|=1}, \qquad g(x) = f\left(\frac{x}{|x|}\right), \tag{2.17}$$

by which we define a K-function on $D(x_0,\gamma)$ as

$$K(f,\delta)_{D,p} := \inf\left\{ \|f-g\|_{D,p} + \delta\left\|\widetilde{\Delta}g\right\|_{D,p} : g,\ \widetilde{\Delta}g \in L^p(D(x_0,\gamma)) \right\}. \tag{2.18}$$

For $f \in L^1(D(x_0,\gamma))$, the translation operator is defined by

$$S_\theta(f)(x) := \frac{1}{|\mathbb{S}^{n-2}|\sin^{n-2}\theta}\int_{x\cdot y=\cos\theta} f^*(y)d\omega'(y), \tag{2.19}$$

where $d\omega'(y)$ denotes the the elementary surface piece on the sphere $\{y \in D(x_0,\gamma) : x \cdot y = \cos\theta\}$. Then we have

$$\int_{D(x_0,\gamma)} f(x)d\omega(x) = \int_0^\pi S_\theta(f)(x_0)\left|\mathbb{S}^{n-2}\right|\sin^{n-2}\theta d\theta. \tag{2.20}$$

The modulus of smoothness of f is defined by

$$\omega^2(f,\delta)_{D,p} := \sup_{0<\theta\leq\delta}\|S_\theta(f)-f\|_{D,p}. \tag{2.21}$$

Using the method of [3], we have

$$C_1\omega^2(f,\delta)_{D,p} \leq K\left(f,\delta^2\right)_{D,p} \leq C_2\omega^2(f,\delta)_{D,p}. \tag{2.22}$$

We introduce mth translation operator in terms of multipliers (see [6, 7, 12])

$$S_\theta^m(f) = \sum_{j=0}^\infty \left(\frac{G_j^\lambda(\cos\theta)}{G_j^\lambda(1)}\right)^m Y_j(f) \equiv \sum_{j=0}^\infty \left(P_j^n(\cos\theta)\right)^m Y_j(f), \quad f \in L^p(D(x_0,\gamma)). \tag{2.23}$$

It has been proved that (see [7])

$$\begin{aligned} S_\theta(f)(x) = \frac{1}{|\mathbb{S}^{n-2}|\sin^{n-2}\theta}\int_{x\cdot y=\cos\theta} f^*(y)d\omega'(y) &= \sum_{j=0}^\infty P_j^n(\cos\theta)Y_j(f)(x) \\ &= S_\theta^1(f)(x). \end{aligned} \tag{2.24}$$

With the help of S_θ^m, we can construct Jackson-type operator on $D(x_0,\gamma)$.

Definition 2.2. For $f \in L^p(D(x_0,\gamma))$, the mth Jackson-type operator of degree k on $D(x_0,\gamma)$ is defined by

$$J_{k,s}^m(f)(x) = \int_0^\gamma S_\theta^m(f)(x)\widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta, \tag{2.25}$$

where $\lambda = (n-2)/2$, and $\widetilde{D}_{k,s}(\theta) = A^{-1}_{\gamma,k,s}(\sin^{2s}(k\theta/2)/\sin^{2s-1}(\theta/2))$ satisfying $\int_0^{\gamma}\widetilde{D}_{k,s}(\theta) \times \sin^{2\lambda}\theta d\theta = 1$.

Remark 2.3. We may notice that $\widetilde{D}_{k,s}(\theta)$ is a bit different from classical Jackson kernel

$$\mathfrak{D}_{k,s}(\theta) = \left(\frac{\sin(k\theta/2)}{\sin(\theta/2)}\right)^{2s}. \tag{2.26}$$

