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Research Article

Perturbed Iterative Approximation of Solutions for Nonlinear General A-Monotone Operator Equations in Banach Spaces

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We introduce and study a new class of nonlinear general A-monotone operator equations with multivalued operator. By using Alber's inequalities, Nalder's results, and the new proximal mapping technique, we construct some new perturbed iterative algorithms with mixed errors for solving the nonlinear general A-monotone operator equations and study the approximation-solvability of the nonlinear operator equations in Banach spaces. The results presented in this paper improve and generalize the corresponding results on strongly monotone quasivariational inclusions and nonlinear implicit quasivariational inclusions.

1. Introduction

Let $\mathcal{X}$ be a real Banach space with the topological dual space of $\mathcal{X}^*$, let $\langle x, y\rangle$ be the pairing between $x \in \mathcal{X}^*$ and $y \in \mathcal{X}$, let $2^{\mathcal{X}^*}$ denote the family of all subsets of $\mathcal{X}^*$, and let $CB(\mathcal{X})$ denote the family of all nonempty closed bounded subsets of $\mathcal{X}$. We denote by the $\langle z, x\rangle = z(x)$ for all $x \in \mathcal{X}$ and $z \in \mathcal{X}^*$. Let $f : \mathcal{X} \to \mathcal{X}^*$, $T : \mathcal{X} \to 2^{\mathcal{X}^*}$, $g : \mathcal{X} \to \mathcal{X}$, and $A : \mathcal{X} \to \mathcal{X}^*$ be nonlinear operators, and let $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a general A-monotone operator such that $g(\mathcal{X}) \cap \operatorname{dom} M(\cdot) \neq \emptyset$. We will consider the following nonlinear general A-monotone operator equation with multivalued operator.

Find $x \in \mathcal{X}$ such that $u \in T(x)$ and

$$\epsilon g(x) = P_M^A\left[A\left(g(x)\right) - \rho\left(f(x) + u\right)\right], \tag{1.1}$$

where $\epsilon \in (0,1]$ is a constant and $P_M^A = (A+\rho M)^{-1}$ is the proximal mapping associated with the general A-monotone operator M due to Cui et al. [1].

It is easy to see that the problem (1.1) is equivalent to the problem of finding $x \in \mathcal{X}$ such that

$$\epsilon g(x) \in P_M^A\left[A(g(x)) - \rho(f(x) + T(x))\right]. \tag{1.2}$$

Example 1.1. If $\epsilon \equiv 1$, then the problem (1.1) is equivalent to finding $x \in \mathcal{X}$ such that $u \in T(x)$ and

$$g(x) = P_M^A\left[A(g(x)) - \rho(f(x) + u)\right]. \tag{1.3}$$

Based on the definition of the proximal mapping P_M^A, (1.3) can be written as

$$0 \in f(x) + u + M(g(x)). \tag{1.4}$$

Example 1.2. If $T : \mathcal{X} \to \mathcal{X}^*$ is a single-valued operator, then a special case of the problem (1.3) is to determine element $x \in \mathcal{X}$ such that

$$g(x) - P_M^A\left[A(g(x)) - \rho Q(x)\right] = 0, \tag{1.5}$$

where $Q : \mathcal{X} \to \mathcal{X}^*$ is defined by $Q(x) = f(x) + T(x)$ for all $x \in \mathcal{X}$. The problem (1.5) was studied by Xia and Huang [2] when M is a general H-monotone mapping. Further, the problem (1.5) was studied by Peng et al. [3] if $g = I$, the identity operator, and M is a multivalued maximal monotone mapping.

Example 1.3. If $g = I, T : \mathcal{X} \to \mathcal{X}^*$, and $N : \mathcal{X} \times \mathcal{X} \to \mathcal{X}^*$ are single-valued operators, and $f(x) + T(x) = N(x,x)$ for all $x \in \mathcal{X}$, then the problem (1.3) reduces to finding an element $x \in \mathcal{X}$ such that

$$x - P_M^A\left[A(x) - \rho N(x,x)\right] = 0, \tag{1.6}$$

which was considered by Verma [4, 5].

