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Research Article

Existence of Solutions for Hyperbolic System of Second Order Outside a Domain

Jie Xin and Xiuyan Sha

School of Mathematics and Information, Ludong University, Yantai, Shandong 264025, China

Correspondence should be addressed to Jie Xin, fdxinjie@sina.com

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We study the mixed initial-boundary value problem for hyperbolic system of second order outside a closed domain. The existence of solutions to this problem is proved and the estimate for the regularity of solutions is given. The application of the existence theorem to elastrodynamics is discussed.

1. Introduction

This paper is concerned with the exterior problem for hyperbolic system of second order. Let $\mathcal{K}$ be a closed domain with smooth boundary in $\mathbb{R}^3$ and let the origin belong to $\mathcal{K}$. Consider the following exterior problem for the hyperbolic system of second order:

$$\begin{gathered}\partial_t^2 u^i - \sum_{j,k,l=1}^{3} a_{ijkl}(t,x)\partial_j\partial_l u^k = b^i, \quad i = 1,2,3,\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \\ u(0,x) = f(x), \qquad \partial_t u(0,x) = g(x), \\ u(t,x) = 0, \quad x \in \partial\mathcal{K}, \end{gathered} \tag{1.1}$$

where $a_{ijkl}(t,x) \in C_B^2([0,\infty) \times \mathbb{R}^3 \setminus \mathcal{K})$ and $b = (b^1, b^2, b^3)$. We assume that $a_{ijkl}(t,x)$ satisfies

$$\sum_{j,k,l=1}^{3} a_{ijkl}(t,x)e_{ij}e_{kl} \geq \alpha |E|^2, \quad (\alpha > 0), \tag{1.2}$$

for all symmetric matrixes $E = (e_{ij})$, where $e_{ij} = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, $|E|^2 = \sum_{i,j=1}^{3} e_{ij}^2$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}$.

Let $v = \partial_t u$. The system (1.1) can be written as an evolution system in the form

$$\frac{d}{dt}U = A(t)U + B, \tag{1.3}$$

where

$$\begin{gathered} U = \left(u^1, u^2, u^3, \partial_t u^1, \partial_t u^2, \partial_t u^3\right)^T = (u,v)^T, \qquad B = (0,b)^T, \\ A(t) = \begin{pmatrix} 0 & I_{3\times 3} \\ a(t) & 0 \end{pmatrix}_{6\times 6}, \\ a(t) = \left(\sum_{j,l=1}^{3} a_{ijkl} \partial_j \partial_l \right)_{3\times 3}. \end{gathered} \tag{1.4}$$

Ikawa considered in [1] the mixed problem of a hyperbolic equation of second-order. The existence theorem is known for the obstacle free problem in [2]. Dafermos and Hrusa proved in [3] the local existence of the Dirichlet problem for the hyperbolic system inside a domain by energy method.

In this paper, we deal with the exterior problem for the second order hyperbolic system. In Section 2, we show the existence of the exterior problem for the problem (1.1) by the semigroup theory. In Section 3, we prove the regularity for the solutions of the exterior problem (1.1) and give the estimate for the regularity of solutions. In Section 4, we discuss the application of the existence theorem to elastrodynamics.

2. Existence of the Exterior Problem for Hyperbolic System of Second Order

Note that $H(t) = H_0^1(\mathbb{R}^3 \setminus \mathcal{K}) \times L^2(\mathbb{R}^3 \setminus \mathcal{K})$ with the inner product

$$(U_1, U_2)_{H(t)} = ((u_1, v_1), (u_2, v_2))_{H(t)} = \sum_{i,j,k,l=1}^{3} \left(a_{ijkl}(t,x) \partial_j u_1^i, \partial_l u_2^k \right) + (v_1, v_2). \tag{2.1}$$

By (1.2) and Korn inequality (cf. [4, 5]), we have

Lemma 2.1. *For some $M > 0$, we have*

$$\frac{1}{M}\left(\|u\|^2_{H_0^1(\mathbb{R}^3\setminus\mathcal{K})} + \|v\|^2_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \right) \le \|U\|^2_{H(t)} \le M\left(\|u\|^2_{H_0^1(\mathbb{R}^3\setminus\mathcal{K})} + \|v\|^2_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \right). \tag{2.2}$$

Then $H(t)$ is a Hilbert space with the inner product defined as above. We define the operator (without loss of generality, we still write this operator as $A(t)$) in $H(t)$ by

$$\begin{aligned} A(t) : D &\longrightarrow H(t), \\ U &\longrightarrow A(t)U, \end{aligned} \tag{2.3}$$

where $D = (H^2(\mathbb{R}^3 \setminus \mathcal{K}) \cap H_0^1(\mathbb{R}^3 \setminus \mathcal{K})) \times H_0^1(\mathbb{R}^3 \setminus \mathcal{K})$. It is obvious that $A(t)$ is a densely defined operator.

Lemma 2.2. *There exists a constant $c > 0$ such that for any $U \in D$,*

$$\left|(A(t)U, U)_{H(t)}\right| \leq c(U, U)_{H(t)} \tag{2.4}$$

holds.