This difference will help us to prove the converse inequality for $J^m_{k,s}$. For sake of ensuring that the converse inequality for $J^m_{k,s}$ holds, γ has to be no more than $\pi/2$. Particularly, for $m = 1$, we have

$$\begin{aligned} J^1_{k,s}(f)(x) &= \int_0^{\gamma} S^1_{\theta}(f)(x)\widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta \\ &= \int_0^{\gamma}\frac{1}{|\mathbb{S}^{n-2}|\sin^{2\lambda}\theta}\int_{x\cdot y=\cos\theta} f^*(y)d\omega'(y)\widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta \\ &= \frac{1}{|\mathbb{S}^{n-2}|}\int_{D(x_0,\gamma)} f^*(y)\widetilde{D}_{k,s}(\arccos(x\cdot y))d\omega(y) \\ &= \frac{1}{|\mathbb{S}^{n-2}|}\int_{D(x_0,\gamma)\bigcap D(x,\gamma)} f(y)\widetilde{D}_{k,s}(\arccos(x\cdot y))d\omega(y). \end{aligned} \tag{2.27}$$

Finally, we introduce the definition of saturation for operators (see [13]).

Definition 2.4. Let $\varphi(\rho)$ be a positive function with respect to ρ, $0 < \rho < \infty$, tending monotonely to zero as $\rho \to \infty$. For $\rho > 0$, I_ρ is a sequence of operators. If there exists $\mathcal{K} \subsetneqq L^p(D(x_0,\gamma))$ such that:

(i) If $\|I_\rho(f) - f\|_{D,p} = o(\varphi(\rho))$, then $I_\rho(f) = f$;

(ii) $\|I_\rho(f) - f\|_{D,p} = O(\varphi(\rho))$ if and only if $f \in \mathcal{K}$;

then I_ρ is said to be saturated on $L^p(D(x_0,\gamma))$ with order $O(\varphi(\rho))$ and $\mathcal{K}$ is called its saturation class.

3. Some Lemmas

In this section, we show some lemmas on both S^m_θ and $J^m_{k,s}$ as the preparation for the main results. For S^m_θ, we have the following.

Lemma 3.1. *For $f \in L^p(D(x_0,\gamma))$, $1 \le p \le \infty$, $0 < \theta \le \pi$,*

(i) *for $1 \le m_1 < m$, $S^m_\theta = S^{m_1}_\theta S^{m-m_1}_\theta$;*

(ii) *for $m \ge 1$, $\|S^m_\theta(f)\|_{D,p} \le \|f\|_{D,p}$;*

(iii) *for $m \ge 1$, $\|S^m_\theta(f) - f\|_{D,p} \le m\|S_\theta(f) - f\|_{D,p}$;*

(iv) *for $m > 2([n/2]+3)/(n-2)$, $0 < \theta \le \pi/2$, $\|\widetilde{\Delta} S^m_\theta(f)\|_{D,p} \le C_m\theta^{-2}\|f\|_{D,p}$, where $C_m \to 0$, as $m \to \infty$;*

(v) *for $m \ge 1$, and f which satisfies $\widetilde{\Delta} f \in L^p(D(x_0,\gamma))$, $\|\widetilde{\Delta} S^m_\theta(f)\|_{D,p} \le \|\widetilde{\Delta} f\|_{D,p}$.*

Proof. (i), (ii) and (iii) are clear. Using [2, Remark 3.5], we can obtain (iv). For (v), we have

$$\widetilde{\Delta} S_\theta(f) = -\sum_{j=0}^{\infty} j(j+2\lambda) P_j^n(\cos\theta) Y_j(f) = S_\theta\left(\widetilde{\Delta} f\right), \tag{3.1}$$

which implies

$$\left\| \widetilde{\Delta} S_\theta^m(f) \right\|_{D,p} = \left\| S_\theta\left(\widetilde{\Delta} S_\theta^{m-1}(f)\right) \right\|_{D,p} \le \left\| \widetilde{\Delta} S_\theta^{m-1}(f) \right\|_{D,p} \le \cdots \le \left\| \widetilde{\Delta} f \right\|_{D,p}. \tag{3.2}$$

□

We need the following lemma.

Lemma 3.2. *For $\beta \ge -2$, $2s > \beta + 2\lambda + 1$, $0 < \gamma \le \pi$, $n \ge 3$, and $k \ge 1$, one has*

$$\int_0^\gamma \theta^\beta \widetilde{D}_{k,s}(\theta) \sin^{2\lambda}\theta \, d\theta \approx k^{-\beta}, \tag{3.3}$$

where $\lambda = (n-2)/2$.

