Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 458134, 20 pages
doi:10.1155/2009/458134

Research Article

Self-Adaptive Implicit Methods for Monotone Variant Variational Inequalities

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Received 26 January 2009; Accepted 24 February 2009

Recommended by Ram U. Verma

The efficiency of the implicit method proposed by He (1999) depends on the parameter β heavily; while it varies for individual problem, that is, different problem has different "suitable" parameter, which is difficult to find. In this paper, we present a modified implicit method, which adjusts the parameter β automatically per iteration, based on the message from former iterates. To improve the performance of the algorithm, an inexact version is proposed, where the subproblem is just solved approximately. Under mild conditions as those for variational inequalities, we prove the global convergence of both exact and inexact versions of the new method. We also present several preliminary numerical results, which demonstrate that the self-adaptive implicit method, especially the inexact version, is efficient and robust.

1. Introduction

Let Ω be a closed convex subset of $\mathcal{R}^n$ and let F be a mapping from $\mathcal{R}^n$ into itself. The so-called finite-dimensional variant variational inequalities, denoted by $\mathrm{VVI}(\Omega, F)$, is to find a vector $u \in \mathcal{R}^n$, such that

$$F(u) \in \Omega, \quad (v - F(u))^{\top} u \geq 0, \quad \forall v \in \Omega, \tag{1.1}$$

while a classical variational inequality problem, abbreviated by $\mathrm{VI}(\Omega, f)$, is to find a vector $x \in \Omega$, such that

$$(x' - x)^{\top} f(x) \geq 0, \quad \forall x' \in \Omega, \tag{1.2}$$

where f is a mapping from $\mathcal{R}^n$ into itself.

Both $\mathrm{VVI}(\Omega, F)$ and $\mathrm{VI}(\Omega, f)$ serve as very general mathematical models of numerous applications arising in economics, engineering, transportation, and so forth. They include some widely applicable problems as special cases, such as mathematical programming problems, system of nonlinear equations, and nonlinear complementarity problems, and so forth. Thus, they have been extensively investigated. We refer the readers to the excellent monograph of Faccinei and Pang [1, 2] and the references therein for theoretical and algorithmic developments on $\mathrm{VI}(\Omega, f)$, for example, [3–10], and [11–16] for $\mathrm{VVI}(\Omega, F)$.

It is observed that if F is invertible, then by setting $f = F^{-1}$, the inverse mapping of F, $\mathrm{VVI}(\Omega, F)$ can be reduced to $\mathrm{VI}(\Omega, f)$. Thus, theoretically, all numerical methods for solving $\mathrm{VI}(\Omega, f)$ can be used to solve $\mathrm{VVI}(\Omega, F)$. However, in many practical applications, the inverse mapping F^{-1} may not exist. On the other hand, even if it exists, it is not easy to find it. Thus, there is a need to develop numerical methods for $\mathrm{VVI}(\Omega, F)$ and recently, the Goldstein's type method was extended from solving $\mathrm{VI}(\Omega, f)$ to $\mathrm{VVI}(\Omega, F)$ [12, 17].

In [11], He proposed an implicit method for solving general variational inequality problems. A general variational inequality problem is to find a vector $u \in \mathcal{R}^n$, such that

$$F(u) \in \Omega, \quad (v - F(u))^{\top} G(u) \geq 0, \quad \forall v \in \Omega. \tag{1.3}$$

When G is the identity mapping, it reduces to $\mathrm{VVI}(\Omega, F)$ and if F is the identity mapping, it reduces to $\mathrm{VI}(\Omega, G)$. He's implicit method is as follows.

(S0) Given $u^0 \in R^n$, $\beta > 0$, $\gamma \in (0, 2)$, and a positive definite matrix M.

(S1) Find u^{k+1} via

$$\theta_k(u) = 0, \tag{1.4}$$

where

$$\begin{aligned} \theta_k(u) = {} & F(u) + \beta G(u) - F\left(u^k\right) - \beta G\left(u^k\right) \\ & + \gamma \rho\left(u^k, M, \beta\right) M^{-1} e\left(u^k, \beta\right), \end{aligned} \tag{1.5}$$

$$\begin{aligned} \rho\left(u^k, M, \beta\right) &= \frac{\left\| e(u^k, \beta) \right\|^2}{e(u^k, \beta)^{\top} M^{-1} e(u^k, \beta)}, \\ e(u, \beta) &:= F(u) - P_{\Omega}[F(u) - \beta G(u)], \end{aligned} \tag{1.6}$$

with P_{Ω} being the projection from $\mathcal{R}^n$ onto Ω, under the Euclidean norm.

He's method is attractive since it solves the general variational inequality problem, which is essentially equivalent to a system of nonsmooth equations

$$e(u, \beta) = 0, \tag{1.7}$$

via solving a series of smooth equations (1.4). The mapping in the subproblem is well conditioned and many efficient numerical methods, such as Newton's method, can be applied to solve it. Furthermore, to improve the efficiency of the algorithm, He [11] proposed to solve the subproblem approximately. That is, at Step 1, instead of finding a zero of θ_k, it only needs to find a vector u^{k+1} satisfying

$$\left\|\theta_k\left(u^{k+1}\right)\right\| \leq \eta_k \left\|e\left(u^k, \beta\right)\right\|, \tag{1.8}$$

where $\{\eta_k\}$ is a nonnegative sequence. He proved the global convergence of the algorithm under the condition that the error tolerance sequence $\{\eta_k\}$ satisfies

$$\sum_{k=0}^{\infty} {\eta_k}^2 < +\infty. \tag{1.9}$$

In the above algorithm, there are two parameters $\beta > 0$ and $\gamma \in (0,2)$, which affect the efficiency of the algorithm. It was observed that nearly for all problems, γ close to 2 is a better choice than smaller γ, while different problem has different *optimal*β. A suitable parameter β is thus difficult to find for an individual problem. For solving variational inequality problems, He et al. [18] proposed to choose a sequence of parameters $\{\beta_k\}$, instead of a fixed parameter β, to improve the efficiency of the algorithm. Under the same conditions as those in [11], they proved the global convergence of the algorithm. The numerical results reported there indicated that for any given initial parameter β_0, the algorithm can find a suitable parameter self-adaptively. This improves the efficiency of the algorithm greatly and makes the algorithm easy and robust to implement in practice.