We note that for appropriate and suitable choices of ϵ, g, A, M, f, T, and $\mathcal{X}$, it is easy to see that the problem (1.1) includes a number of quasivariational inclusions, generalized quasivariational inclusions, quasivariational inequalities, implicit quasivariational inequalities, complementarity problems, and equilibrium problems studied by many authors as special cases; see, for example, [4–7] and the references therein.

The study of such types of problems is motivated by an increasing interest to study the behavior and approximation of the solution sets for many important nonlinear problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional structural, transportation, elasticity, engineering, and various applied sciences in a general and unified framework. It is well known that many authors have studied a number of nonlinear variational inclusions and many systems of variational inequalities, variational inclusions, complementarity problems, and equilibrium problems by using the resolvent operator technique, which is a very important method to find solutions of variational inequality and variational inclusion problems; see, for example, [1–15] and the references therein.

On the other hand, Verma [4, 5] introduced the concept of A-monotone mappings, which generalizes the well-known general class of maximal monotone mappings and originates way back from an earlier work of the Verma [7]. Furthermore, motivated and inspired by the works of Xia and Huang [2], Cui et al. [1] introduced first a new class of general A-monotone operators in Banach spaces, studied some properties of general A-monotone operator, and defined a new proximal mapping associated with the general A-monotone operator.

Inspired and motivated by the research works going on this field, the purpose of this paper is to introduce the new class of nonlinear general A-monotone operator equation with multivalued operator. By using Alber's inequalities, Nalder's results, and the new proximal mapping technique, some new perturbed iterative algorithms with mixed errors for solving the nonlinear general A-monotone operator equations will be constructed, and applications of general A-monotone operators to the approximation-solvability of the nonlinear operator equations in Banach spaces will be studied. The results presented in this paper improve and extend some corresponding results in recent literature.

2. Preliminaries

In this paper, we will use the following definitions and lemmas.

Definition 2.1. Let $A : \mathcal{X} \to \mathcal{X}^*$, $g : \mathcal{X} \to \mathcal{X}$, and $f : \mathcal{X} \to \mathcal{X}$ be single-valued operators. Then

(i) A is r-strongly monotone, if there exists a positive constant r such that

$$\langle A(x) - A(y), x - y \rangle \geq r \|x - y\|^2, \quad \forall x,\ y \in \mathcal{X}; \tag{2.1}$$

(ii) A is τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|A(x) - A(y)\| \leq \tau \|x - y\|^2, \quad \forall x,\ y \in \mathcal{X}; \tag{2.2}$$

(iii) g is k-strongly accretive if for any $x,\ y \in \mathcal{X}$, there exist $j(x-y) \in J(x-y)$ and a positive constant k such that

$$\langle j(x-y), g(x) - g(y) \rangle \geq k \|x - y\|^2, \tag{2.3}$$

where the generalized duality mapping $J_q : \mathcal{X} \to 2^{\mathcal{X}^*}$ is defined by

$$J_q(x) = \left\{ f^* \in \mathcal{X}^* : \langle x, f^* \rangle = \|x\|^q,\ \|f^*\| = \|x\|^{q-1} \right\}, \quad \forall x \in \mathcal{X}; \tag{2.4}$$

(iv) f is (γ, μ)-relaxed cocoercive with respect to A, if for all $x, y \in \mathcal{X}$, there exist positive constants γ and μ such that

$$\langle A(x) - A(y), f(x) - f(y) \rangle \geq -\gamma \|f(x) - f(y)\|^2 + \mu \|x - y\|^2. \tag{2.5}$$

Example 2.2 (see [11, 12]). (1) Consider an r-strongly monotone (and hence r-expanding) operator $T : \mathcal{X} \to \mathcal{X}$. Then T is $(r + r^2, 1)$-relaxed cocoercive with respect to I.

(2) Very m-cocoercive operator is m-relaxed cocoercive, while each r-strongly monotone mapping is $(r + r^2, 1)$-relaxed cocoercive with respect to I.

Remark 2.3. The notion of the cocoercivity is applied in several directions, especially to solving variational inequality problems using the auxiliary problem principle and projection methods [5], while the notion of the relaxed cocoercivity is more general than the strong monotonicity as well as cocoercivity. Several classes of relaxed cocoercive variational inequalities have been studied in [4, 5].