Proof. Let $U = (u, v) \in D$.

$$\begin{aligned} \left|(A(t)U, U)_{H(t)}\right| &= \left|\sum_{i,j,k,l=1}^{3} \left(a_{ijkl}\partial_j v^i, \partial_l u^k\right) + (a(t)u, v)\right| \\ &= \left|\sum_{i,j,k,l=1}^{3} \int_{\partial\mathcal{K}} a_{ijkl}\partial_l u^k v^i \nu_j \, d\Gamma - \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3\setminus\mathcal{K}} v^i \partial_j a_{ijkl}\partial_l u^k \, dx \right. \\ &\quad \left. - \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3\setminus\mathcal{K}} v^i a_{ijkl}\partial_j\partial_l u^k \, dx + (a(t)u, v)\right| \\ &= \left|- \sum_{i,j,k,l=1}^{3} \int_{\mathbb{R}^3\setminus\mathcal{K}} v^i \partial_j a_{ijkl}\partial_l u^k \, dx - (v, a(t)u) + (a(t)u, v)\right| \\ &\leq C\left(\|u\|^2_{H_0^1(\mathbb{R}^3\setminus\mathcal{K})} + \|v\|^2_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\right) \\ &\leq c\|U\|^2_{H(t)}. \end{aligned} \tag{2.5}$$

□

Corollary 2.3. *For all real λ such that $|\lambda| > 2c$, the estimate*

$$\|(\lambda I - A(t))U\|_{H(t)} \geq (|\lambda| - c)\|U\|_{H(t)} \tag{2.6}$$

holds for any $U \in D$.

Proof. By (2.4),

$$\begin{aligned}
&((\lambda I - A(t))U, (\lambda I - A(t))U)_{H(t)} \\
&\quad = |\lambda|^2 (U,U)_{H(t)} - \lambda\Big((U, A(t)U)_{H(t)} + (A(t)U,U)_{H(t)}\Big) + (A(t)U, A(t)U)_{H(t)} \\
&\quad \geq |\lambda|^2 (U,U)_{H(t)} - 2|\lambda| c (U,U)_{H(t)} \\
&\quad = \Big((|\lambda| - 2c)^2 + 2c(|\lambda| - 2c)\Big)\|U\|^2_{H(t)} \\
&\quad \geq (|\lambda| - 2c)^2 \|U\|^2_{H(t)}.
\end{aligned} \tag{2.7}$$

□

The estimate of the resolvent operator $(\lambda I - A(t))^{-1}$ is the following.

Lemma 2.4. *There exists a constant $\delta > 0$ such that for all λ real and $|\lambda| > \delta$,*

$$\lambda I - A(t) : D \longrightarrow H(t) \tag{2.8}$$

is a bijective mapping. Moreover, we have

$$\left\|(\lambda I - A(t))^{-1}\right\|_{H(t)} \leq \frac{1}{|\lambda| - \delta}. \tag{2.9}$$

Proof. Consider the system

$$(\lambda I - A(t))U = P, \tag{2.10}$$

namely,

$$\begin{aligned}
\lambda u - v &= p \\
-a(t)u + \lambda v &= q,
\end{aligned} \tag{2.11}$$

where $(p, q) \in H_0^1(\mathbb{R}^3 \setminus \mathcal{K}) \times L^2(\mathbb{R}^3 \setminus \mathcal{K}) = H(t)$.

The substitution of the first relation

$$v = \lambda u - p \tag{2.12}$$

in the second of (2.11) gives

$$\left(-a(t) + \lambda^2\right)u = \lambda p + q = w \in L^2\left(\mathbb{R}^3 \setminus \mathcal{K}\right). \tag{2.13}$$

By the well-known variation method, there exists a solution $u \in H^2(\mathbb{R}^3 \setminus \mathcal{K}) \cap H_0^1(\mathbb{R}^3 \setminus \mathcal{K})$ of the elliptic system (2.13) for any $w \in L^2(\mathbb{R}^3 \setminus \mathcal{K})$. Defining v by (2.12), we have a solution

$$(u, v) \in \left(H^2\left(\mathbb{R}^3 \setminus \mathcal{K}\right) \cap H_0^1\left(\mathbb{R}^3 \setminus \mathcal{K}\right)\right) \times H_0^1\left(\mathbb{R}^3 \setminus \mathcal{K}\right) = D \tag{2.14}$$

of (2.10). Therefore, $\lambda I - A(t)$ is a surjection.