Proof. A simple calculation gives, for $\beta \ge -2$,

$$\begin{aligned} &\int_0^\gamma \theta^\beta \frac{\sin^{2s}(k\theta/2)}{\sin^{2s-1}(\theta/2)} \sin^{2\lambda}\theta d\theta \le C\left(\int_0^\infty \frac{\sin^{2s}}{\theta^{2s-(2\lambda+\beta+1)}} d\theta\right) k^{2s-(2\lambda+\beta+2)}, \\ &\int_0^\gamma \theta^\beta \frac{\sin^{2s}(k\theta/2)}{\sin^{2s-1}(\theta/2)} \sin^{2\lambda}\theta d\theta \ge C\left(\int_0^{k\gamma/4} \frac{\sin^{2s}}{\theta^{2s-(2\lambda+\beta+1)}} d\theta\right) k^{2s-(2\lambda+\beta+2)}. \end{aligned} \tag{3.4}$$

Therefore,

$$\int_0^\gamma \theta^\beta \widetilde{D}_{k,s}(\theta) \sin^{2\lambda}\theta d\theta = \frac{\int_0^\gamma \theta^\beta \sin^{2s}(k\theta/2)/\sin^{2s-1}(\theta/2) \sin^{2\lambda}\theta d\theta}{\int_0^\gamma \sin^{2s}(k\theta/2)/\sin^{2s-1}(\theta/2) \sin^{2\lambda}\theta d\theta} \approx k^{-\beta}. \tag{3.5}$$

□

For Jackson-type operator, we have the following lemma.

Lemma 3.3. *For $f \in L^p(D(x_0,\gamma))$, $1 \le p < \infty$, there hold*

(i) $\|J_{k,s}^m(f)\|_{D,p} \le \|f\|_{D,p}$;

(ii) *for f which satisfies $\widetilde{\Delta} f \in L^p(D(x_0,\gamma))$,* $\|\widetilde{\Delta} J_{k,s}^m(f)\|_{D,p} \le \|\widetilde{\Delta} f\|_{D,p}$;

(iii) *for $n \ge 3$, $m > 2([n/2]+3)/(n-2)$, and $0 < \gamma \le \pi/2$,* $\|\widetilde{\Delta} J_{k,s}^m(f)\|_{D,p} \le C_m k^2 \|f\|_{D,p}$.

Proof. From the definition and (ii) and (v) of Lemma 3.1, (i) and (ii) are clear. We just have to add the proof of (iii). In fact, using Minkowski inequality, (iv) of Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\left\|\widetilde{\Delta} J_{k,s}^{m}(f)\right\|_{D,p} &\leq \int_{0}^{\gamma}\left\|\widetilde{\Delta} S_{\theta}^{m}(f)\right\|_{D,p}\widetilde{D}_{k,s}(\theta)\sin^{2}\theta d\theta \\ &\leq C_{m}\|f\|_{D,p}\int_{0}^{\gamma}\theta^{-2}\widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta \\ &\approx C_{m,\gamma}k^{2}\|f\|_{D,p},\end{aligned} \tag{3.6}$$

where the constant in the approximation is independent of m and k. □

The following lemma is useful in the proof of the converse inequality for Jackson-type operator.

Lemma 3.4 (see [14]). *Suppose that for nonnegative sequences $\{\sigma_k\}_{k=1}^{\infty}$, $\{\tau_k\}_{k=1}^{\infty}$ with $\sigma_1 = 0$ the inequality*

$$\sigma_n \leq \left(\frac{k}{n}\right)^{p}\sigma_k + \tau_k, \quad p > 0,\ 1 \leq k \leq n, \tag{3.7}$$

is satisfied for any positive integer n. Then one has

$$\sigma_n \leq C_p n^{-p}\sum_{k=1}^{n}k^{p-1}\tau_k. \tag{3.8}$$

The following lemma gives the multiplier representation of $J_{k,s}^{m}(f)$, which follows from Definition 2.2 and (2.23).