Definition 2.4. A multivalued operator $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ is said to be

(i) maximal monotone if, for any $x \in \mathcal{X}, u \in M(x)$,

$$\langle u - v, x - y \rangle \geq 0 \quad \text{implies} \quad y \in \mathcal{X}, \ v \in M(y); \tag{2.6}$$

(ii) m-relaxed monotone if, for any $x, y \in \mathcal{X}$, $u \in M(x)$, and $v \in M(y)$, there exists a positive constant m such that

$$\langle u - v, x - y \rangle \geq -m \|x - y\|^2; \tag{2.7}$$

(iii) ξ-$\widehat{\mathbf{H}}$-Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$\widehat{\mathbf{H}}(M(x), M(y)) \leq \xi \|x - y\|, \quad \forall x, y \in \mathcal{X}, \tag{2.8}$$

where $\widehat{\mathbf{H}} : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff pseudometric, that is,

$$\widehat{\mathbf{H}}(D, E) = \max\left\{ \sup_{x \in D} \inf_{y \in E} \|x - y\|, \sup_{x \in E} \inf_{y \in D} \|x - y\| \right\}, \quad \forall D, E \in 2^{\mathcal{X}^*}. \tag{2.9}$$

Note that if the domain of $\widehat{\mathbf{H}}$ is restricted to closed bounded subsets $CB(\mathcal{X})$, then $\widehat{\mathbf{H}}$ is the Hausdorff metric.

Definition 2.5. A single-valued operator $g : \mathcal{X} \to \mathcal{X}^*$ is said to be

(i) coercive if

$$\lim_{\|x\| \to \infty} \frac{\langle g(x), x \rangle}{\|x\|} = +\infty; \tag{2.10}$$

(ii) hemicontinuous if, for any fixed $x, y, z \in \mathcal{X}$, the function $t \to \langle g(x+ty), z\rangle$ is continuous at 0^+.

We remark that the uniform convexity of the space $\mathcal{X}$ means that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathcal{X}$, $\|x\| \le 1$, $\|y\| \le 1$ and $\|x-y\| = \varepsilon$ ensure the inequality $\|x+y\| \le 2(1-\delta)$. The function

$$\delta_{\mathcal{X}}(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1, \|y\| = 1, \|x-y\| = \varepsilon\right\} \tag{2.11}$$

is called the modulus of the convexity of the space $\mathcal{X}$.

The uniform smoothness of the space $\mathcal{X}$ means that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $(1/2)(\|x+y\| + \|x-y\|) - 1 \le \varepsilon\|y\|$ holds. The function $\varrho_X : [0,\infty) \to [0,\infty)$ defined by

$$\varrho_{\mathcal{X}}(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le t\right\} \tag{2.12}$$

is called the modulus of the smoothness of the space $\mathcal{X}$.

We also remark that the space $\mathcal{X}$ is uniformly convex if and only if $\delta_{\mathcal{X}}(\varepsilon) > 0$ for all $\varepsilon > 0$, and it is uniformly smooth if and only if $\lim_{t\to 0}(\varrho_{\mathcal{X}}(t))/t = 0$. Moreover, $\mathcal{X}$ is uniformly convex if and only if $\mathcal{X}$ is uniformly smooth. In this case, $\mathcal{X}$ is reflexive by the Milman theorem. A Hilbert space is uniformly convex and uniformly smooth. The proof of the following inequalities can be found, for example, in page 24 of Alber [16].

Lemma 2.6. *Let $\mathcal{X}$ be a uniformly smooth Banach space, and let J be the normalized duality mapping from $\mathcal{X}$ into $\mathcal{X}^*$. Then, for all $x, y \in \mathcal{X}$, we have*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle$;

(ii) $\langle x-y, J(x)-J(y)\rangle \le 2d^2\varrho_{\mathcal{X}}(4\|x-y\|/d)$, *where* $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

Definition 2.7. Let $\mathcal{X}$ be a Banach space with the dual space $\mathcal{X}^*$, $A : \mathcal{X} \to \mathcal{X}^*$ be a nonlinear operator, and $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a multivalued operator. The map M is said to be general A-monotone if M is m-relaxed monotone and $R(A+\rho M) = \mathcal{X}^*$ holds for every $\rho > 0$.

This is equivalent to stating that M is general A-monotone if.