From (2.6), it follows that the existence of $(\lambda I - A(t))^{-1}$ and the estimate

$$\left\| (\lambda I - A(t))^{-1} U \right\|_{H(t)} \leq \frac{1}{|\lambda| - 2c} \|U\|_{H(t)}. \tag{2.15}$$

Let $\delta = 2c$, we have (2.9). □

For $U = (u, v) \in H^p(\mathbb{R}^3 \setminus \mathcal{K}) \times H^{p-1}(\mathbb{R}^3 \setminus \mathcal{K})$, we define the following norm:

$$\|U\|_p^2 = \|u\|^2_{H^p(\mathbb{R}^3 \setminus \mathcal{K})} + \|v\|^2_{H^{p-1}(\mathbb{R}^3 \setminus \mathcal{K})}. \tag{2.16}$$

Suppose that $a_{ijkl}(t, x) \in C^p([0, \infty) \times \mathbb{R}^3 \setminus \mathcal{K})$, we have

Corollary 2.5. *For the real number $\lambda_0 > \delta$ (λ_0 fixed) and the integer $p \geq 1$, where δ is as in Lemma 2.4, there exists $d_p > 0$ such that for any $U \in D \cap (H^p(\mathbb{R}^3 \setminus \mathcal{K}) \times H^{p-1}(\mathbb{R}^3 \setminus \mathcal{K}))$,*

$$\|U\|_p < d_p \|(\lambda_0 I - A(t))U\|_{p-1}. \tag{2.17}$$

Proof. From Lemma 2.4,

$$\lambda_0 I - A(t) : D \cap \left(H^p\left(\mathbb{R}^3 \setminus \mathcal{K}\right) \times H^{p-1}\left(\mathbb{R}^3 \setminus \mathcal{K}\right)\right) \longrightarrow H(t) \cap \left(H^{p-1}\left(\mathbb{R}^3 \setminus \mathcal{K}\right) \times H^{p-2}\left(\mathbb{R}^3 \setminus \mathcal{K}\right)\right) \tag{2.18}$$

is a bijective continuous mapping, then $\lambda_0 I - A(t)$ is a closed operator. It implies that $(\lambda_0 I - A(t))^{-1}$ is also a closed operator. By Banach's closed graph theorem, $(\lambda_0 I - A(t))^{-1}$ is continuous. So for any $U \in D \cap (H^p(\mathbb{R}^3 \setminus \mathcal{K}) \times H^{p-1}(\mathbb{R}^3 \setminus \mathcal{K}))$, we have

$$\|U\|_p = \left\| (\lambda_0 I - A(t))^{-1} (\lambda_0 I - A(t)) U \right\|_p \leq d_p \|(\lambda_0 I - A(t))U\|_{p-1}. \tag{2.19}$$

□

Definition 2.6. Let X be a *Banach* space. A family $\{A(t)\}_{t \in [0,T]}$ of infinitesimal generators of C_0 semigroups on X is called stable if there are constants $M \geq 1$ and δ (called the stability constants) such that

$$\rho(A(t)) \supset (\delta, \infty), \quad \forall t \in [0, T],$$
$$\left\| \prod_{j=1}^{k} (\lambda I - A(t_j))^{-1} \right\| \leq M(\lambda - \delta)^{-k}, \quad \forall \lambda > \delta, \tag{2.20}$$

for every finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$, $k = 1, 2, \ldots$.

Lemma 2.7. *For $t \in [0,T]$, let $A(t)$ be the infinitesimal generators of C_0 semigroups $S_t(s)$ on the* Banach *X. The family of generators $\{A(t)\}_{t\in[0,T]}$ is stable if and only if there are constants $M \ge 1$ and δ such that*

$$\rho(A(t)) \supset (\delta, \infty), \quad \forall t \in [0,T],$$
$$\left\| \prod_{j=1}^{k} S_{t_j}(s_j) \right\| \le M \exp\left\{ \delta \sum_{j=1}^{k} s_j \right\}, \tag{2.21}$$

for any finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$, $k = 1, 2, \ldots$.

Lemma 2.8. *Let $\{A(t)\}_{t\in[0,T]}$ be a stable family of infinitesimal generators of C_0 semigroups $S_t(s)$ on the* Banach *space X such that $D(A(t)) = D$ is independent of t and for every $U_0 \in D$, $A(t)U_0$ is continuously differentiable in X. If $B(t) \in C^1([0,T]; X)$, then*

$$\frac{d}{dt}U(t) = A(t)U(t) + B(t) \tag{2.22}$$

has a unique classical solution $U(t) \in C^1([0,T]; X) \cap C([0,T]; D)$ such that $U(0) = U_0$.

The proofs of Lemmas 2.7 and 2.8 are in [6]. The straightforward application of the semigroup theory to the system (1.3) gives the following proposition.

Proposition 2.9. *Given $U_0 \in D$ and $B(t) \in C^1([0,T], H_0^1(\mathbb{R}^3 \setminus \mathcal{K}) \times L^2(\mathbb{R}^3 \setminus \mathcal{K}))$, then there exists one and only one solution $U(t) \in C^1([0,T]; H_0^1(\mathbb{R}^3 \setminus \mathcal{K}) \times L^2(\mathbb{R}^3 \setminus \mathcal{K})) \cap C([0,T]; D)$ of* (1.3) *such that $U(0) = U_0$.*