In this paper, in a similar theme as [18], we suggest a general rule for choosing suitable parameter in the implicit method for solving VVI(Ω, F). By replacing the constant factor β in (1.4) and (1.5) with a self-adaptive variable positive sequence $\{\beta_k\}$, the efficiency of the algorithm can be improved greatly. Moreover, it is also robust to the initial choice of the parameter β_0. Thus, for any given problems, we can choose a parameter β_0 arbitrarily, for example, $\beta_0 = 1$ or $\beta_0 = 0.1$. The algorithm chooses a suitable parameter self-adaptively, based on the information from the former iteration, which makes it able to add a little additional computational cost against the original algorithm with fixed parameter β. To further improve the efficiency of the algorithm, we also admit approximate computation in solving the subproblem per iteration. That is, per iteration, we just need to find a vector u^{k+1} that satisfies (1.8).

Throughout this paper, we make the following assumptions.

Assumption A. The solution set of VVI(Ω, F), denoted by Ω^*, is nonempty.

Assumption B. The operator F is monotone, that is, for any $u, v \in \mathcal{R}^n$,

$$(u - v)^\top (F(u) - F(v)) \geq 0. \tag{1.10}$$

The rest of this paper is organized as follows. In Section 2, we summarize some basic properties which are useful in the convergence analysis of our method. In Sections 3 and 4, we describe the exact version and inexact version of the method and prove their global convergence, respectively. We report our preliminary computational results in Section 5 and give some final conclusions in the last section.

2. Preliminaries

For a vector $x \in \mathcal{R}^n$ and a symmetric positive definite matrix $M \in \mathcal{R}^{n\times n}$, we denote $\|x\| = \sqrt{x^\top x}$ as the Euclidean-norm and $\|x\|_M$ as the matrix-induced norm, that is, $\|x\|_M := (x^\top M x)^{1/2}$.

Let Ω be a nonempty closed convex subset of $\mathcal{R}^n$, and let $P_\Omega(\cdot)$ denote the projection mapping from $\mathcal{R}^n$ onto Ω, under the matrix-induced norm. That is,

$$P_\Omega(x) := \arg\min\{\|x-y\|_M, \, y \in \Omega\}. \tag{2.1}$$

It is known [12, 19] that the variant variational inequality problem (1.1) is equivalent to the projection equation

$$F(u) = P_\Omega\left[F(u) - \beta M^{-1} u\right], \tag{2.2}$$

where β is an arbitrary positive constant. Then, we have the following lemma.

Lemma 2.1. *u^* is a solution of $VVI(\Omega, F)$ if and only if $e(u,\beta) = 0$ for any fixed constant $\beta > 0$, where*

$$e(u,\beta) := F(u) - P_\Omega\left[F(u) - \beta M^{-1} u\right] \tag{2.3}$$

is the residual function of the projection equation (2.2).

Proof. See [11, Theorem 1]. □

The following lemma summarizes some basic properties of the projection operator, which will be used in the subsequent analysis.

Lemma 2.2. *Let Ω be a closed convex set in $\mathcal{R}^n$ and let P_Ω denote the projection operator onto Ω under the matrix-induced norm, then one has*

$$(w - P_\Omega(v))^\top M (v - P_\Omega(v)) \le 0, \quad \forall v \in \mathcal{R}^n, \forall w \in \Omega, \tag{2.4}$$

$$\|P_\Omega(u) - P_\Omega(v)\|_M \le \|u - v\|_M, \quad \forall u, v \in \mathcal{R}^n. \tag{2.5}$$

The following lemma plays an important role in convergence analysis of our algorithm.

Lemma 2.3. *For a given $u \in \mathcal{R}^n$, let $\tilde{\beta} \ge \beta > 0$. Then it holds that*

$$\left\|e\left(u,\tilde{\beta}\right)\right\|_M \ge \|e(u,\beta)\|_M. \tag{2.6}$$

Proof. See [20] for a simple proof. □

Lemma 2.4. *Let $u^* \in \Omega^*$, then for all $u \in \mathcal{R}^n$ and $\beta > 0$, one has*

$$\{[F(u) - F(u^*)] + \beta M^{-1}(u - u^*)\}^\top Me(u,\beta) \ge \|e(u,\beta)\|_M^2. \tag{2.7}$$

Proof. It follows from the definition of VVI(Ω, F) (see (1.1)) that

$$\{P_\Omega[F(u) - \beta M^{-1}u] - F(u^*)\}^\top \beta u^* \ge 0. \tag{2.8}$$

By setting $v := F(u) - \beta M^{-1}u$ and $w := F(u^*)$ in (2.4), we obtain

$$\{P_\Omega[F(u) - \beta M^{-1}u] - F(u^*)\}^\top M\left\{e(u,\beta) - \beta M^{-1}u\right\} \ge 0. \tag{2.9}$$

Adding (2.8) and (2.9), and using the definition of $e(u, \beta)$ in (2.3), we get

$$\{F(u) - F(u^*) - e(u,\beta)\}^\top M\left\{e(u,\beta) - \beta M^{-1}(u - u^*)\right\} \ge 0, \tag{2.10}$$

that is,

$$\begin{aligned}(F(u) - F(u^*) + \beta M^{-1}(u - u^*))^\top Me(u,\beta)& \\ \ge \|e(u,\beta)\|_M^2 + \beta(F(u) - F(u^*))^\top(u - u^*)& \\ \ge \|e(u,\beta)\|_M^2,&\end{aligned} \tag{2.11}$$

where the last inequality follows from the monotonicity of F (Assumption B). This completes the proof. □

3. Exact Implicit Method and Convergence Analysis

We are now in the position to describe our algorithm formally.