(i) M is m-relaxed monotone;

(ii) $A+\rho M$ is maximal monotone for every $\rho > 0$.

Remark 2.8. (1) If $m = 0$, that is, M is 0-relaxed monotone, then the general A-monotone operators reduce to general H-monotone operators (see, e.g., [1, 2]).

(2) If $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then the general A-monotone operator reduces to the A-monotone operator in Verma [7]. Therefore, the class of general A-monotone operators provides a unifying frameworks for classes of maximal monotone operators, H-monotone operators, A-monotone operators, and general H-monotone operators. For details about these operators, we refer the reader to [1, 2, 7] and the references therein.

Example 2.9. Let $\mathcal{X}$ be a reflexive Banach space with the dual space $\mathcal{X}^*$, $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ a maximal monotone mapping, and $A : \mathcal{X} \to \mathcal{X}$ a bounded, coercive, hemicontinuous, and relaxed monotone mapping. Then for any given $\rho > 0$, it follows from Theorem 3.1 in page 401 of Guo [10] that $(A+\rho M)(\mathcal{X}) = \mathcal{X}^*$. This shows that M is a general A-monotone operator.

Example 2.10 (see [4]). Let $\mathcal{X}$ be a reflexive Banach space with $\mathcal{X}^*$ its dual, and let $A : \mathcal{X} \to \mathcal{X}^*$ be r-strongly monotone. Let $f : \mathcal{X} \to R$ be locally Lipschitz such that ∂f is m-relaxed monotone. Then ∂f is A-monotone, which is equivalent to stating that $A + \partial f$ is pseudomonotone (and in fact, maximal monotone).

Lemma 2.11 (see [1]). *Let $\mathcal{X}$ be a reflexive Banach space with the dual space $\mathcal{X}^*$, let $A : \mathcal{X} \to \mathcal{X}^*$ be a nonlinear operator, and let $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a general A-monotone operator. Then the proximal mapping P_M^A is*

(i) $(1/(r - \rho m))$*-Lipschitz continuous when A is r-strongly monotone with $r > m$ and $\rho \in (0, r/m)$;*

(ii) $(1/\rho m)$*-Lipschitz continuous if A is a strictly monotone operator and M is an m-strongly monotone operator.*

3. Perturbed Algorithms and Convergence

Now we will consider some new perturbed algorithms for solving the nonlinear general A-monotone operator equation problem (1.1) or (1.2) by using the proximal mapping technique associated with the general A-monotone operators and the convergence of the sequences given by the algorithms.

Lemma 3.1. *Let ϵ, g, A, M, f, and T be the same as in* (1.1). *Then the following propositions are equivalent.*

(1) (x, u) *is a solution of the problem* (1.1), *where $x \in \mathcal{X}$ and $u \in T(x)$.*

(2) *x is the fixed-point of the function F defined by*

$$F(x) = \bigcup_{u \in T(u)} \left\{ x - \epsilon g(x) + P_M^A\left[A\big(g(x)\big) - \rho\big(f(x) + u\big)\right] \right\}, \tag{3.1}$$

where $\rho > 0$ is a constant.

(3) (x, u, z) *is a solution of the following equation system:*

$$\begin{gathered} \epsilon g(x) - P_M^A(z) = 0, \\ z = A\big(g(x)\big) - \rho\big(f(x) + u\big), \end{gathered} \tag{3.2}$$

where $x \in \mathcal{X}$, $u \in T(x)$ and $z \in \mathcal{X}^*$.

Lemma 3.2 (see [17]). *Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition. There exists a natural number n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0, \tag{3.3}$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0 \ (n \to \infty)$.

Algorithm 3.3.
Step 1. Choose an arbitrary initial point $x_0 \in \mathcal{X}$.
Step 2. Take any $\{u_n\} \subset \{T(x_n)\} \subset \mathcal{X}$ for $n = 0, 1, 2, \ldots$.
Step 3. Choose sequences $\{\alpha_n\}$, $\{d_n\}$, $\{e_n\}$, and $\{\omega_n\}$ such that for $n \geq 0$, $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $(0, 1]$ and $\sum_{i=0}^{\infty} \alpha_i = \infty$; $\{d_n\}, \{e_n\}$, and $\{\omega_n\}$ are error sequences in $\mathcal{X}$ to take into account a possible inexact computation of the operator point, which satisfies the following conditions:

(i) $d_n = d'_n + d''_n$;

(ii) $\lim_{n\to\infty} \|d'_n\| = \lim_{n\to\infty} \|e_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|d''_n\| < \infty$, $\sum_{n=0}^{\infty} \|\omega_n\| < \infty$.