Proof. Let $X = H(t)$. For given $t > 0$, $A(t)$ is an infinitesimal generator of C_0 semigroups $S_t(s)$ on X. For any $U \in D$, it is easy to know that

$$\|S_t(s)U\|_{H(t)} \le e^{\delta s}\|U\|_{H(t)}. \tag{2.23}$$

Then for any $U \in D$, $t_1, t_2 > 0$, we have

$$\begin{aligned}
\|U\|_{H(t_1)}^2 &= \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{i,j,k,l=1}^{3} a_{ijkl}(t_1)\partial_j u^i \partial_l u^k \, dx + (v, v) \\
&= \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{i,j,k,l=1}^{3} a_{ijkl}(t_2)\partial_j u^i \partial_l u^k \, dx + (v, v) + \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{i,j,k,l=1}^{3} (a_{ijkl}(t_1) - a_{ijkl}(t_2))\partial_j u^i \partial_l u^k \, dx \\
&\le \|U\|_{H(t_2)}^2 + C|t_1 - t_2|\|U\|_{H(t_2)}^2,
\end{aligned} \tag{2.24}$$

namely,

$$\|U\|_{H(t_1)} \le (1 + C_1|t_1 - t_2|)^{1/2}\|U\|_{H(t_2)}. \tag{2.25}$$

For any finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$ and any s_j, $j = 1, 2, \ldots, k$,

$$\begin{aligned}
&\|S_{t_k}(s_k)S_{t_{k-1}}(s_{k-1})\cdots S_{t_1}(s_1)U\|_{H(t)} \\
&\quad\le C\|S_{t_k}(s_k)S_{t_{k-1}}(s_{k-1})\cdots S_{t_1}(s_1)U\|_{H(t_k)} \\
&\quad\le Ce^{\delta s_k}\|S_{t_{k-1}}(s_{k-1})\cdots S_{t_1}(s_1)U\|_{H(t_k)} \\
&\quad\le Ce^{\delta s_k}(1 + C_1(t_k - t_{k-1}))^{1/2}\|S_{t_{k-1}}(s_{k-1})\cdots S_{t_1}(s_1)U\|_{H(t_{k-1})} \\
&\quad\le Ce^{\delta(s_k+s_{k-1}+\cdots+s_2+s_1)}(1+C_1(t_k-t_{k-1}))^{1/2}(1+C_1(t_{k-1}-t_{k-2}))^{1/2}\cdots(1+C_1(t_2-t_1))^{1/2}\|U\|_{H(t_1)} \\
&\quad\le C\exp\left(\delta\sum_{j=1}^{k}s_j\right)\left(\frac{k + C_1t_k}{k}\right)^{k/2}\|U\|_{H(t)} \\
&\quad\le C\exp\left(\delta\sum_{j=1}^{k}s_j\right)e^{C_1T/2}\|U\|_{H(t)} \\
&\quad\le M\exp\left(\delta\sum_{j=1}^{k}s_j\right)\|U\|_{H(t)},
\end{aligned} \tag{2.26}$$

where $M \ge 1$. From Lemma 2.4, for any $t \in [0, T]$, $(\delta, \infty) \subset \rho(A(t))$. Then by Lemma 2.7, $\{A(t)\}_{t\in[0,T]}$ is a stable family. Obviously, $A(t)U_0$ is continuously differentiable in X. So Proposition 2.9 follows from Lemma 2.8. □

From Proposition 2.9, we obtain the existence of solutions to the problem (1.1).

Theorem 2.10. *Given $(f, g) \in D$ and $b \in C^1([0, T]; L^2(\mathbb{R}^3 \setminus \mathcal{K}))$, then there exists one and only one solution $u(t, x)$ of* (1.1) *such that*

$$\begin{aligned}
u(t, x) \in C\Big([0, T]; H^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \cap H_0^1\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big) \\
\cap C^1\Big([0, T]; H_0^1\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big) \cap C^2\Big([0, T]; L^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big).
\end{aligned} \tag{2.27}$$

Proof. Let $U_0 = (f, g)^T$, $B = (0, b)^T$. By Proposition 2.9, there exists a solution $U(t) \in C^1([0, T]; H_0^1(\mathbb{R}^3 \setminus \mathcal{K}) \times L^2(\mathbb{R}^3 \setminus \mathcal{K})) \cap C([0, T]; D)$ of problem (1.3) such that $U(0) = U_0$. Let $u(t, x)$ denote the forgoing three components of $U(t)$, then $u(t, x)$ is the solution of problem (1.1) and satisfies (2.27). □

3. Regularity of Solutions for the Exterior Problem

First, we show the energy inequalities for our problem. These inequalities play an important role in the proof of the regularity of solutions.

Proposition 3.1. *Suppose that*

$$u(t,x) \in C\Big([0,T]; H^2\big(\mathbb{R}^3 \setminus \mathcal{K}\big) \cap H_0^1\big(\mathbb{R}^3 \setminus \mathcal{K}\big)\Big) \cap C^1\Big([0,T]; H_0^1\big(\mathbb{R}^3 \setminus \mathcal{K}\big)\Big) \cap C^2\Big([0,T]; L^2\big(\mathbb{R}^3 \setminus \mathcal{K}\big)\Big) \tag{3.1}$$

is a solution of problem (1.1) *and that* $b(t,x) \in C^1([0,T]; L^2(\mathbb{R}^3 \setminus \mathcal{K}))$*, then for any given* $t \in [0,T]$*, we have*

