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Research Article

Schur-Convexity for a Class of Symmetric Functions and Its Applications

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For $x = (x_1, x_2, \ldots, x_n) \in R_+^n$, the symmetric function $\phi_n(x, r)$ is defined by $\phi_n(x, r) = \phi_n(x_1, x_2, \ldots, x_n; r) = \prod_{1\le i_1<i_2\cdots<i_r\le n}\left(\sum_{j=1}^{r}(x_{i_j}/(1+x_{i_j}))\right)^{1/r}$, where $r = 1, 2, \ldots, n$ and $i_1, i_2, \ldots, i_n$ are positive integers. In this article, the Schur convexity, Schur multiplicative convexity and Schur harmonic convexity of $\phi_n(x, r)$ are discussed. As applications, some inequalities are established by use of the theory of majorization.

1. Introduction

Throughout this paper we use R^n to denote the n-dimensional Euclidean space over the field of real numbers and $R_+^n = \{(x_1, x_2, \ldots, x_n) : x_i > 0,\ i = 1, 2, \ldots, n\}$. In particular, we use R to denote R^1.

For the sake of convenience, we use the following notation system.

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in R_+^n$, and $\alpha > 0$, let

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

$$xy = (x_1y_1, x_2y_2, \ldots, x_ny_n),$$

$$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n),$$

$$x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha),$$

$$\frac{1}{x} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right),$$

$$\log x = (\log x_1, \log x_2, \ldots, \log x_n),$$
$$e^x = (e^{x_1}, e^{x_2}, \ldots, e^{x_n}). \tag{1.1}$$

The notion of Schur convexity was first introduced by Schur in 1923 [1]. It has many important applications in analytic inequalities [2–7], combinatorial optimization [8], isoperimetric problem for polytopes [9], linear regression [10], graphs and matrices [11], gamma and digamma functions [12], reliability and availability [13], and other related fields. The following definition for Schur convex or concave can be found in [1, 3, 7] and the references therein.

Definition 1.1. Let $E \subseteq R^n$ $(n \geq 2)$ be a set, a real-valued function F on E is called a Schur convex function if

$$F(x_1, x_1, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n) \tag{1.2}$$

for each pair of n-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ on E, such that x is majorized by y (in symbols $x \prec y$), that is,

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1,$$
$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{1.3}$$

where $x_{[i]}$ denotes the ith largest component in x. F is called Schur concave if $-F$ is Schur convex.

The notation of multiplicative convexity was first introduced by Montel [14]. The Schur multiplicative convexity was investigated by Niculescu [15], Guan [7], and Chu et al. [16].

Definition 1.2 (see [7, 16]). Let $E \subseteq R_+^n$ $(n \geq 2)$ be a set, a real-valued function $F : E \to R_+$ is called a Schur multiplicatively convex function on E if

$$F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n) \tag{1.4}$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ on E, such that x is logarithmically majorized by y (in symbols $\log x \prec \log y$), that is,

$$\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1,$$
$$\prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}. \tag{1.5}$$

However F is called Schur multiplicatively concave if $1/F$ is Schur multiplicatively convex.

In paper [17], Anderson et al. discussed an attractive class of inequalities, which arise from the notion of harmonic convex functions. Here, we introduce the notion of Schur harmonic convexity.

Definition 1.3. Let $E \subseteq R_+^n$ $(n \geq 2)$ be a set. A real-valued function F on E is called a Schur harmonic convex function if

$$F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n) \tag{1.6}$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ on E, such that $1/x \prec 1/y$. F is called a Schur harmonic concave function on E if (1.6) is reversed.

The main purpose of this paper is to discuss the Schur convexity, Schur multiplicative convexity, and Schur harmonic convexity of the following symmetric function:

$$\phi_n(x, r) = \phi_n(x_1, x_2, \ldots, x_n; r) = \prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^{r} \frac{x_{i_j}}{1 + x_{i_j}} \right)^{1/r}, \tag{1.7}$$

where $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ $(n \geq 2)$, $r = 1, 2, \ldots, n$, and $i_1, i_2, \ldots, i_r$ are positive integers. As applications, some inequalities are established by use of the theory of majorization.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

The following lemma is so-called Schur's condition which is very useful for determining whether or not a given function is Schur convex or Schur concave.