Step 4. Let $\{(x_n, z_n, u_n)\} \subset \mathcal{X} \times \mathcal{X}^* \times \mathcal{X}$ satisfy

$$\begin{gathered} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\{x_n - \epsilon g(x_n) + P_M^A(z_n)\} + \alpha_n d_n + \omega_n, \\ z_n = A\big(g(x_n)\big) - \rho\big(f(x_n) + u_n\big) + e_n, \end{gathered} \tag{3.4}$$

where $\rho > 0$ is a constant.
Step 5. If x_n, z_n, u_n, ω_n, d_n, and e_n $(n = 0, 1, 2, \ldots)$ satisfy (3.4) to sufficient accuracy, stop; otherwise, set $k := k + 1$ and return to Step 2.

Algorithm 3.4. *For any $x_0 \in \mathcal{X}$, $u_0 \in T(x_0)$, compute the iterative sequence $\{x_n\}$ by*

$$\begin{gathered} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\Big\{x_n - g(x_n) + P_M^A(z_n)\Big\} + \alpha_n d_n + \omega_n, \\ z_n = A\big(g(x_n)\big) - \rho\big(f(x_n) + u_n\big) + e_n, \quad \forall u_n \in T(x_n), \\ n = 0, 1, 2, \ldots, \end{gathered} \tag{3.5}$$

where ρ, α_n, d_n, ω_n, and e_n are the same as in Algorithm 3.3.

Theorem 3.5. *Let $\mathcal{X}$ be a uniformly smooth Banach space with $\varrho_{\mathcal{X}}(t) \leq Ct^2$ for some $C > 0$, and let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$. Let $A : \mathcal{X} \to \mathcal{X}^*$ be r-strongly monotone and τ-Lipschitz continuous, and let $T : \mathcal{X} \to CB(\mathcal{X})$ be ξ-$\widehat{\mathbf{H}}$-Lipschitz continuous. Suppose that $f : \mathcal{X} \to \mathcal{X}$ is (γ, μ)-relaxed cocoercive with respect to g_1 and π-Lipschitz continuous, $g : \mathcal{X} \to \mathcal{X}$ is δ-strongly monotone and σ-Lipschitz continuous, and $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ is general A-monotone, where $g_1 : \mathcal{X} \to \mathcal{X}$ is defined by*

$g_1(x) = A \circ g(x) = A(g(x))$ for all $x \in \mathcal{X}$. If, in addition, there exist constants $\rho > 0$ and $m \in (0,\ r)$ such that

$$k = \sqrt{1 - 2\epsilon\delta + 64C\epsilon\sigma^2} < 1, \quad \tau\sigma > r(1-k),$$
$$h = \xi + m(1-k) < 8\pi\sqrt{C}, \quad \rho < \min\left\{\frac{r}{m}, \frac{r(1-k)}{h}\right\},$$
$$\mu\pi^2 > \gamma\tau^2\sigma^2 + rh(1-k) + \sqrt{(64C\pi^2\mu^2 - h^2)\left[\tau^2\sigma^2 - r^2(1-k)^2\right]},$$
$$\left|\rho - \frac{\mu\pi^2 - \gamma\tau^2\sigma^2 - rh(1-k)}{64C\pi^2 - h^2}\right| < \frac{\sqrt{[\mu\pi^2 - \gamma\tau^2\sigma^2 - rh(1-k)]^2 - (64C\pi^2 - h^2)\left[\tau^2\sigma^2 - r^2(1-k)^2\right]}}{64C\pi^2 - h^2}, \tag{3.6}$$

then the following results hold:

(1) *the solution set of the problem* (1.1) *is nonempty;*

(2) *the iterative sequence $\{(x_n, u_n)\}$ generated by Algorithm 3.3 converges strongly to the solution (x^*, u^*) of the problem* (1.1).