Lemma 3.5. *For $f \in L^p(D(x_0,\gamma))$, $J_{k,s}^{m}(f)$ has the representation*

$$J_{k,s}^{m}(f)(x) = \sum_{j=0}^{\infty}\xi_{k}^{m}(j)Y_j(f)(x), \tag{3.9}$$

where

$$\xi_{k}^{m}(j) = \int_{0}^{\gamma}\widetilde{D}_{k,s}(\theta)\left(P_j^{n}(\cos\theta)\right)^{m}\sin^{2\lambda}\theta\ d\theta, \quad j = 0,1,2,\ldots. \tag{3.10}$$

The following lemma is useful for determining the saturation order. It can be deduced by the methods of [13, 15].

Lemma 3.6. *Suppose that* $\{I_\rho\}_{\rho>0}$ *is a sequence of operators on* $L^p(D(x_0,\gamma))$*, and there exists function series* $\{\lambda_\rho(j)\}_{j=1}^{\infty}$ *with respect to* ρ*, such that*

$$I_\rho(f)(x) = \sum_{j=0}^{\infty} \lambda_\rho(j) Y_j(f)(x) \tag{3.11}$$

for every $f \in L^p(D(x_0,\gamma))$*. If for any* $j = 0,1,2,\ldots$*, there exists* $\varphi(\rho) \to 0+ (\rho \to \rho_0)$ *such that*

$$\lim_{\rho\to\rho_0} \frac{1-\lambda_\rho(j)}{\varphi(\rho)} = \tau_j \neq 0, \tag{3.12}$$

then $\{I_\rho\}_{\rho>0}$ *is saturated on* $L^p(D(x_0,\gamma))$ *with the order* $O(\varphi(\rho))$ *and the collection of all constants is the invariant class for* $\{I_\rho\}_{\rho>0}$ *on* $L^p(D(x_0,\gamma))$*.*

4. Main Results and Their Proofs

In this section, we will discuss the main results, that is, the lower and upper bounds as well as the saturation order for Jackson-type operator on $L^p(D(x_0,\gamma))$.

The following theorem gives the Jackson-type inequality for $J_{k,s}^m$.

Theorem 4.1. *For any integer* $m \geq 1$ *and* $0 < \gamma \leq \pi/2$*,* $\{J_{k,s}^m\}_{k=1}^{\infty}$ *is the series of Jackson-type operators on* $L^p(D(x_0,\gamma))$ *defined previously, and* $g \in L_p^{(2)}(D(x_0,\gamma)) := \{f \in L^p(D(x_0,\gamma)) : \widetilde{\Delta} f \in L^p(D(x_0,\gamma))\}, 1 \leq p \leq \infty$*.*

Then

$$\left\| J_{k,s}^m(g) - g \right\|_{D,p} \leq C_{m,\gamma}\ k^{-2} \left\| \widetilde{\Delta} g \right\|_{D,p}. \tag{4.1}$$

Therefore, for $f \in L^p(D(x_0,\gamma))$*,*

$$\left\| J_{k,s}^m(f) - f \right\|_{D,p} \leq C_{m,\gamma}\ \omega^2\left(f, k^{-1}\right)_{D,p}, \tag{4.2}$$

where C is independent of k and f.

Proof. Since $g \in L_p^{(2)}(D(x_0,\gamma))$, we have (see [13])

$$S_\theta(g)(x) - g(x) = \int_0^\theta \sin^{2-n}\nu \int_0^\nu \sin^{n-2}\tau S_\tau\left(\widetilde{\Delta} g\right)(x) d\tau\, d\nu, \tag{4.3}$$

and it is true that

$$\sup_{\theta>0}\left\{\theta^{-2}\int_0^{\theta}\sin^{2-n}\nu\int_0^{\nu}\sin^{n-2}\tau d\tau\, d\nu\right\}<\infty. \tag{4.4}$$