$$\begin{aligned}
&\|u(t,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(t,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\
&\quad \le C(T)\Bigg(\|u(0,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(0,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} \\
&\qquad + \|b(0,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \int_0^t \|\partial_s b(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\, ds\Bigg),
\end{aligned} \tag{3.2}$$

where $C(T)$ *is a constant which depends on* T*.*

Proof. Put $U(t) = (u, \partial_t u)$, then $U(t) \in D$ and satisfies

$$\begin{aligned}
&\frac{d}{dt}U(t) = A(t)U(t) + B(t), \\
&\frac{d}{dt}(U(t),U(t))_{H(t)} \\
&\quad = (U'(t),U(t))_{H(t)} + (U(t),U'(t))_{H(t)} + (U(t),U(t))_{\dot{H}(t)} \\
&\quad = (A(t)U(t)+B(t),U(t))_{H(t)} + (U(t),A(t)U(t)+B(t))_{H(t)} + (U(t),U(t))_{\dot{H}(t)},
\end{aligned} \tag{3.3}$$

where $U'(t) = (d/dt)U(t)$, $(U(t),U(t))_{\dot{H}(t)} = \sum_{i,j,k,l=1}^{3}(\partial_t a_{ijkl}\partial_j u^i, \partial_l u^k)$. Obviously,

$$\left|(U(t),U(t))_{\dot{H}(t)}\right| \le C\|U(t)\|_{H(t)}^2. \tag{3.4}$$

By (2.4),

$$\left|(A(t)U(t),U(t))_{H(t)} + (U(t),A(t)U(t))_{H(t)}\right| \le C\|U(t)\|_{H(t)}^2. \tag{3.5}$$

Thus

$$\frac{d}{dt}\|U(t)\|_{H(t)}^2 \le C\left(\|U(t)\|_{H(t)}^2 + \|B(t)\|_{H(t)}\|U(t)\|_{H(t)}\right),$$
$$\frac{d}{dt}\|U(t)\|_{H(t)} \le C\left(\|U(t)\|_{H(t)} + \|B(t)\|_{H(t)}\right). \tag{3.6}$$

Applying Gronwall's inequality, we get

$$\|U(t)\|_{H(t)} \le e^{Ct}\left(\|U(0)\|_{H(0)} + \int_0^t \|B(s)\|_{H(s)}\,ds\right). \tag{3.7}$$

Without loss of generality, we assume that $\partial_t u(t,x) \in C([0,T]; H^2(\mathbb{R}^3 \setminus \mathcal{K}) \cap H_0^1(\mathbb{R}^3 \setminus \mathcal{K})) \cap C^1([0,T]; H_0^1(\mathbb{R}^3 \setminus \mathcal{K}))$. Then we see

$$U'(t) = \left(\partial_t u, \partial_t^2 u\right) \in D,$$
$$\frac{d}{dt}U'(t) = A(t)U'(t) + A'(t)U(t) + B'(t). \tag{3.8}$$

Applying (3.7) for $U'(t)$, we get

$$\left\|U'(t)\right\|_{H(t)} \le e^{Ct}\left(\left\|U'(0)\right\|_{H(0)} + \int_0^t \left\|A'(s)U(s) + B'(s)\right\|_{H(s)}\,ds\right). \tag{3.9}$$

By (2.17) and (2.2),

$$\begin{aligned}\|U(t)\|_2 + \left\|U'(t)\right\|_1 &\le d_2\|(\lambda_0 I - A(t))U(t)\|_{H(t)} + C\left\|U'(t)\right\|_{H(t)}\\ &\le d_2\left(\lambda_0\|U(t)\|_{H(t)} + \left\|U'(t)\right\|_{H(t)} + \|B(t)\|_{H(t)}\right) + C\left\|U'(t)\right\|_{H(t)}\\ &\le C(T)\left(\|U(0)\|_{H(0)} + \int_0^t \|B(s)\|_{H(s)}\,ds + \|B(t)\|_{H(t)} + \left\|U'(0)\right\|_{H(0)}\right.\\ &\qquad \left. + \int_0^t \left\|A'(s)U(s)\right\|_{H(s)}\,ds + \int_0^t \left\|B'(s)\right\|_{H(s)}\,ds\right).\end{aligned} \tag{3.10}$$

Obviously,

$$\left\|U'(0)\right\|_{H(0)} \le \|A(0)U(0)\|_{H(0)} + \|B(0)\|_{H(0)} \le C\left(\|U(0)\|_{H(0)} + \|b(0,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\right). \tag{3.11}$$

Also we have

$$\|B(t)\|_{H(t)} \le C\left(\int_0^t \|\partial_s b(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\,ds + \|b(0,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\right), \tag{3.12}$$

and for all $t \in [0,T]$,

$$\int_0^t \|B(s)\|_{H(s)}\, ds \le T\left(\int_0^t \|\partial_s b(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\, ds + \|b(0,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\right). \tag{3.13}$$

Inserting these estimates to the above inequality, we get

$$\begin{aligned} &\|U(t)\|_2 + \left\|U'(t)\right\|_1 \\ &\quad\le C(T)\left(\|U(0)\|_2 + \|b(0,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \int_0^t \|\partial_s b(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\, ds + \int_0^t \|U(s)\|_2\, ds\right). \end{aligned} \tag{3.14}$$