Proof. Setting a multivalued function $F : \mathcal{X} \to 2^{\mathcal{X}}$ to be the same as (3.1), then we can prove that F is a multivalued contractive operator.

In fact, for any $x, \hat{x} \in \mathcal{X}$ and any $a \in F(x)$, there exists $u \in T(x)$ such that

$$a = x - \epsilon g(x) + P_M^A\left[A(g(x)) - \rho(f(x) + u)\right]. \tag{3.7}$$

Note that $T(\hat{x}) \in CB(\mathcal{X})$; it follows from Nadler's result [18] that there exists $\hat{u} \in T(\hat{x})$ such that

$$\|u - \hat{u}\| \leq \widehat{\mathbf{H}}(T(x), T(\hat{x})). \tag{3.8}$$

Letting

$$b = \hat{x} - \epsilon g(\hat{x}) + P_M^A\left[A(g(\hat{x})) - \rho(f(\hat{x}) + \hat{u})\right], \tag{3.9}$$

then we have $b \in F(\hat{x})$. The δ-strongly monotonicity and σ-Lipschitz continuity of g, the ξ-$\widehat{\mathbf{H}}$-Lipschitz continuity of T, the (γ, μ)-relaxed cocoercivity with respect to g_1 and π-Lipschitz continuity of f, and the τ-Lipschitz continuity of A, Lemma 2.6, and the inequality (3.8) imply that

$$\begin{aligned}
&\left\| x-\hat{x}-\epsilon[g(x)-g(\hat{x})]\right\|^2\\
&\quad\le \|x-\hat{x}\|^2-2\epsilon\langle j(x-\hat{x}),g(x)-g(\hat{x})\rangle\\
&\qquad+2\epsilon\langle j(x-\hat{x}-[g(x)-g(\hat{x})])-j(x-\hat{x}),-[g(x)-g(\hat{x})]\rangle\\
&\quad\le \|x-\hat{x}\|^2-2\delta\epsilon\|x-\hat{x}\|^2+4\epsilon d^2\varrho_{\mathcal{X}}\left(\frac{4\|g(x)-g(\hat{x})\|}{d}\right)\\
&\quad\le \left(1-2\epsilon\delta+64C\epsilon\sigma^2\right)\|x-\hat{x}\|^2,\\
&\|u-\hat{u}\|\le \widehat{\mathbf{H}}(T(u),T(\hat{u}))\le \xi\|x-\hat{x}\|,\\
&\left\| A(g(x))-A(g(\hat{x}))-\rho[f(x)-f(\hat{x})]\right\|^2\\
&\quad\le \left\| A(g(x))-A(g(\hat{x}))\right\|^2-2\rho\langle j(A(g(x))-A(g(\hat{x}))),f(x)-f(\hat{x})\rangle\\
&\qquad+2\langle j(A(g(x))-A(g(\hat{x}))-\rho[f(x)-f(\hat{x})])-j(A(g(x))-A(g(\hat{x})))-\rho[f(x)-f(\hat{x})]\rangle\\
&\quad\le \tau^2\sigma^2\|x-\hat{x}\|^2-2\left[-\gamma\left\| A(g(x))-A(g(\hat{x}))\right\|^2+\mu\left\| f(x)-f(\hat{x})\right\|^2\right]\\
&\qquad+4d^2\varrho_{\mathcal{X}}\left(\frac{4\|\rho[f(x)-f(\hat{x})]\|}{d}\right)\\
&\quad\le \left(\tau^2\sigma^2-2\rho\mu\pi^2+2\rho\gamma\tau^2\sigma^2+64C\rho^2\pi^2\right)\|x-\hat{x}\|^2.
\end{aligned}\tag{3.10}$$

Thus, it follows from (3.4) and Lemma 2.11 that

$$\begin{aligned}
\|a-b\| &\le \left\| x-\hat{x}-\epsilon[g(x)-g(\hat{x})]\right\|\\
&\quad+\left\| P_M^A[A(g(x))-\rho(f(x)+u)]-P_M^A[A(g(\hat{x}))-\rho(f(\hat{x})+\hat{u})]\right\|\\
&\le \left\| x-\hat{x}-\epsilon[g(x)-g(\hat{x})]\right\|+\frac{\rho}{r-\rho m}\|u-\hat{u}\|\\
&\quad+\frac{1}{r-\rho m}\left\| A(g(x))-A(g(\hat{x}))-\rho[f(x)-f(\hat{x})]\right\|\\
&\le \theta\|x-\hat{x}\|,
\end{aligned}\tag{3.11}$$

where

$$\theta=\sqrt{1-2\epsilon\delta+64C\epsilon\sigma^2}+\frac{\rho\xi+\sqrt{\tau^2\sigma^2-2\rho\mu\pi^2+2\rho\gamma\tau^2\sigma^2+64C\rho^2\pi^2}}{r-\rho m}.\tag{3.12}$$