3.1. Self-Adaptive Exact Implicit Method

(S0) Given $\gamma \in (0, 2)$, $\beta_0 > 0$, $u^0 \in \mathcal{R}^n$ and a positive definite matrix M.

(S1) Compute u^{k+1} such that

$$F\left(u^{k+1}\right) + \beta_k M^{-1}\ u^{k+1} - F\left(u^k\right) - \beta_k M^{-1}\ u^k + \gamma e\left(u^k, \beta_k\right) = 0. \tag{3.1}$$

(S2) If the given stopping criterion is satisfied, then stop; otherwise choose a new parameter $\beta_{k+1} \in [1/(1+\tau_k)\beta_k, (1+\tau_k)\beta_k]$, where τ_k satisfies

$$\sum_{k=0}^{\infty} \tau_k < +\infty, \quad \tau_k \geq 0. \tag{3.2}$$

Set $k := k+1$ and go to Step 1.

From (3.1), we know that u^{k+1} is the (exact) unique zero of

$$\theta_k(u) := F(u) + \beta_k M^{-1} u - F\left(u^k\right) - \beta_k M^{-1} u^k + \gamma e\left(u^k, \beta_k\right). \tag{3.3}$$

We refer to the above method as *the self-adaptive exact implicit method.*

Remark 3.1. According to the assumption $\tau_k \geq 0$ and $\sum_{k=0}^{\infty} \tau_k < +\infty$, we have $\prod_{k=0}^{\infty}(1+\tau_k) < +\infty$. Denote

$$S_\tau := \prod_{k=0}^{\infty}(1+\tau_k). \tag{3.4}$$

Hence, the sequence$\{\beta_k\} \subset [(1/S_\tau)\beta_0, S_\tau \beta_0]$ is bounded. Then, let $\inf\{\beta_k\}_{k=0}^{\infty} := \beta_L > 0$ and $\sup\{\beta_k\}_{k=0}^{\infty} := \beta_U < +\infty$.

Now, we analyze the convergence of the algorithm, beginning with the following lemma.

Lemma 3.2. *Let $\{u^k\}$ be the sequence generated by the proposed self-adaptive exact implicit method. Then for any $u^* \in \Omega^*$ and $k > 0$, one has*

$$\begin{aligned} &\left\| (F(u^{k+1}) - F(u^*)) + \beta_k M^{-1}(u^{k+1} - u^*) \right\|_M^2 \\ &\quad \leq \left\| (F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*) \right\|_M^2 - \gamma(2-\gamma) \left\| e(u^k, \beta_k) \right\|_M^2. \end{aligned} \tag{3.5}$$

Proof. Using (3.1), we get

$$\begin{aligned} &\left\| (F(u^{k+1}) - F(u^*)) + \beta_k M^{-1}(u^{k+1} - u^*) \right\|_M^2 \\ &\quad = \left\| [(F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*)] - \gamma e(u^k, \beta_k) \right\|_M^2 \\ &\quad \leq \left\| (F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*) \right\|_M^2 - 2\gamma \left\| e(u^k, \beta_k) \right\|_M^2 + \gamma^2 \left\| e(u^k, \beta_k) \right\|_M^2 \\ &\quad = \left\| (F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*) \right\|_M^2 - \gamma(2-\gamma) \left\| e(u^k, \beta_k) \right\|_M^2, \end{aligned} \tag{3.6}$$

where the inequality follows from (2.7). This completes the proof. □

Since $0 < \beta_{k+1} \leq (1+\tau_k)\beta_k$ and F is monotone, it follows that

$$\begin{aligned}
&\left\|(F(u^{k+1}) - F(u^*)) + \beta_{k+1}M^{-1}(u^{k+1} - u^*)\right\|_M^2 \\
&\quad = \left\|(F(u^{k+1}) - F(u^*)) + \beta_k M^{-1}(u^{k+1} - u^*) + (\beta_{k+1} - \beta_k)M^{-1}(u^{k+1} - u^*)\right\|_M^2 \\
&\quad = \left\|(F(u^{k+1}) - F(u^*)) + \beta_k M^{-1}(u^{k+1} - u^*)\right\|_M^2 + (\beta_{k+1} - \beta_k)^2\left\|(u^{k+1} - u^*)\right\|_M^2 \\
&\qquad + 2(\beta_{k+1} - \beta_k)(u^{k+1} - u^*)^\top\left[\left(F\left(u^{k+1}\right) - F(u^*)\right) + \beta_k M^{-1}\left(u^{k+1} - u^*\right)\right] \\
&\quad \leq (1+\tau_k)^2\left\|(F(u^{k+1}) - F(u^*)) + \beta_k M^{-1}(u^{k+1} - u^*)\right\|_M^2,
\end{aligned} \tag{3.7}$$

where the inequality follows from the monotonicity of the mapping F. Combining (3.5) and (3.7), we have

$$\begin{aligned}
&\left\|(F(u^{k+1}) - F(u^*)) + \beta_{k+1}M^{-1}(u^{k+1} - u^*)\right\|_M^2 \\
&\quad \leq (1+\tau_k)^2\left\|(F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*)\right\|_M^2 - \gamma(2-\gamma)\left\|e(u^k, \beta_k)\right\|_M^2.
\end{aligned} \tag{3.8}$$

Now, we give the self-adaptive rule in choosing the parameter β_k. For the sake of balance, we hope that

$$\left\|(F(u^{k+1}) - F(u^k))\right\|_M \approx \left\|\beta_k M^{-1}(u^{k+1} - u^k)\right\|_M. \tag{3.9}$$

That is, for given constant $\tau > 0$, if

$$\left\|(F(u^{k+1}) - F(u^k))\right\|_M > (1+\tau)\left\|\beta_k M^{-1}(u^{k+1} - u^k)\right\|_M, \tag{3.10}$$

we should increase β_k in the next iteration; on the other hand, we should decrease β_k when

$$\left\|(F(u^{k+1}) - F(u^k))\right\|_M < \frac{1}{(1+\tau)}\left\|\beta_k M^{-1}(u^{k+1} - u^k)\right\|_M. \tag{3.11}$$