Therefore (explained below),

$$\begin{aligned}\left\|J_{k,s}^m(g)-g\right\|_{D,p}&=\left\|\int_0^{\gamma}\widetilde{D}_{k,s}(\theta)\left(S_\theta^m(g)(x)-g(x)\right)\sin^{n-2}\theta d\theta\right\|_{D,p}\\&\le m\int_0^{\gamma}\widetilde{D}_{k,s}(\theta)\sin^{n-2}\theta\left(\int_0^{\theta}\sin^{2-n}\nu\int_0^{\nu}\sin^{n-2}\tau\left\|S_\tau(\widetilde{\Delta}g)\right\|_{D,p}d\tau d\nu\right)d\theta\\&\le m\sup_{\theta>0}\left\{\theta^{-2}\int_0^{\theta}\sin^{2-n}\nu\int_0^{\nu}\sin^{n-2}\tau d\tau\, d\nu\right\}\left\|\widetilde{\Delta}g\right\|_{D,p}\int_0^{\gamma}\theta^2\widetilde{D}_{k,s}(\theta)\sin^{n-2}\theta\, d\theta\\&\le C_{m,\gamma}k^{-2}\left\|\widetilde{\Delta}g\right\|_{D,p},\end{aligned} \tag{4.5}$$

where the Minkowski inequality, (4.3), and Lemma 3.1 are used in the first inequality, and the second and third one are deduced from (3.3) and Lemma 3.1. From (2.22) and (i) of Lemma 3.3, it is easy to deduce (4.2). □

Next, we prove the Bernstein-type inequality for $J_{k,s}^m(f)(x)$ for $f\in L^p(D(x_0,\gamma))$.

Theorem 4.2. *Assume that* $\{J_{k,s}^m\}_{k=1}^{\infty}$, $m>2([n/2]+3)/(n-2)$ *are mth Jackson-type operators on* $D(x_0,\gamma)$. *For* $f\in L^p(D(x_0,\gamma))$, $0<\gamma\le\pi/2$, *then there exits a constant C independent of k and f such that*

$$\omega^2\left(f,k^{-1}\right)_{D,p}\le C\max_{v\ge k}\left\|J_{v,s}^m(f)-f\right\|_{D,p} \tag{4.6}$$

holds for every $f\in L^p(D(x_0,\gamma))$ *and every integer k.*

Proof. Li and Yang [4] have proved the Marchaud-Stečkin inequality for Jackson operator on the sphere. Following the method in [4], we first prove the Marchaud-Stečkin inequality for $J_{k,s}^m$:

$$\omega^2\left(f,k^{-1}\right)_{D,p}\le C_1k^{-2}\sum_{v=1}^{k}v\left\|J_{v,s}^m(f)-f\right\|_{D,p}. \tag{4.7}$$

Let

$$\sigma_k=k^{-2}\left\|\widetilde{\Delta}J_{k,s}^m(f)\right\|_{D,p},\qquad \tau_k=\left\|J_{k,s}^m(f)-f\right\|_{D,p}, \tag{4.8}$$

then for $v = 1, 2, \ldots, k$, by Lemma 3.3,

$$\begin{aligned}
\sigma_k &= k^{-2}\left\| \widetilde{\Delta} J^m_{k,s}(f) - \widetilde{\Delta} J^m_{k,s}(J^m_{v,s}(f)) + \widetilde{\Delta} J^m_{k,s}(J^m_{v,s}(f))\right\|_{D,p} \\
&= k^{-2}\left\| \widetilde{\Delta} J^m_{k,s}(f - J^m_{v,s}(f))\right\|_{D,p} + \left\| \widetilde{\Delta} J^m_{k,s}(J^m_{v,s}(f))\right\|_{D,p} \\
&\leq k^{-2}\left(Ck^2 \| f - J^m_{v,s}(f)\|_{D,p} + \left\| \widetilde{\Delta} J^m_{v,s}(f)\right\|_{D,p}\right) \\
&= C\| f - J^m_{v,s}(f)\|_{D,p} + k^{-2}\left\| \widetilde{\Delta} J^m_{v,s}(f)\right\|_{D,p} \\
&= C\tau_v + \left(\frac{v}{k}\right)^2 \sigma_v,
\end{aligned} \tag{4.9}$$

so we can deduce from Lemma 3.4 (where p is set to be 2) that

$$\sigma_k \leq Ck^{-2}\sum_{v=1}^{k} v\tau_v, \tag{4.10}$$

that is,

$$\left\| \widetilde{\Delta} J^m_{k,s}(f)\right\|_{D,p} \leq C \sum_{v=1}^{k} v\| J^m_{v,s}(f) - f\|_{D,p}. \tag{4.11}$$