It follows from condition (3.6) that $\theta < 1$. Hence, from (3.11), we get

$$d(a, F(\hat{x})) = \inf_{b \in F(\hat{x})} \|a - b\| \leq \theta \|x - \hat{x}\|. \tag{3.13}$$

Since $a \in F(x)$ is arbitrary, we obtain $\sup_{a \in F(x)} d(a, F(\hat{x})) \leq \theta \|x - \hat{x}\|$. By using same argument, we can prove $\sup_{b \in F(\hat{x})} d(F(x), b) \leq \theta \|x - \hat{x}\|$. It follows from the definition of the Hausdorff metric $\widehat{\mathbf{H}}$ on $CB(\mathcal{X})$ that

$$\widehat{\mathbf{H}}(F(x), F(\hat{x})) \leq \theta \|x - \hat{x}\|, \quad \forall x,\ \hat{x} \in \mathcal{X}, \tag{3.14}$$

and so F is a multivalued contractive mapping. By a fixed-point theorem of Nadler [18], the definition of F and (3.2), now we know that F has a fixed-point x^*, that is, $x^* \in F(x^*)$, and there exists $u^* \in T(x^*)$ such that

$$\epsilon g(x^*) = P_M^A\left[A\left(g(x^*)\right) - \rho\left(f(x^*) + u^*\right)\right]. \tag{3.15}$$

Hence, it follows from Lemma 3.1 that (x^*, u^*) is a solution of the problem (1.1), that is, the solution set of the problem (1.1) is nonempty.

Next, we prove the conclusion (2). Let (x^*, u^*) be a solution of problem (1.1). Then for all $n \geq 0$, we have

$$x^* = (1 - \alpha_n)x^* + \alpha_n\left\{x^* - \epsilon g(x^*) + P_M^A(z^*)\right\}, \quad z^* = A\left(g(x^*)\right) - \rho\left(f(x^*) + u^*\right). \tag{3.16}$$

From Algorithm 3.3, the assumptions of the theorem 3.5 and Lemma 2.11, it follows that

$$\|z_n - z^*\| \leq \left\|A\left(g(x_n)\right) - A\left(g(x^*)\right) - \rho\left(f(x_n) - f(x^*)\right)\right\| + \rho\|u_n - u^*\| + \|e_n\|, \tag{3.17}$$

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\left\|x_n - x^* - \epsilon\left(g(x_n) - g(x^*)\right)\right\| \\
&\quad + \alpha_n\left\|P_M^A(z_n) - P_M^A(z^*)\right\| + \alpha_n\|d_n\| + \|\omega_n\| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\left\|x_n - x^* - \epsilon\left(g(x_n) - g(x^*)\right)\right\| \\
&\quad + \frac{\alpha_n}{r - \rho m}\|z_n - z^*\| + \alpha_n\left\|d_n'\right\| + \left(\left\|d_n''\right\| + \|\omega_n\|\right).
\end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad \leq (1 - \alpha_n)\|x_n - x^*\| + \theta\alpha_n\|x_n - x^*\| + \frac{\alpha_n}{r - \rho m}\|e_n\| + \alpha_n\left\|d_n'\right\| + \left(\left\|d_n''\right\| + \|\omega_n\|\right) \\
&\quad \leq [1 - \alpha_n(1 - \theta)]\|x_n - x^*\| + \alpha_n\left(\left\|d_n'\right\| + \frac{1}{r - \rho m}\|e_n\|\right) + \left(\left\|d_n''\right\| + \|\omega_n\|\right),
\end{aligned} \tag{3.19}$$

where θ is the same as in (3.11). Since $\theta < 1$, we know that $1-\theta > 0$ and (3.19) implies

$$\|u_{n+1}-u^*\| \\ \leq [1-\alpha_n(1-\theta)]\|x_n-x^*\| + \alpha_n(1-\theta)\cdot\frac{1}{1-\theta}\left(\|d_n'\| + \frac{1}{r-\rho m}\|e_n\|\right) + \left(\|d_n''\| + \|\omega_n\|\right). \tag{3.20}$$