Let

$$\omega_k = \frac{\left\|(F(u^{k+1}) - F(u^k))\right\|_M}{\left\|\beta_k M^{-1}(u^{k+1} - u^k)\right\|_M}. \tag{3.12}$$

Then we give

$$\beta_{k+1} := \begin{cases} (1+\tau_k)\beta_k, & \text{if } \omega_k > (1+\tau), \\ \dfrac{1}{(1+\tau_k)}\beta_k, & \text{if } \omega_k < \dfrac{1}{(1+\tau)}, \\ \beta_k, & \text{otherwise.} \end{cases} \tag{3.13}$$

Such a self-adaptive strategy was adopted in [18, 21–24] for solving variational inequality problems, where the numerical results indicated its efficiency and robustness to the choice of the initial parameter β_0. Here we adopted it for solving variant variational inequality problems.

We are now in the position to give the convergence result of the algorithm, the main result of this section.

Theorem 3.3. *The sequence $\{u^k\}$ generated by the proposed self-adaptive exact implicit method converges to a solution of VVI(Ω, F).*

Proof. Let $\xi_k := 2\tau_k + \tau_k{}^2$. Then from the assumption that $\sum_{k=0}^{\infty}\tau_k < +\infty$, we have that $\sum_{k=0}^{\infty}\xi_k < +\infty$, which means that $\prod_{k=0}^{\infty}(1+\xi_k) < +\infty$. Denote

$$C_s := \sum_{i=0}^{\infty}\xi_i, \qquad C_p := \prod_{i=0}^{\infty}(1+\xi_i). \tag{3.14}$$

From (3.8), for any $u^* \in \Omega^*$, that is, an arbitrary solution of VVI(Ω, F), we have

$$\begin{aligned} &\left\|(F(u^{k+1}) - F(u^*)) + \beta_{k+1}M^{-1}(u^{k+1} - u^*)\right\|_M^2 \\ &\quad\le (1+\xi_k)\left\|(F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*)\right\|_M^2 &(3.15) \\ &\quad\le \left(\prod_{i=0}^{k}(1+\xi_i)\right)\left\|(F(u^0) - F(u^*)) + \beta_0 M^{-1}(u^0 - u^*)\right\|_M^2 \\ &\quad\le C_p\left\|(F(u^0) - F(u^*)) + \beta_0 M^{-1}(u^0 - u^*)\right\|_M^2 &(3.16) \\ &\quad< +\infty. \end{aligned}$$

This, together with the monotonicity of the mapping F, means that the generated sequence $\{u^k\}$ is bounded.

Also from (3.8), we have

$$\begin{aligned}
\gamma(2-\gamma)\left\|e(u^k,\beta_k)\right\|_M^2 &\le (1+\tau_k)^2\left\|(F(u^k)-F(u^*))+\beta_k M^{-1}(u^k-u^*)\right\|_M^2 \\
&\quad -\left\|(F(u^{k+1})-F(u^*))+\beta_{k+1}M^{-1}(u^{k+1}-u^*)\right\|_M^2 \\
&= \left\|(F(u^k)-F(u^*))+\beta_k M^{-1}(u^k-u^*)\right\|_M^2 \\
&\quad -\left\|(F(u^{k+1})-F(u^*))+\beta_{k+1}M^{-1}(u^{k+1}-u^*)\right\|_M^2 \\
&\quad +\xi_k\left\|(F(u^k)-F(u^*))+\beta_k M^{-1}(u^k-u^*)\right\|_M^2.
\end{aligned} \tag{3.17}$$

Adding both sides of the above inequality, we obtain

$$\begin{aligned}
&\gamma(2-\gamma)\sum_{k=k_0}^{\infty}\left\|e(u^k,\beta_k)\right\|_M^2 \\
&\quad\le \left\|(F(u^0)-F(u^*))+\beta_0 M^{-1}(u^0-u^*)\right\|_M^2 \\
&\qquad +\sum_{k=0}^{\infty}\xi_k\left\|(F(u^k)-F(u^*))+\beta_k M^{-1}(u^k-u^*)\right\|_M^2 \\
&\quad\le \left\|(F(u^0)-F(u^*))+\beta_0 M^{-1}(u^0-u^*)\right\|_M^2 \\
&\qquad +\left(\sum_{k=0}^{\infty}\xi_k\right)C_p\left\|(F(u^0)-F(u^*))+\beta_0 M^{-1}(u^0-u^*)\right\|_M^2 \\
&\quad= (1+C_sC_p)\left\|(F(u^0)-F(u^*))+\beta_0 M^{-1}(u^0-u^*)\right\|_M^2 \\
&\quad< +\infty,
\end{aligned} \tag{3.18}$$

where the second inequality follows from (3.15). Thus, we have

$$\lim_{k\to\infty}\left\|e(u^k,\beta_k)\right\|_M = 0, \tag{3.19}$$

which, from Lemma 2.3, means that

$$\lim_{k\to\infty}\left\|e(u^k,\beta_L)\right\|_M \le \lim_{k\to\infty}\left\|e(u^k,\beta_k)\right\|_M = 0. \tag{3.20}$$

Since $\{u^k\}$ is bounded, it has at least one cluster point. Let $\overline{u}$ be a cluster point of $\{u^k\}$ and let $\{u^{k_j}\}$ be the subsequence converging to $\overline{u}$. Since $e(u, \beta_L)$ is continuous, taking limit in (3.20) along the subsequence, we get

$$\|e(\overline{u}, \beta_L)\|_M = \lim_{j \to \infty} \left\|e(u^{k_j}, \beta_L)\right\|_M = 0. \tag{3.21}$$

Thus, from Lemma 2.1, $\overline{u}$ is a solution of VVI(Ω, F).