Lemma 2.1 (see [6, 7, 18]). *Let $f : R_+^n = (0, \infty)^n \to R_+ = (0, \infty)$ be a continuous symmetric function. If f is differentiable in R_+^n, then f is Schur convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right) \geq 0 \tag{2.1}$$

for all $i, j = 1, 2, \ldots, n$ and $x = (x_1, \ldots, x_n) \in R_+^n$. Also f is Schur concave if and only if (2.1) *is reversed for all $i, j = 1, 2, \ldots, n$ and $x = (x_1, \ldots, x_n) \in R_+^n$. Here, f is a symmetric function in R_+^n meaning that $f(Px) = f(x)$ for any $x \in R_+^n$ and any $n \times n$ permutation matrix P.*

Remark 2.2. Since f is symmetric, the Schur's condition in Lemma 2.1, that is, (2.1) can be reduced to

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) \geq 0. \tag{2.2}$$

Lemma 2.3 (see [7, 16]). *Let $f : R_+^n \to R_+$ be a continuous symmtric function. If f is differentiable in R_+^n, then f is Schur multiplicatively convex if and only if*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.3}$$

for all $x = (x_1, x_2, \ldots, x_n) \in R_+^n$. Also f is Schur multiplicatively concave if and only if (2.3) *is reversed.*

Lemma 2.4. *Let $f : R_+^n \to R_+$ be a continuous symmetric function. If f is differentiable in R_+^n, then f is Schur harmonic convex if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial f}{\partial x_1} - x_2^2 \frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.4}$$

for all $x = (x_1, x_2, \ldots, x_n) \in R_+^n$. Also f is Schur harmonic concave if and only if (2.4) *is reversed.*

Proof. From Definitions 1.1 and 1.3, we clearly see the fact that $f : R_+^n \to R_+$ is Schur harmonic convex if and only if $F(x) = 1/f(1/x) : R_+^n \to R_+$ is Schur concave.

This fact, Lemma 2.1 and Remark 2.2 together with elementary calculation imply that Lemma 2.4 is true. □

Lemma 2.5 (see [5, 6]). *Let $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq s$, then*

$$\frac{c-x}{nc/s-1} = \left(\frac{c-x_1}{nc/s-1}, \frac{c-x_2}{nc/s-1}, \ldots, \frac{c-x_n}{nc/s-1}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{2.5}$$

Lemma 2.6 (see [6]). *Let $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq 0$, then*

$$\frac{c+x}{nc/s+1} = \left(\frac{c+x_1}{nc/s+1}, \frac{c+x_2}{nc/s+1}, \ldots, \frac{c+x_n}{nc/s+1}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{2.6}$$

Lemma 2.7 (see [19]). *Suppose that $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $\sum_{i=1}^n x_i = s$. If $0 \leq \lambda \leq 1$, then*

$$\frac{s-\lambda x}{n-\lambda} = \left(\frac{s-\lambda x_1}{n-\lambda}, \frac{s-\lambda x_2}{n-\lambda}, \ldots, \frac{s-\lambda x_n}{n-\lambda}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{2.7}$$

3. Main Results

Theorem 3.1. *For $r \in \{1, 2, \ldots, n\}$, the symmetric function $\phi_n(x, r)$ is Schur concave in R_+^n.*

Proof. By Lemma 2.1 and Remark 2.2, we only need to prove that

$$(x_1 - x_2)\left(\frac{\partial \phi_n(x,r)}{\partial x_1} - \frac{\partial \phi_n(x,r)}{\partial x_2}\right) \leq 0 \tag{3.1}$$

for all $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $r = 1, 2, \ldots, n$.

The proof is divided into four cases.