Since $\sum_{i=0}^{n}\alpha_i = \infty$, it follows from Lemma 3.2 that the sequence x_n strongly converges to x^*. By $u_n \in T(x_n)$, $u^* \in T(u^*)$, and the $\widehat{\mathbf{H}}$-Lipschitz continuity of T, we obtain

$$\|u_n-u^*\| \leq \widehat{\mathbf{H}}(S(u_n),S(u^*)) \leq \xi\|u_n-u^*\|, \qquad \|y_n-y^*\| \leq \widehat{\mathbf{H}}(T(u_n),T(u^*)) \leq \zeta\|u_n-u^*\|. \tag{3.21}$$

Thus, $\{u_n\}$ is also strongly converges to u^*. Therefore, the iterative sequence $\{(x_n,u_n)\}$ generated by Algorithm 3.3 converges strongly to the solution (x^*,u^*) of the problem (1.1) or (1.2). This completes the proof. □

Based on Theorem 3.3 in [2], we have the following comment.

Remark 3.6. If $\epsilon \equiv 1$, g is k-strongly accretive and δ-Lipschitz continuous, A is a strictly monotone and s-Lipschitz continuous operator, M is a general H-monotone and β-strongly monotone operator, T is a single-valued operator and $Q = f + T$ is α-Lipschitz continuous, and $\rho > 0$ is some constant such that

$$\rho > \frac{s\delta}{\beta-\alpha-\beta\sqrt{1-2k+64C\delta^2}}, \quad \sqrt{1-2k+64C\delta^2}+\alpha\beta^{-1} < 1, \tag{3.22}$$

then (3.6) holds.

Theorem 3.7. *Assume that A, T, f, g, M, and $\mathcal{X}$ are the same as in Theorem 3.5. If there exist constants $\rho > 0$ and $m \in (0,\ r)$such that*

$$\begin{gathered}
k = \sqrt{1-2\delta+64C\sigma^2} < 1, \quad \tau\sigma > r(1-k), \\
h = \xi + m(1-k) < 8\pi\sqrt{C}, \quad \rho < \min\left\{\frac{r}{m}, \frac{r(1-k)}{h}\right\}, \\
\mu\pi^2 > \gamma\tau^2\sigma^2 + rh(1-k) + \sqrt{\left(64C\pi^2\mu^2-h^2\right)\left[\tau^2\sigma^2-r^2(1-k)^2\right]}, \\
\left|\rho - \frac{\mu\pi^2-\gamma\tau^2\sigma^2-rh(1-k)}{64C\pi^2-h^2}\right| < \frac{\sqrt{\left[\mu\pi^2-\gamma\tau^2\sigma^2-rh(1-k)\right]^2-\left(64C\pi^2-h^2\right)\left[\tau^2\sigma^2-r^2(1-k)^2\right]}}{64C\pi^2-h^2},
\end{gathered} \tag{3.23}$$

then there exists $x^* \in \mathcal{X}$ such that x^* is a solution of the problem (1.3)*, and the iterative sequence $\{x_n\}$ generated by Algorithm 3.4 converges strongly to the solution x^* of the problem* (1.3).

Remark 3.8. If $\alpha_n = \omega_n = e_n \equiv 0$ for $n \geq 0$ in Algorithm 3.4, T and Q as the same in the problem (1.5), then the results of Theorem 3.4 obtained by Xia and Huang [2] also hold. For details, we can refer to [1, 2, 4, 5].

Remark 3.9. If $d_n = 0$ or $e_n = 0$ or $\omega_n = 0$ $(n \geq 0)$ in Algorithms 3.3 and 3.4, then the conclusions of Theorems 3.5 and 3.7 also hold, respectively. The results of Theorems 3.5 and 3.7 improve and generalize the corresponding results of [2, 4–9, 15, 17]. For other related works, we refer to [1–16] and the references therein.

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