In the following we prove that the sequence $\{u^k\}$ has exactly one cluster point. Assume that $\hat{u}$ is another cluster point of $\{u^k\}$, which is different from $\overline{u}$. Because $\overline{u}$ is a cluster point of the sequence $\{u^k\}$ and F is monotone, there is a $k_0 > 0$ such that

$$\left\|F(u^{k_0}) - F(\overline{u}) + \beta_{k_0} M^{-1}(u^{k_0} - \overline{u})\right\|_M \le \frac{\delta}{2C_p}, \tag{3.22}$$

where

$$\delta := \left\|(F(\hat{u}) - F(\overline{u})) + \beta_{k_0} M^{-1}(\hat{u} - \overline{u})\right\|_{\mathrm{M}}. \tag{3.23}$$

On the other hand, since $\overline{u} \in \Omega^*$ and u^* is an arbitrary solution, by setting $u^* := \overline{u}$ in (3.15), we have for all $k \ge k_0$,

$$\begin{aligned}
&\left\|(F(u^k) - F(\overline{u})) + \beta_k M^{-1}(u^k - \overline{u})\right\|_M^2 \\
&\quad \le \prod_{i=k_0}^{k}(1+\xi_i)\left\|(F(u^i) - F(\overline{u})) + \beta_i M^{-1}(u^i - \overline{u})\right\|_M^2 \\
&\quad \le C_p\left\|(F(u^{k_0}) - F(\overline{u})) + \beta_{k_0} M^{-1}(u^{k_0} - \overline{u})\right\|_M^2,
\end{aligned} \tag{3.24}$$

that is,

$$\begin{aligned}
&\left\|(F(u^k) - F(\overline{u})) + \beta_k M^{-1}(u^k - \overline{u})\right\|_M \\
&\quad \le C_p^{1/2}\left\|(F(u^{k_0}) - F(\overline{u})) + \beta_{k_0} M^{-1}(u^{k_0} - \overline{u})\right\|_M \\
&\quad \le \frac{\delta}{2C_p^{1/2}}.
\end{aligned} \tag{3.25}$$

Then,

$$\begin{aligned}
&\left\|(F(u^k)-F(\hat{u}))+\beta_k M^{-1}(u^k-\hat{u})\right\|_M \\
&\quad\geq \left\|(F(\hat{u})-F(\overline{u}))+\beta_k M^{-1}(\hat{u}-\overline{u})\right\|_M \\
&\qquad-\left\|(F(u^k)-F(\overline{u}))+\beta_k M^{-1}(u^k-\overline{u})\right\|_M.
\end{aligned}\tag{3.26}$$

Using the monotonicity of F and the choosing rule of β_k, we have

$$\begin{aligned}
&\left\|(F(\overline{u})-F(\hat{u}))+\beta_k M^{-1}(\overline{u}-\hat{u})\right\|_M^2 \\
&\quad=\left\|(F(\overline{u})-F(\hat{u}))+\beta_{k-1} M^{-1}(\overline{u}-\hat{u})+(\beta_k-\beta_{k-1})M^{-1}(\overline{u}-\hat{u})\right\|_M^2 \\
&\quad=\left\|(F(\overline{u})-F(\hat{u}))+\beta_{k-1} M^{-1}(\overline{u}-\hat{u})\right\|_M^2+\left\|(\beta_k-\beta_{k-1})M^{-1}(\overline{u}-\hat{u})\right\|_M^2 \\
&\qquad+2(\beta_k-\beta_{k-1})(\overline{u}-\hat{u})^T\left[(F(\overline{u})-F(\hat{u}))+\beta_{k-1}M^{-1}(\overline{u}-\hat{u})\right] \\
&\quad\geq \frac{1}{(1+\tau_{k-1})^2}\left\|(F(\overline{u})-F(\hat{u}))+\beta_{k-1} M^{-1}(\overline{u}-\hat{u})\right\|_M^2 \\
&\quad\geq \frac{1}{C_p}\left\|(F(\overline{u})-F(\hat{u}))+\beta_{k_0} M^{-1}(\overline{u}-\hat{u})\right\|_M^2.
\end{aligned}\tag{3.27}$$

Combing (3.25)–(3.27), we have that for any $k \geq k_0$,

$$\begin{aligned}
&\left\|(F(u^k)-F(\hat{u}))+\beta_k M^{-1}(u^k-\hat{u})\right\|_M \\
&\quad\geq \frac{\delta}{C_p^{1/2}}-\frac{\delta}{2C_p^{1/2}} \\
&\quad=\frac{\delta}{2C_p^{1/2}}>0,
\end{aligned}\tag{3.28}$$

which means that $\hat{u}$ cannot be a cluster point of $\{u^k\}$. Thus, $\{u^k\}$ has just one cluster point. □

4. Inexact Implicit Method and Convergence Analysis

The main task at each iteration of the implicit exact algorithm in the last section is to solve a system of nonlinear equations. To solve it exactly per iteration is time consuming, and there is little justification to solve it exactly, especially when the iterative point is far away from the solution set. Thus, in this section, we propose to solve the subproblem approximately. That is, for a given u^k, instead of finding the exact solution of (3.1), we would accept u^{k+1} as the new iterate if it satisfies

$$\left\|F(u^{k+1}) - F(u^k) + \beta_k M^{-1}(u^{k+1} - u^k) + \gamma e(u^k, \beta_k)\right\|_M \leq \eta_k \left\|e(u^k, \beta_k)\right\|_M, \tag{4.1}$$

where $\{\eta_k\}$ is a nonnegative sequence with $\sum_{k=0}^{\infty} {\eta_k}^2 < +\infty$. If (3.1) is replaced by (4.1), the modified method is called *inexact implicit method*.

We now analyze the convergence of the inexact implicit method.