Case 1. If $r = 1$, then (1.7) leads to

$$\phi_n(x,1) = \phi_n(x_1, x_2, \ldots, x_n; 1) = \prod_{i=1}^{n} \frac{x_i}{1+x_i}. \tag{3.2}$$

However (3.2) and elementary computation lead to

$$(x_1 - x_2)\left(\frac{\partial \phi_n(x,1)}{\partial x_1} - \frac{\partial \phi_n(x,1)}{\partial x_2}\right) = -\frac{(x_1-x_2)^2(1+x_1+x_2)}{x_1x_2(1+x_1)(1+x_2)}\phi_n(x,1) \le 0. \tag{3.3}$$

Case 2. If $n \ge 2$ and $r = n$, then (1.7) yields

$$\phi_n(x,n) = \phi_n(x_1, x_2, \ldots, x_n; n) = \left(\sum_{i=1}^{n} \frac{x_i}{1+x_i}\right)^{1/n}. \tag{3.4}$$

From (3.4) and elementary computation, we have

$$(x_1 - x_2)\left(\frac{\partial \phi_n(x,n)}{\partial x_1} - \frac{\partial \phi_n(x,n)}{\partial x_2}\right) = -\frac{(x_1-x_2)^2(2+x_1+x_2)}{n(1+x_1)^2(1+x_2)^2}\left(\sum_{i=1}^{n} \frac{x_i}{1+x_i}\right)^{1/n-1} \le 0. \tag{3.5}$$

Case 3. If $n \ge 3$ and $r = 2$, then by (1.7) we have

$$\begin{aligned}
\phi_n(x,2) &= \phi_n(x_1, x_2, \cdots, x_n; 2) \\
&= \left(\frac{x_1}{1+x_1} + \frac{x_2}{1+x_2}\right)^{1/2}\left[\prod_{j=3}^{n}\left(\frac{x_1}{1+x_1} + \frac{x_j}{1+x_j}\right)^{1/2}\right]\phi_{n-1}(x_2, x_3, \ldots, x_n; 2) \\
&= \left(\frac{x_2}{1+x_2} + \frac{x_1}{1+x_1}\right)^{1/2}\left[\prod_{j=3}^{n}\left(\frac{x_2}{1+x_2} + \frac{x_j}{1+x_j}\right)^{1/2}\right]\phi_{n-1}(x_1, x_3, \ldots, x_n; 2).
\end{aligned} \tag{3.6}$$

Elementary computation and (3.6) yield

$$
\begin{aligned}
&(x_1 - x_2)\left(\frac{\partial\phi_n(x,2)}{\partial x_1} - \frac{\partial\phi_n(x,2)}{\partial x_2}\right)\\
&\quad = -\frac{(x_1 - x_2)^2}{(1+x_1)(1+x_2)}\frac{\phi_n(x,2)}{2}\\
&\qquad \times \left[\frac{2+x_1+x_2}{x_1+x_2+2x_1x_2} + \sum_{j=3}^{n}\frac{\left[(1+x_1+x_2)+(3+2x_1+2x_2)x_j\right](1+x_j)}{(x_1+x_j+2x_1x_j)(x_2+x_j+2x_2x_j)}\right] \le 0.
\end{aligned}
\tag{3.7}
$$