Since there exists $k/2 \leq r \leq k$ such that

$$\| J^m_{r,s}(f) - f\|_{D,p} = \min_{k/2\leq v\leq k} \| J^m_{v,s}(f) - f\|_{D,p} \tag{4.12}$$

then,

$$\begin{aligned}
K\left(f, k^{-2}\right)_{D,p} &\leq \| J^m_{r,s}(f) - f\|_{D,p} + k^{-2}\left\| \widetilde{\Delta} J^m_{r,s}(f)\right\|_{D,p} \\
&\leq 4k^{-2}\sum_{k/2\leq v\leq k} v\| J^m_{v,s}(f) - f\|_{D,p} + Ck^{-2}\sum_{v=1}^{k} v\| J^m_{v,s}(f) - f\|_{D,p} \\
&\leq Ck^{-2}\sum_{v=1}^{k} v\| J^m_{v,s}(f) - f\|_{D,p}.
\end{aligned} \tag{4.13}$$

By (2.22), we obtain that

$$\begin{aligned}
\omega^2\left(f, k^{-1}\right)_{D,p} &\leq CK\left(f, k^{-2}\right)_{D,p} \\
&\leq Ck^{-2}\sum_{v=1}^{k} v\| J^m_{v,s}(f) - f\|_{D,p}.
\end{aligned} \tag{4.14}$$

So (4.7) holds, and it implies that

$$\omega^2\left(f,k^{-1}\right)_{D,p} \le C_1 k^{-1-1/2}\sum_{v=1}^{k} v^{1/2}\left\| J_{v,s}^m(f)-f\right\|_{D,p}. \tag{4.15}$$

In order to prove (4.6), we have to show that

$$\begin{aligned}\omega^2\left(f,k^{-1}\right)_{D,p} &\approx \frac{1}{k^2}\max_{1\le v\le k} v^2\left\| J_{v,s}^m(f)-f\right\|_{D,p} \\ &\approx \frac{1}{k^{2+1/4}}\max_{1\le v\le k} v^{2+1/4}\left\| J_{v,s}^m(f)-f\right\|_{D,p}.\end{aligned} \tag{4.16}$$

We first prove

$$\omega^2\left(f,k^{-1}\right)_{D,p} \approx \frac{1}{k^2}\max_{1\le v\le k} v^2\left\| J_{v,s}^m(f)-f\right\|_{D,p}. \tag{4.17}$$

It follows from (4.2) and (4.15) that

$$\begin{aligned}\omega^2\left(f,k^{-1}\right)_{D,p} &\le C_1 k^{-2}k^{1/2}\sum_{v=1}^{k} v^{-3/2}\left(v^2\left\| J_{v,s}^m(f)-f\right\|_{D,p}\right) \\ &\le C_1 k^{-2}\left(\max_{1\le v\le k} v^2\left\| J_{v,s}^m(f)-f\right\|_{D,p}\right)\left(k^{1/2}\sum_{v=1}^{k} v^{-3/2}\right) \\ &\le C_3 k^{-2}\max_{1\le v\le k} v^2\left\| J_{v,s}^m(f)-f\right\|_{D,p} \\ &\le C_4 k^{-2}\max_{1\le v\le k} v^2\ \omega^2\left(f,v^{-1}\right)_{D,p} \\ &\le C_5\left(k^{-2}\max_{1\le v\le k} v^2\left(1+\left(\frac{k}{v}\right)^2\right)\right)\omega^2\left(f,k^{-1}\right)_{D,p} \\ &\le 2C_5\omega^2\left(f,k^{-1}\right)_{D,p}.\end{aligned} \tag{4.18}$$

Then we prove

$$\omega^2\left(f,k^{-1}\right)_{D,p} \approx \frac{1}{k^{2+1/4}}\max_{1\le v\le k} v^{2+1/4}\left\| J_{v,s}^m(f)-f\right\|_{D,p}. \tag{4.19}$$