An application of Gronwall's inequality implies

$$\|U(t)\|_2 + \left\|U'(t)\right\|_1 \le C(T)\left(\|U(0)\|_2 + \|b(0,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \int_0^t \|\partial_s b(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\, ds\right). \tag{3.15}$$

Namely,

$$\begin{aligned} &\|u(t,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \|v(t,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(t,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t v(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\ &\quad= \|u(t,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(t,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(t,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^2 u(t,\cdot)\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\ &\quad\le C(T)\Bigg(\|u(0,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(0,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \|b(0,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\ &\qquad\qquad + \int_0^t \|\partial_s b(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\, ds\Bigg). \end{aligned} \tag{3.16}$$

This completes the proof of (3.2). □

Theorem 3.2. *For $h > 2$, suppose that $a_{ijkl}(t,x) \in C_B^h([0,T] \times \mathbb{R}^3 \setminus \mathcal{K})$, $f \in H^h(\mathbb{R}^3 \setminus \mathcal{K})$, $g \in H^{h-1}(\mathbb{R}^3 \setminus \mathcal{K})$, and*

$$\begin{gathered} b \in C^\beta\left([0,T]; H^{h-2-\beta}\left(\mathbb{R}^3 \setminus \mathcal{K}\right)\right), \quad 0 \le \beta \le h-2, \\ \partial_t^{h-1} b \in L^1\left([0,T]; L^2\left(\mathbb{R}^3 \setminus \mathcal{K}\right)\right). \end{gathered} \tag{3.17}$$

If the compatibility conditions of order $h-1$ are satisfied, then problem (1.1) *has a solution u such that*

$$u(t,x) \in C^{\beta}\Big([0,T];H^{h-\beta}\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big), \quad 0 \le \beta \le h,$$

$$\sup_{|\alpha|\le h}\|\partial^{\alpha}u(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\Bigg(\|f\|_{H^h(\mathbb{R}^3\setminus\mathcal{K})} + \|g\|_{H^{h-1}(\mathbb{R}^3\setminus\mathcal{K})} + \sup_{0\le r\le t}\sup_{|\alpha|\le h-2}\|\partial^{\alpha}b(r,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \int_0^t \Big\|\partial_r^{h-1}b(r,\cdot)\Big\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}dr\Bigg), \quad \forall t \ge 0. \tag{3.18}$$

Proof. At first we prove

$$u(t,x) \in C^{h-2}\Big([0,T];H^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big) \cap C^{h-1}\Big([0,T];H^1\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big) \cap C^{h}\Big([0,T];L^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big). \tag{3.19}$$

Let $\phi_0 = f$ and $\phi_1 = g$. We define ϕ_p^i by

$$\phi_p^i = \sum_{j,k,l=1}^{3}\sum_{n=0}^{p-2}\binom{p-2}{n}\partial_t^{p-2-n}a_{ijkl}\partial_j\partial_l\phi_n^k + \partial_t^{p-2}b^i(0,x), \quad i=1,2,3,\ p=2,3,\ldots,h-1, \tag{3.20}$$

then $(\phi_p,\phi_{p+1}) \in D$, $p=1,2,\ldots,h-2$.

We consider the following problem:

$$\begin{gathered}
\partial_t^2 v_{q+1}^i - \sum_{j,k,l=1}^{3}a_{ijkl}(t,x)\partial_j\partial_l v_{q+1}^k \\
= \sum_{j,k,l=1}^{3}\sum_{n=0}^{h-3}\binom{h-2}{n}\Big(\partial_t^{h-2-n}a_{ijkl}\Big)\Big(\partial_t^n\partial_j\partial_l u_q^k\Big) + \partial_t^{h-2}b^i, \quad i=1,2,3, \\
v_{q+1}(0,x) = \phi_{h-2}(x), \qquad \partial_t v_{q+1}(0,x) = \phi_{h-1}(x), \\
v_{q+1}(t,x) = 0, \quad x \in \partial\mathcal{K},
\end{gathered} \tag{3.21}$$

where

$$u_q(t,x) = \phi_0(x) + t\phi_1(x) + \cdots + \frac{t^{h-3}}{(h-3)!}\phi_{h-3}(x) + \int_0^t \frac{(t-r)^{h-3}}{(h-3)!}v_q(r,x)dr, \tag{3.22}$$

here $u_0 \equiv 0$.