Case 4. If $n \ge 4$ and $3 \le r \le n-1$, then from (1.7), we have

$$
\begin{aligned}
\phi_n(x,r) &= \phi_n(x_1, x_2, \ldots, x_n; r)\\
&= \phi_{n-1}(x_2, x_3, \ldots, x_n; r)\prod_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\left(\frac{x_1}{1+x_1} + \sum_{j=1}^{r-1}\frac{x_{i_j}}{1+x_{i_j}}\right)^{1/r}\\
&\quad \times \prod_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\left(\frac{x_1}{1+x_1} + \frac{x_2}{1+x_2} + \sum_{j=1}^{r-2}\frac{x_{i_j}}{1+x_{i_j}}\right)^{1/r}\\
&= \phi_{n-1}(x_1, x_3, \ldots, x_n; r)\prod_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\left(\frac{x_2}{1+x_2} + \sum_{j=1}^{r-1}\frac{x_{i_j}}{1+x_{i_j}}\right)^{1/r}\\
&\quad \times \prod_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\left(\frac{x_1}{1+x_1} + \frac{x_2}{1+x_2} + \sum_{j=1}^{r-2}\frac{x_{i_j}}{1+x_{i_j}}\right)^{1/r},
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&(x_1 - x_2)\left(\frac{\partial\phi_n(x,r)}{\partial x_1} - \frac{\partial\phi_n(x,r)}{\partial x_2}\right)\\
&\quad = -\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}\\
&\qquad \times \left[\sum_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\frac{2+x_1+x_2}{(x_1/(1+x_1)) + (x_2/(1+x_2)) + \sum_{j=1}^{r-2}\left(x_{i_j}/\left(1+x_{i_j}\right)\right)}\right.\\
&\qquad\quad \left. + \sum_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\frac{1+x_1+x_2+(2+x_1+x_2)\sum_{j=1}^{r-1}\left(x_{i_j}/1+x_{i_j}\right)}{\left(x_1/(1+x_1) + \sum_{j=1}^{r-1}\left(x_{i_j}/\left(1+x_{i_j}\right)\right)\right)\left(x_2/(1+x_2) + \sum_{j=1}^{r-1}\left(x_{i_j}/\left(1+x_{i_j}\right)\right)\right)}\right]\\
&\qquad \times \frac{\phi_n(x,r)}{r} \le 0.
\end{aligned}
\tag{3.9}
$$

Therefore, (3.1) follows from Cases 1–4 and the proof of Theorem 3.1 is completed. □

For the Schur multiplicative convexity or concavity of $\phi_n(x,r)$, we have the following theorem

Theorem 3.2. *It holds that $\phi_n(x,r)$ is Schur multiplicatively concave in* $[1,\infty)^n$.

Proof. According to Lemma 2.3 we only need to prove that

$$(\log x_1-\log x_2)\left(x_1\frac{\partial\phi_n(x,r)}{\partial x_1}-x_2\frac{\partial\phi_n(x,r)}{\partial x_2}\right)\le 0 \tag{3.10}$$

for all $x=(x_1,x_2,\ldots,n)\in[1,\infty)^n$ and $r=1,2,\ldots,n$. Then proof is divided into four cases.

Case 1. If $r=1$, then (3.2) leads to

$$(\log x_1-\log x_2)\left(x_1\frac{\partial\phi_n(x,1)}{\partial x_1}-x_2\frac{\partial\phi_n(x,1)}{\partial x_2}\right)=-\frac{(\log x_1-\log x_2)(x_1-x_2)}{(1+x_1)(1+x_2)}\phi_n(x,1)\le 0. \tag{3.11}$$

Case 2. If $r=n$, $n\ge 2$, then (3.4) yields

$$\begin{aligned}&(\log x_1-\log x_2)\left(x_1\frac{\partial\phi_n(x,n)}{\partial x_1}-x_2\frac{\partial\phi_n(x,n)}{\partial x_2}\right)\\&\quad=\frac{(\log x_1-\log x_2)(x_1-x_2)}{n(1+x_1)^2(1+x_2)^2}(1-x_1x_2)\left(\sum_{i=1}^{n}\frac{x_i}{1+x_i}\right)^{1/n-1}\le 0.\end{aligned} \tag{3.12}$$

Case 3. If $n\ge 3$ and $r=2$, then (3.6) implies

$$\begin{aligned}&(\log x_1-\log x_2)\left(x_1\frac{\partial\phi_n(x,2)}{\partial x_1}-x_2\frac{\partial\phi_n(x,2)}{\partial x_2}\right)\\&\quad=\frac{\phi_n(x,2)}{2}\frac{(\log x_1-\log x_2)(x_1-x_2)}{(1+x_1)^2(1+x_2)^2}\\&\qquad\times\left[\frac{1-x_1x_2}{x_1/(1+x_1)+x_2/(1+x_2)}\right.\\&\qquad\qquad\left.+\sum_{j=3}^{n}\frac{-x_1x_2+(1-x_1x_2)(x_j/(1+x_j))}{(x_1/(1+x_1)+x_j/(1+x_j))(x_2/(1+x_2)+x_j/(1+x_j))}\right]\le 0.\end{aligned} \tag{3.13}$$