In fact, the proof is similar to that of (4.17)

$$\begin{aligned}\omega^2\left(f,k^{-1}\right)_{D,p} &\leq C_1 k^{-2-1/4} k^{3/4} \sum_{k=1}^{k} v^{-7/4}\left(v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p}\right)\\ &\leq C_1 k^{-2-1/4}\left(\max_{1\leq v\leq k} v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p}\right)\left(k^{3/4}\sum_{v=1}^{k} v^{-7/4}\right)\\ &\leq C_6 k^{-2-1/4}\max_{1\leq v\leq k} v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p}\\ &\leq C_7 k^{-2-1/4}\max_{1\leq v\leq k} v^{2+1/4}\omega^2\left(f,v^{-1}\right)_{D,p}\\ &\leq C_8\left(k^{-2-1/4}\max_{1\leq v\leq k} v^{2+1/4}\left(1+\left(\frac{k}{v}\right)^2\right)\right)\omega^2\left(f,k^{-1}\right)_{D,p}\leq 2C_8\omega^2\left(f,k^{-1}\right)_{D,p}.\end{aligned} \tag{4.20}$$

Hence,

$$\omega^2\left(f,k^{-1}\right)_{D,p} \approx \frac{1}{k^2}\max_{1\leq v\leq k} v^2\left\|J_{v,s}^m(f)-f\right\|_{D,p} \approx \frac{1}{k^{2+1/4}}\max_{1\leq v\leq k} v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p}. \tag{4.21}$$

Now we can complete the proof of (4.6). Let

$$\max_{1\leq v\leq k} v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p} = k_1^{2+1/4}\left\|J_{k_1,s}^m(f)-f\right\|_{D,p}, \quad 1\leq k_1\leq k. \tag{4.22}$$

It follows from (4.16) that

$$\begin{aligned}k^{-2}k_1^2\left\|J_{k_1,s}^m(f)-f\right\|_{D,p} &\leq k^{-2}\max_{1\leq v\leq k} v^2\left\|J_{v,s}^m(f)-f\right\|_{D,p}\\ &\leq \frac{C_9}{k^{2+1/4}}\max_{1\leq v\leq k} v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p}\\ &= \frac{C_9}{k^{2+1/4}}k_1^{2+1/4}\left\|J_{k_1,s}^m(f)-f\right\|_{D,p}.\end{aligned} \tag{4.23}$$

Thus, $C_9^{-4}k \leq k_1$. Since $k_1 \leq k$, then $k \approx k_1$, we obtain from (4.16) that

$$\begin{aligned}\omega^2\left(f,k^{-1}\right)_{D,p} &\leq C_{10}k^{-2-1/4}\max_{1\leq v\leq k} v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p}\\ &= C_{10}k^{-2-1/4}\left(k_1^{2+1/4}\left\|J_{k_1,s}^m(f)-f\right\|_{D,p}\right)\\ &\leq C_{10}k^{-2-1/4}\max_{k_1\leq v\leq k} v^{2+1/4}\left\|J_{v,s}^m(f)-f\right\|_{D,p}\\ &\leq C_{10}\max_{k_1\leq v\leq k}\left\|J_{v,s}^m(f)-f\right\|_{D,p}.\end{aligned} \tag{4.24}$$

Noticing that $k \approx k_1$, we may rewrite the previous inequality as

$$\omega^2\left(f, k^{-1}\right)_{D,p} \le C_{11} \max_{v \ge k} \left\| J^m_{v,s}(f) - f \right\|_{D,p}. \tag{4.25}$$

This completes the proof. □

We thus obtain the corollary of Theorems 4.1 and 4.2.

Corollary 4.3. *Suppose that* $\{J^m_{k,s}\}_{k=1}^{\infty}$, $m > 2([n/2]+3)/(n-2)$, *are Jackson-type operators on the spherical cap* $D(x_0, \gamma)$, $0 < \gamma \le \pi/2$, *then the following are equivalent for any* $f \in L^p(D(x_0,\gamma))$, $\alpha > 0$,

(i) $\|J^m_{k,s}(f) - f\|_{D,p} = O(k^{-\alpha})$, $k \to \infty$;

(ii) $\omega^2(f,\delta)_{D,p} = O(\delta^{\alpha})$, $\delta \to 0+$.