Lemma 4.1. *Let $\{u^k\}$ be the sequence generated by the inexact implicit method. Then there exists a $k_0 > 0$ such that for any $k \geq k_0$ and $u^* \in \Omega^*$,*

$$\begin{aligned} &\left\|(F(u^{k+1}) - F(u^*)) + \beta_k M^{-1}(u^{k+1} - u^*)\right\|_M^2 \\ &\quad\leq \left(1 + \frac{4{\eta_k}^2}{\gamma(2-\gamma)}\right)\left\|(F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*)\right\|_M^2 \\ &\quad\quad - \frac{1}{2}\gamma(2-\gamma)\left\|e(u^k, \beta_k)\right\|_M^2. \end{aligned} \tag{4.2}$$

Proof. Denote

$$\theta_k(u) := F(u) - F\left(u^k\right) + \beta_k M^{-1}\left(u - u^k\right) + \gamma e\left(u^k, \beta_k\right). \tag{4.3}$$

Then (4.1) can be rewritten as

$$\left\|\theta_k(u^{k+1})\right\|_M \leq \eta_k \left\|e(u^k, \beta_k)\right\|_M. \tag{4.4}$$

According to (4.3) and (2.7),

$$\begin{aligned} &\left\|F(u^{k+1}) - F(u^*) + \beta_k(u^{k+1} - u^*)\right\|_M^2 \\ &\quad= \left\|[(F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*)] - [\gamma e(u^k, \beta_k) - \theta_k(u^{k+1})]\right\|_M^2 \\ &\quad\leq \left\|F(u^k) - F(u^*) + \beta_k M^{-1}(u^k - u^*)\right\|_M^2 - 2\gamma\left\|e(u^k, \beta_k)\right\|_M^2 \\ &\quad\quad + 2\{(F(u^k) - F(u^*)) + \beta_k M^{-1}(u^k - u^*)\}^\top M\theta_k\left(u^{k+1}\right) \\ &\quad\quad + \left\|\gamma e(u^k, \beta_k) - \theta_k(u^{k+1})\right\|_M^2. \end{aligned} \tag{4.5}$$

Using Cauchy-Schwarz inequality and (4.4), we have

$$\begin{aligned}
&2\{(F(u^k)-F(u^*))+\beta_k M^{-1}(u^k-u^*)\}^{\top} M\theta_k\left(u^{k+1}\right)\\
&\quad\le \frac{4\eta_k{}^2}{\gamma(2-\gamma)}\left\|(F(u^k)-F(u^*))+\beta_k M^{-1}(u^k-u^*)\right\|_M^2+\frac{\gamma(2-\gamma)}{4\eta_k{}^2}\left\|\theta_k(u^{k+1})\right\|_M^2\\
&\quad\le \frac{4\eta_k{}^2}{\gamma(2-\gamma)}\left\|(F(u^k)-F(u^*))+\beta_k M^{-1}(u^k-u^*)\right\|_M^2+\frac{\gamma(2-\gamma)}{4}\left\|e(u^k,\beta_k)\right\|_M^2,
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
&\left\|\gamma e(u^k,\beta_k)-\theta_k(u^{k+1})\right\|_M^2\\
&\quad=\gamma^2\left\|e(u^k,\beta_k)\right\|_M^2-2\gamma e\left(u^k,\beta_k\right)_M^T\theta_k\left(u^{k+1}\right)+\left\|\theta_k(u^{k+1})\right\|_M^2\\
&\quad\le \gamma^2\left\|e(u^k,\beta_k)\right\|_M^2+\frac{\gamma(2-\gamma)}{8}\left\|e(u^k,\beta_k)\right\|_M^2+\frac{8\gamma}{(2-\gamma)}\left\|\theta_k(u^{k+1})\right\|_M^2+\left\|\theta_k(u^{k+1})\right\|_M^2\\
&\quad\le \gamma^2\left\|e(u^k,\beta_k)\right\|_M^2+\frac{\gamma(2-\gamma)}{8}\left\|e(u^k,\beta_k)\right\|_M^2+\left(1+\frac{8\gamma}{(2-\gamma)}\right)\eta_k^2\left\|e(u^k,\beta_k)\right\|_M^2.
\end{aligned}\tag{4.7}$$

Since $\sum_{k=0}^{\infty}\eta_k^2<+\infty$, there is a constant $k_0\ge 0$, such that for all $k\ge k_0$,

$$\left(1+\frac{8\gamma}{(2-\gamma)}\right)\eta_k^2\le\frac{\gamma(2-\gamma)}{8},\tag{4.8}$$

and (4.7) becomes that for all $k\ge k_0$,

$$\left\|\gamma e(u^k,\beta_k)-\theta_k(u^{k+1})\right\|_M^2\le\gamma^2\left\|e(u^k,\beta_k)\right\|_M^2+\frac{\gamma(2-\gamma)}{4}\left\|e(u^k,\beta_k)\right\|_M^2.\tag{4.9}$$

Substituting (4.6) and (4.9) into (4.5), we complete the proof. □

In a similar way to (3.7), by using the monotonicity and the assumption that $0<\beta_{k+1}\le(1+\tau_k)\beta_k$ and (4.2), we obtain that for all $k\ge k_0$

$$\begin{aligned}
&\left\|\left(F\left(u^{k+1}\right)-F(u^*)\right)+\beta_{k+1}M^{-1}\left(u^{k+1}-u^*\right)\right\|_M^2\\
&\quad\le(1+\tau_k)^2\left(1+\frac{4\eta_k{}^2}{\gamma(2-\gamma)}\right)\left\|\left(F\left(u^k\right)-F(u^*)\right)+\beta_k M^{-1}\left(u^k-u^*\right)\right\|_M^2\\
&\qquad-\frac{1}{2}\gamma(2-\gamma)\left\|e\left(u^k,\beta_k\right)\right\|_M^2.
\end{aligned}\tag{4.10}$$

Now, we prove the convergence of the inexact implicit method.