Case 4. If $n \geq 4$ and $3 \leq r \leq n-1$, then from (3.8) we have

$$\begin{aligned}
&(\log x_1 - \log x_2)\left(x_1 \frac{\partial \phi_n(x,r)}{\partial x_1} - x_2 \frac{\partial \phi_n(x,r)}{\partial x_2}\right) \\
&\quad = \frac{(\log x_1 - \log x_2)(x_1 - x_2)}{(1+x_1)^2(1+x_2)^2} \\
&\qquad \times \Bigg[\sum_{3 \leq i_1 < i_2 < \cdots < i_{r-2} \leq n} \frac{1 - x_1 x_2}{x_1/(1+x_1) + x_2/(1+x_2) + \sum_{j=1}^{r-2}\left(x_{i_j}/\left(1+x_{i_j}\right)\right)} \\
&\qquad + \sum_{3 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n} \frac{-x_1 x_2 + (1 - x_1 x_2)\sum_{j=1}^{r-1}\left(x_{i_j}/\left(1+x_{i_j}\right)\right)}{\left(x_1/(1+x_1) + \sum_{j=1}^{r-1}\left(x_{i_j}/\left(1+x_{i_j}\right)\right)\right)\left(x_2/(1+x_2) + \sum_{j=1}^{r-1}\left(x_{i_j}/\left(1+x_{i_j}\right)\right)\right)} \Bigg] \\
&\qquad \times \frac{\phi_n(x,r)}{r} \leq 0.
\end{aligned} \tag{3.14}$$

Therefore, Theorem 3.2 follows from (3.10) and Cases 1–4. □

Remark 3.3. From (3.11) and (3.12) we know that $\phi_n(x,1)$ is Schur multiplicatively concave in $(0,\infty)^n$ and $\phi_n(x,n)$ is Schur multiplicatively convex in $(0,1]^n$.

Theorem 3.4. *For $r \in \{1,2,\ldots,n\}$, the symmetric function $\phi_n(x,r)$ is Schur harmonic convex in R_+^n.*

Proof. According to Lemma 2.4 we only need to prove that

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \phi_n(x,r)}{\partial x_1} - x_2^2 \frac{\partial \phi_n(x,r)}{\partial x_2}\right) \geq 0 \tag{3.15}$$

for all $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $r = 1,2,\ldots,n$.

The proof is divided into four cases.

Case 1. If $r = 1$, then from (3.2) we have

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \phi_n(x,1)}{\partial x_1} - x_2^2 \frac{\partial \phi_n(x,1)}{\partial x_2}\right) = \frac{(x_1 - x_2)^2}{(1+x_1)(1+x_2)}\phi_n(x,1) \geq 0. \tag{3.16}$$

Case 2. If $n \geq 2$ and $r = n$, then (3.4) leads to

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \phi_n(x,n)}{\partial x_1} - x_2^2 \frac{\partial \phi_n(x,n)}{\partial x_2}\right) = \frac{(x_1 - x_2)^2(x_1 + x_2 + 2x_1 x_2)}{n(1+x_1)^2(1+x_2)^2}\left(\sum_{i=1}^{n} \frac{x_i}{1+x_i}\right)^{1/n-1} \geq 0. \tag{3.17}$$

Case 3. If $n \geq 3$ and $r = 2$, then (3.6) yields

$$
\begin{aligned}
&(x_1 - x_2)\left(x_1^2 \frac{\partial \phi_n(x,2)}{\partial x_1} - x_2^2 \frac{\partial \phi_n(x,2)}{\partial x_2}\right) \\
&\quad = \frac{(x_1 - x_2)^2}{(1 + x_1)(1 + x_2)} \frac{\phi_n(x,2)}{2} \\
&\qquad \times \left[1 + \sum_{j=3}^{n} \frac{(x_1 x_2 + x_1 x_j + x_2 x_j + 3x_1 x_2 x_j)(1 + x_j)}{(x_1 + x_j + 2x_1 x_j)(x_2 + x_j + 2x_2 x_j)}\right] \\
&\quad \geq 0.
\end{aligned}
\tag{3.18}
$$