Theorem 4.4. *Suppose that* $\{J^m_{k,s}\}_{k=1}^{\infty}$, $m \ge 1$ *are Jackson-type operators on the spherical cap* $D(x_0,\gamma)$, $0 < \gamma \le \pi/2$. *Then* $\{J^m_{k,s}\}_{k=1}^{\infty}$ *are saturated on* $L^p(D(x_0,\gamma))$ *with order* k^{-2} *and the collection of constants is their invariant class.*

Proof. We obtain from Lemma 3.2 that, for $v = 1, 2, \ldots,$

$$1 - \xi_k^m(1) = \int_0^{\gamma} \widetilde{D}_{k,s}(\theta)\left(1 - \left(\frac{G_1^{\lambda}(\cos\theta)}{G_1^{\lambda}(1)}\right)^m\right)\sin^{2\lambda}\theta \, d\theta \approx \int_0^{\gamma} \widetilde{D}_{k,s}(\theta)\sin^2\frac{\theta}{2}\sin^{2\lambda}\theta \, d\theta \approx k^{-2}. \tag{4.26}$$

By Lemma 3.6, if it is true that for $j = 0, 1, 2, \ldots,$

$$\lim_{k\to\infty} \frac{1 - \xi_k^m(j)}{1 - \xi_k^m(1)} = \frac{j(j+2\lambda)}{2\lambda + 1}, \tag{4.27}$$

then the proof is completed.

In fact, for any $0 < \delta < \gamma$, it follows from (3.3) that

$$\begin{aligned} \int_{\delta}^{\gamma} \widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta \, d\theta &\le \int_{\delta}^{\gamma} \left(\frac{\theta}{\delta}\right)^3 \widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta \, d\theta \\ &\le \delta^{-3}\int_0^{\gamma} \theta^3 \widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta \, d\theta \le C_{\delta,s}k^{-3}. \end{aligned} \tag{4.28}$$

We deduce from (2.15) that, for any $\varepsilon > 0$, there exists $\delta > 0$, for $0 < \theta < \delta$, it holds that

$$\left| \left(1 - \left(P_j^n(\cos\theta)\right)^m\right) - \frac{j(j+2\lambda)}{2\lambda+1}\left(1 - \left(P_1^n(\cos\theta)\right)^m\right) \right| \le \varepsilon\left(1 - \left(P_1^n(\cos\theta)\right)^m\right). \tag{4.29}$$

Then it follows that

$$
\begin{aligned}
&\left|\left(1-\xi_k^m(j)\right)-\frac{j(j+2\lambda)}{2\lambda+1}\left(1-\xi_k^m(1)\right)\right| \\
&\quad=\left|\int_0^{\gamma}\widetilde{D}_{k,s}(\theta)\left(1-\left(P_j^n(\cos\theta)\right)^m\right)\sin^{2\lambda}\theta\,d\theta\right. \\
&\qquad\left.-\int_0^{\gamma}\widetilde{D}_{k,s}(\theta)\left(1-\left(P_1^n(\cos\theta)\right)^m\right)\frac{j(j+2\lambda)}{2\lambda+1}\sin^{2\lambda}\theta\;d\theta\right| \\
&\quad=\left|\int_0^{\gamma}\widetilde{D}_{k,s}(\theta)\left(\left(1-\left(P_j^n(\cos\theta)\right)^m\right)-\left(1-\left(P_1^n(\cos\theta)\right)^m\right)\frac{j(j+2\lambda)}{2\lambda+1}\right)\sin^{2\lambda}\theta\;d\theta\right| \\
&\quad\leq\int_0^{\delta}\widetilde{D}_{k,s}(\theta)\;\varepsilon\;\sin^{2\lambda}\theta\;d\theta+2\int_{\delta}^{\gamma}\widetilde{D}_{k,s}(\theta)\sin^{2\lambda}\theta\left(1+\frac{j(j+2\lambda)}{2\lambda+1}\right)d\theta \\
&\quad\leq\varepsilon k^{-2}+C_{\delta,s}k^{-3}.
\end{aligned}
\tag{4.30}
$$

So,

$$
\lim_{k\to\infty}\frac{1-\xi_k(j)}{1-\xi_k(1)}=\frac{j(j+2\lambda)}{2\lambda}\neq 0.
\tag{4.31}
$$

Therefore, we obtain by Lemma 3.6 that the saturation order for $J_{k,s}^m$ is $O(1-\xi_k^m(1))\approx k^{-2}$. □

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