From (3.21),

$$\partial_t^2\left(v_{q+1}^i - v_q^i\right) - \sum_{j,k,l=1}^{3} a_{ijkl}(t,x)\partial_j\partial_l\left(v_{q+1}^k - v_q^k\right)$$
$$= \sum_{j,k,l=1}^{3}\sum_{n=0}^{h-3}\binom{h-2}{n}\left(\partial_t^{h-2-n}a_{ijkl}\right)\partial_t^n\partial_j\partial_l\left(u_q^k - u_{q-1}^k\right)$$
$$= \sum_{j,k,l=1}^{3}\sum_{n=0}^{h-3}\binom{h-2}{n}\left(\partial_t^{h-2-n}a_{ijkl}\right)\partial_j\partial_l\int_0^t \frac{(t-r)^{h-3-n}}{(h-3-n)!}\left(v_q^k - v_{q-1}^k\right)dr. \quad (3.23)$$

By (3.2), we have

$$\left\|v_{q+1} - v_q\right\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t v_{q+1} - \partial_t v_q\right\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^2 v_{q+1} - \partial_t^2 v_q\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}$$
$$\leq C(T)\int_0^t \left\|v_q - v_{q-1}\right\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} dr, \quad q = 2,3,\ldots, \quad (3.24)$$

thus

$$\left\|v_{q+1} - v_q\right\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t v_{q+1} - \partial_t v_q\right\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^2 v_{q+1} - \partial_t^2 v_q\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \leq C\frac{(C(T)t)^q}{q!}. \quad (3.25)$$

This implies that v_q converges to some v in $C([0,T];H^2(\mathbb{R}^3\setminus\mathcal{K}))\cap C^1([0,T];H^1(\mathbb{R}^3\setminus\mathcal{K}))\cap C^2([0,T];L^2(\mathbb{R}^3\setminus\mathcal{K}))$. Set

$$u(t,x) = \phi_0(x) + t\phi_1(x) + \cdots + \frac{t^{h-3}}{(h-3)!}\phi_{h-3}(x) + \int_0^t \frac{(t-r)^{h-3}}{(h-3)!}v(r,x)dr, \quad (3.26)$$

then u_q tends to u in $C([0,T];H^{h-2}(\mathbb{R}^3\setminus\mathcal{K}))\cap C^{h-1}([0,T];H^1(\mathbb{R}^3\setminus\mathcal{K}))\cap C^h([0,T];L^2(\mathbb{R}^3\setminus\mathcal{K}))$. The passage to the limit of (3.21) shows

$$\partial_t^2\partial_t^{h-2}u^i - \sum_{j,k,l=1}^{3} a_{ijkl}(t,x)\partial_j\partial_l\partial_t^{h-2}u^k$$
$$= \sum_{j,k,l=1}^{3}\sum_{n=0}^{h-3}\binom{h-2}{n}\left(\partial_t^{h-2-n}a_{ijkl}\right)\left(\partial_t^n\partial_j\partial_l u^k\right) + \partial_t^{h-2}b^i, \quad i = 1,2,3, \quad (3.27)$$

namely,

$$\frac{d^{h-2}}{dt^{h-2}}\left(\partial_t^2 u^i - \sum_{j,k,l=1}^{3} a_{ijkl}(t,x)\partial_j\partial_l u^k\right) = \partial_t^{h-2}b^i, \quad i = 1,2,3. \quad (3.28)$$

Taking account of the definition of ϕ_p, we see

$$\frac{d^p}{dt^p}\left(\partial_t^2 u^i - \sum_{j,k,l=1}^{3} a_{ijkl}(t,x)\partial_j\partial_l u^k\right)\Bigg|_{t=0} = \partial_t^p b^i(0,x), \quad i = 1,2,3, \quad p = 0,1,2,\ldots,h-2. \tag{3.29}$$

Therefore u is the solution of problem (1.1) and satisfies (3.19).

We now prove (3.18) by induction. When $h = 2$, (3.18) follows from (3.2). For $h > 2$, suppose that (3.18) holds for $h-1$. We show that it still holds for h.

Applying (3.2) to (3.27), we conclude from the inductive hypothesis that

$$\left\|\partial_t^{h-2}u\right\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^{h-1}u\right\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^{h}u\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le \text{the right-hand side of (3.19)}. \tag{3.30}$$

In a similar way, we can obtain

$$\sup_{|\alpha|\le h-2}\left(\left\|\partial_t^{\alpha}u\right\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^{\alpha+1}u\right\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^{\alpha+2}u\right\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\right) \le \text{the right-hand side of (3.19)}. \tag{3.31}$$

Set $U(t) = \{u, \partial_t u\}$, then $U(t)$ is the solution of (1.3) and

$$U(t) \in C^{h-2}\left([0,T]; H^2\left(\mathbb{R}^3\setminus\mathcal{K}\right)\times H^1\left(\mathbb{R}^3\setminus\mathcal{K}\right)\right). \tag{3.32}$$

Now

$$(\lambda_0 I - A(t))U(t) = \lambda_0 U(t) - U'(t) + B(t) \in C\left([0,T]; H^2\left(\mathbb{R}^3\setminus\mathcal{K}\right)\times H^1\left(\mathbb{R}^3\setminus\mathcal{K}\right)\right), \tag{3.33}$$

then by (2.17) (taking $p = 3$), we see

$$\begin{aligned}
&U(t) \in C\left([0,T]; H^3\left(\mathbb{R}^3\setminus\mathcal{K}\right)\times H^2\left(\mathbb{R}^3\setminus\mathcal{K}\right)\right),\\
&\|u(t,\cdot)\|_{H^3(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(t,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad = \|U(t)\|_3 \le \|(\lambda_0 I - A(t))U(t)\|_2\\
&\quad \le C\left(\|U(t)\|_2 + \left\|U'(t)\right\|_2 + \|B(t)\|_2\right)\\
&\quad \le C\left(\|u(t,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \|\partial_t u(t,\cdot)\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} + \left\|\partial_t^2 u(t,\cdot)\right\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})} + \|b(t,\cdot)\|_{H^1(\mathbb{R}^3\setminus\mathcal{K})}\right)\\
&\quad \le \text{the right-hand side of (3.19)}.
\end{aligned} \tag{3.34}$$