Theorem 4.2. *The sequence $\{u^k\}$ generated by the proposed self-adaptive inexact implicit method converges to a solution point of $VVI(\Omega, F)$.*

Proof. Let

$$\xi_k := 2\tau_k + \tau_k^2 + \frac{4\eta_k^2}{\gamma(2-\gamma)} + \frac{8\tau_k\eta_k^2}{\gamma(2-\gamma)} + \frac{4\tau_k^2\eta_k^2}{\gamma(2-\gamma)}. \tag{4.11}$$

Then, it follows from (4.10) that for all $k \geq k_0$,

$$\begin{aligned} &\left\| \left(F\left(u^{k+1}\right) - F(u^*)\right) + \beta_{k+1}M^{-1}\left(u^{k+1} - u^*\right) \right\|_M^2 \\ &\quad \leq (1+\xi_k)\left\| (F\left(u^k\right) - F(u^*)) + \beta_k M^{-1}\left(u^k - u^*\right) \right\|_M^2 \\ &\quad - \frac{1}{2}\gamma(2-\gamma)\left\| e\left(u^k, \beta_k\right) \right\|_M^2. \end{aligned} \tag{4.12}$$

From the assumptions that

$$\sum_{k=0}^{\infty} \tau_k < +\infty, \qquad \sum_{k=0}^{\infty} {\eta_k}^2 < +\infty, \tag{4.13}$$

it follows that

$$C_s := \sum_{i=0}^{\infty} \xi_i, \qquad C_p := \prod_{i=0}^{\infty} (1+\xi_i), \tag{4.14}$$

are finite. The rest of the proof is similar to that of Theorem 3.3 and is thus omitted here. □

5. Computational Results

In this section, we present some numerical results for the proposed self-adaptive implicit methods. Our main interests are two folds: the first one is to compare the proposed method with He's method [11] in solving a simple nonlinear problem, showing the numerical advantage; the second one is to indicate that the strategy is rather insensitive to the initial point, the initial choice of the parameter, as well as the size of the problems. All codes were written in Matlab and run on a AMD 3200+ personal computer. In the following tests, the parameter β_k is changed when

$$\frac{\|(F(u^{k+1}) - F(u^k))\|_M}{\|\beta_k M^{-1}(u^{k+1} - u^k)\|_M} > 2, \qquad \frac{\|(F(u^{k+1}) - F(u^k))\|_M}{\|\beta_k M^{-1}(u^{k+1} - u^k)\|_M} < \frac{1}{2}. \tag{5.1}$$

That is, we set $\tau = 1$ in (3.13). We set $M = I$, and the matrix-induced norm projection is just the projection under Euclidean norm, which is very easy to implement when Ω has some special structure. For example, when Ω is the nonnegative orthant,

$$\{x \in \mathcal{R}^n \mid x \geq 0\}, \tag{5.2}$$

then

$$\left(P_\Omega[y]\right)_j = \begin{cases} y_j, & \text{if } y_j \geq 0, \\ 0, & \text{otherwise;} \end{cases} \tag{5.3}$$

when Ω is a box,

$$\{x \in \mathcal{R}^n \mid l \leq x \leq h\}, \tag{5.4}$$

then

$$\left(P_\Omega[y]\right)_j = \begin{cases} u_j, & \text{if } y_j \geq u_j, \\ y_j, & \text{if } u_j \geq y_j \geq l_j, \\ l_j, & \text{otherwise;} \end{cases} \tag{5.5}$$

when Ω is a ball,

$$\{x \in \mathcal{R}^n \mid \|x\| \leq r\}, \tag{5.6}$$

then

$$\left(P_\Omega[y]\right) = \begin{cases} y, & \text{if } \|y\| \leq r, \\ \dfrac{ry}{\|y\|}, & \text{otherwise.} \end{cases} \tag{5.7}$$

At each iteration, we use Newton's method [25, 26] to solve the system of nonlinear equations

$$(\textbf{SNLE}) \qquad \beta_k M^{-1} u + F(u) = \beta_k M^{-1} u^k + F\left(u^k\right) - \gamma e\left(u^k, \beta_k\right) \tag{5.8}$$

approximately; that is, we stop the iteration of Newton's method as soon as the current iterative point satisfies (4.1), and adopt it as the next iterative point, where

$$\eta_k = \begin{cases} 0.3, & \text{if } k \le k_{\max}, \\ \dfrac{1}{k - k_{\max}}, & \text{otherwise}, \end{cases} \tag{5.9}$$

with $k_{\max} = 50$.

In our first test problem , we take

$$F(u) = \arctan\,(u) + AA^{\top}u + Ac, \tag{5.10}$$

where the matrix A is constructed by $A := W\Sigma V$. Here

$$W = I_m - 2\frac{ww^{\top}}{w^{\top}w}, \qquad V = I_n - 2\frac{vv^{\top}}{v^{\top}v} \tag{5.11}$$

are Householder matrices and $\Sigma = \operatorname{diag}(\sigma_i)$, $i = 1, \ldots n$, is a diagonal matrix with $\sigma_i = \cos(i\pi/n + 1)$. The vectors w, v, and c contain pseudorandom numbers:

$$\begin{aligned} w_1 &= 13846, & w_i &= (31416w_{i-1} + 13846) \mod 46261, & i &= 2, \ldots m, \\ v_1 &= 13846, & v_i &= (42108v_{i-1} + 13846) \mod 46273, & i &= 2, \ldots n, \\ c_1 &= 13846, & c_i &= (45278c_{i-1} + 13846) \mod 46219, & i &= 2, \ldots n. \end{aligned} \tag{5.12}$$

The closed convex set Ω in this problem is defined as

$$\Omega := \{z \in \mathcal{R}^m, \ \|z\| \le \alpha\} \tag{5.13}$$

with different prescribed α. Note that in the case $\|Ac\| > \alpha$, $\|\arctan\,(u^*) + AA^{\top}u^* + Ac\| = \alpha$ (otherwise $u^* = 0$ is the trivial solution). Therefore, we test the problem with $\alpha = \kappa\|Ac\|$ and $\kappa \in (0,1)$. In the test we take $\gamma = 1.85$, $\tau_k = 0.85$, $u^0 = 0$, and $\beta_0 = 0.1$. The stopping criterion is

$$\frac{\|e(u^k, \beta_k)\|}{\alpha} \le 10^{-8}. \tag{5.14}$$

The results in Table 1 show that $\beta_0 = 0.1$ is a "proper" parameter for the problem with $\kappa = 0.05$, while for the other two cases with larger $\kappa = 0.5$ and with smaller $\kappa = 0.01$, it is not. For any of these three cases, the method with self-adaptive strategy rule is efficient.