Differentiation of (3.33) with respect to t gives

$$(\lambda_0 I - A(t))U'(t) = \lambda_0 U'(t) - U'(t) + B'(t) - U''(t) + A'(t)U(t), \tag{3.35}$$

and by the above result $A'(t)U(t) \in C([0,T]; H^2(\mathbb{R}^3 \setminus \mathcal{K}) \times H^1(\mathbb{R}^3 \setminus \mathcal{K}))$,

$$\text{the right-hand side of (3.36)} \in C\Big([0,T]; H^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^1\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big), \tag{3.36}$$

from which it follows that

$$\begin{gathered} U'(t) \in C\Big([0,T]; H^3\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big), \\ \|\partial_t u(t,\cdot)\|_{H^3(\mathbb{R}^3\setminus\mathcal{K})} + \Big\|\partial_t^2 u(t,\cdot)\Big\|_{H^2(\mathbb{R}^3\setminus\mathcal{K})} = \|U'(t)\|_3 \le \text{the right-hand side of (3.19)}. \end{gathered} \tag{3.37}$$

Repeating this process, we get

$$\begin{gathered} U(t) \in C^{h-3}\Big([0,T]; H^3\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big), \\ \sup_{|\alpha| \le h-3} \|\partial_t^\alpha u(t,\cdot)\|_{H^3(\mathbb{R}^3\setminus\mathcal{K})} \le \text{the right-hand side of (3.19)}. \end{gathered} \tag{3.38}$$

Using this, we see the right-hand side of (3.33) $\in C([0,T]; H^3(\mathbb{R}^3 \setminus \mathcal{K}) \times H^2(\mathbb{R}^3 \setminus \mathcal{K}))$, then by (2.17) (taking $p = 4$)

$$U(t) \in C\Big([0,T]; H^4\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^3\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big). \tag{3.39}$$

This assures that the right-hand side of (3.35) $\in C([0,T]; H^3(\mathbb{R}^3 \setminus \mathcal{K}) \times H^2(\mathbb{R}^3 \setminus \mathcal{K}))$, then

$$\begin{gathered} U'(t) \in C\Big([0,T]; H^4\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^3\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big), \\ \|\partial_t u(t,\cdot)\|_{H^4(\mathbb{R}^3\setminus\mathcal{K})} \le \text{the right} - \text{hand side of (3.19)}. \end{gathered} \tag{3.40}$$

Repeating this process, we get

$$\begin{gathered} U(t) \in C^{h-4}\Big([0,T]; H^4\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^3\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big), \\ \sup_{|\alpha| \le h-4} \|\partial_t^\alpha u(t,\cdot)\|_{H^4(\mathbb{R}^3\setminus\mathcal{K})} \le \text{the right} - \text{hand side of (3.19)}. \end{gathered} \tag{3.41}$$

Step by step, finally, we get

$$\begin{aligned} U(t) \in C\Big([0,T]; H^h\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^{h-1}\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big) \cap C^1\Big([0,T]; H^{h-1}\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^{h-2}\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big) \\ \cap \cdots \cap C^{h-2}\Big([0,T]; H^2\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big) \times H^1\Big(\mathbb{R}^3 \setminus \mathcal{K}\Big)\Big) \end{aligned} \tag{3.42}$$

and (3.18). □

4. Application to Elastrodynamics

It is well known that the displacement $u = (u^1, u^2, u^3) = u(t, x)$ of an isotropic, homogeneous, hyperelastic material without the action of external force satisfies the following hyperbolic system (cf. [4, 5]):

$$Lu = \partial_t^2 u - c_2^2 \Delta u - \left(c_1^2 - c_2^2\right) \nabla \operatorname{div} u = F(t, x), \tag{4.1}$$

where $F = (F^1, F^2, F^3)$, and c_1, c_2 are given by the Lamé constants λ, μ:

$$c_1^2 = \lambda + 2\mu, \qquad c_2^2 = \mu. \tag{4.2}$$

We assume that $\mu > 0$, $\lambda + \mu > 0$.

From [5], system (4.1) can be written as

$$\partial_t^2 u^i - \sum_{j,k,l=1}^{3} a_{ijkl}(t, x) \partial_j \partial_l u^k = 0, \quad i = 1, 2, 3, \tag{4.3}$$

where $A = (a_{ijkl}(t, x))$ stands for the elastic tensor.

The system (4.3) is the special case of the system (1.1). So by the existence Theorem 3.2, we derive the existence of solutions for the initial-boundary problem to the elastrodynamic system (4.3) outside a domain.

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