The second example considered here is the variant mixed complementarity problem for short VMCP, with $\Omega = \{u \in \mathcal{R}^n \mid l_i \le u_i(x) \le h_i, \ i = 1, \ldots, n\}$, where $l_i \in (5, 10)$ and $h_i \in (1000, 2000)$ are randomly generated parameters. The mapping F is taken as

$$F(u) = D(u) + Mu + q, \tag{5.15}$$

Table 1: Comparison of the proposed method and He's method [11].

	$m = 100$ $n = 50$				$m = 500$ $n = 300$			
	Proposed method		He's method		Proposed method		He's method	
	It. no.	CPU	It. no.	CPU	It. no.	CPU	It. no.	CPU
$0.5\|Ac\|$	25	0.3910	100	1.0780	34	50.4850	—	—
$0.05\|Ac\|$	20	0.3120	37	0.4850	25	39.8440	17	25.0940
$0.01\|Ac\|$	26	0.4060	350	5.8750	33	61.4070	—	—

"—" means iteration numbers >200 and CPU >2000 (sec).

Table 2: Numerical results for VMCP with dimension $n = 50$.

β	Proposed method		He's method	
	It. no.	CPU	It. no.	CPU
10^5	69	0.0780	—	—
10^4	65	0.1250	7335	6.1250
10^3	61	0.0790	485	0.4530
10^2	59	0.0620	60	4.0780
10	60	0.0780	315	0.3280
1	66	0.0110	2672	2.500
10^{-1}	70	0.0940	22541	21.0320
10^{-2}	73	0.0780	—	—

"—" means iteration numbers >3000 and CPU >300 (sec).

where $D(u)$ and $Mu + q$ are the nonlinear part and the linear part of $F(u)$, respectively. We form the linear part $Mu+q$ similarly as in [27]. The matrix $M = A^{\top}A+B$, where A is an $n\times n$ matrix whose entries are randomly generated in the interval $(-5, 5)$, and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, 0)$. In $D(u)$, the nonlinear part of $F(u)$, the components are $D_j(u) = a_j * \arctan(u_j)$, and a_j is a random variable in $(0, 1)$. The numerical results are summarized in Tables 2–5, where the initial iterative point is $u^0 = 0$ in Tables 2 and 3 and u^0 is randomly generated in $(0, 1)$ in Tables 4 and 5, respectively. The other parameters are the same: $\gamma = 1.85$ and $\tau_k = 0.85$ for $k \le 40$ and $\tau_k = 1/k$ otherwise. The stopping criterion is

$$\left\| e\left(u^k, \beta_k\right) \right\|_{\infty} \le 10^{-7}. \tag{5.16}$$

As the results in Table 1, the results in Tables 2 to 5 indicate that the number of iterations and CPU time are rather insensitive to the initial parameter β_0, while He's method is efficient for proper choice of β. The results also show that the proposed method, as well as He's method, is very stable and efficient to the choice of the initial point u^0.

6. Conclusions

In this paper, we proposed a self-adaptive implicit method for solving monotone variant variational inequalities. The proposed self-adaptive adjusting rule avoids the difficult task of choosing a "suitable" parameter, which makes the method efficient for initial parameter. Our self-adaptive rule adds only a tiny amount of computation than the method with fixed parameter, while the efficiency is enhanced greatly. To make the method more efficient and

Table 3: Numerical results for VMCP with dimension $n = 200$.

β	Proposed method		He's method	
	It. no.	CPU	It. no.	CPU
10^5	82	1.6090	—	—
10^4	74	1.4850	1434	28.3750
10^3	64	1.2660	199	3.8910
10^2	63	1.2500	174	3.4060
10	68	1.3500	1486	30.4840
1	75	1.4850	—	—
10^{-1}	75	1.5000	—	—
10^{-2}	86	1.7030	—	—

"—" means iteration numbers >3000 and CPU >300 (sec).

Table 4: Numerical results for VMCP with dimension $n = 50$.

β	Proposed method		He's method	
	It. no.	CPU	It. no.	CPU
10^5	61	0.0620	—	—
10^4	61	0.0940	3422	3.7190
10^3	60	0.0790	684	0.6410
10^2	67	0.0780	59	0.0620
10	65	0.0940	309	0.2970
1	69	0.0940	2637	2.3750
10^{-1}	72	0.0940	21949	18.9220
10^{-2}	75	0.1250	—	—

"—" means iteration numbers >3000 and CPU >300 (sec).

Table 5: Numerical results for VMCP with dimension $n = 200$.

β	Proposed method		He's method	
	It. no.	CPU	It. no.	CPU
10^5	61	1.2500	—	—
10^4	64	1.2810	1527	29.8750
10^3	64	1.2660	150	2.9220
10^2	64	1.2810	222	4.3440
10	89	1.7920	1922	37.6250
1	70	1.3910	—	—
10^{-1}	88	1.7340	—	—
10^{-2}	84	1.6560	—	—

"—" means iteration numbers >5000 and CPU >300 (sec).

practical, an approximate version of the algorithm was proposed. The global convergence of both the exact version and the inexact version of the new algorithm was proved under mild assumptions; that is, the underlying mapping of $\mathrm{VVI}(\Omega, F)$ is monotone and there is at least one solution of the problem. The reported preliminary numerical results verified our assertion.

Acnowledgments

This research was supported by the NSFC Grants 10501024, 10871098, and NSF of Jiangsu Province at Grant no. BK2006214. D. Han was also supported by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

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