Case 4. If $n \geq 4$ and $3 \leq r \leq n - 1$, then (3.8) implies

$$
\begin{aligned}
&(x_1 - x_2)\left(x_1^2 \frac{\partial \phi_n(x,r)}{\partial x_1} - x_2^2 \frac{\partial \phi_n(x,r)}{\partial x_2}\right) \\
&= \frac{(x_1 - x_2)^2}{(1 + x_1)^2(1 + x_2)^2} \frac{\phi_n(x,r)}{r} \\
&\quad \times \left[\sum_{3 \leq i_1 < i_2 < \cdots < i_{r-2} \leq n} \frac{x_1 + x_2 + 2x_1 x_2}{x_1/(1 + x_1) + x_2/(1 + x_2) + \sum_{j=1}^{r-2}\left(x_{i_j}/\left(1 + x_{i_j}\right)\right)} \right. \\
&\qquad \left. + \sum_{3 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n} \frac{x_1 x_2 + (x_1 + x_2 + 2x_1 x_2) \sum_{j=1}^{r-2}\left(x_{i_j}/\left(1 + x_{i_j}\right)\right)}{\left(x_1/(1 + x_1) + \sum_{j=1}^{r-1}\left(x_{i_j}/\left(1 + x_{i_j}\right)\right)\right)\left(x_2/(1 + x_2) + \sum_{j=1}^{r-1}\left(x_{i_j}/\left(1 + x_{i_j}\right)\right)\right)} \right] \\
&\geq 0.
\end{aligned}
\tag{3.19}
$$

Therefore, (3.15) follows from Cases 1–4 and the proof of Theorem 3.4 is completed. □

4. Applications

In this section, we establish some inequalities by use of Theorems 3.1, 3.2 and 3.4 and the theory of majorization.

Theorem 4.1. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, $s = \sum_{i=1}^{n} x_i, H_n(x) = n/\sum_{i=1}^{n}(1/x_i)$, *and* $r \in \{1, 2, \ldots, n\}$, *then*

(1) $\phi_n(x,r) \leq \phi_n((c - x)/(nc/s - 1), r)$ *for* $c \geq s$;

(2) $\phi_n(x,r) \geq \phi_n((cH_n(x) - 1)/(cx - 1)x, r)$ *for* $c \geq \sum_{i=1}^{n}(1/x_i)$;

(3) $\phi_n(x,r) \leq \phi_n((c + x)/(nc/s + 1), r)$ *for* $c \geq 0$;

(4) $\phi_n(x,r) \geq \phi_n((cH_n(x) + 1)/(cx + 1)x, r)$ *for* $c \geq 0$;

(5) $\phi_n(x,r) \leq \phi_n((s - \lambda x)/(n - \lambda), r)$ *for* $0 \leq \lambda \leq 1$;

(6) $\phi_n(x,r) \geq \phi_n((n-\lambda)/\sum_{i=1}^{n}(1/x_i - \lambda/x),r)$ *for* $0 \leq \lambda \leq 1$;

(7) $\phi_n(x,r) \leq \phi_n((s+\lambda x)/(n+\lambda),r)$ *for* $0 \leq \lambda \leq 1$;

(8) $\phi_n(x,r) \geq \phi_n((n+\lambda)/(\sum_{i=1}^{n}(1/x_i + \lambda/x)),r)$ *for* $0 \leq \lambda \leq 1$.

Proof. Theorem 4.1 follows from Theorem 3.1, Theorem 3.4 and Lemmas 2.5–2.7 together with the fact that

$$\frac{s+\lambda x}{n+\lambda} = \left(\frac{s+\lambda x_1}{n+\lambda}, \frac{s+\lambda x_2}{n+\lambda}, \ldots, \frac{s+\lambda x_n}{n+\lambda}\right) \prec (x_1, x_2, \ldots, x_n) = x. \tag{4.1}$$

□

Theorem 4.2. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, $A_n(x) = (1/n)\sum_{i=1}^{n} x_i$, *and* $r \in \{1, 2, \ldots, n\}$, *then*

$$\text{(i)} \quad \prod_{1\leq i_1<i_2<\cdots<i_r\leq n}\left(\sum_{j=1}^{r}\frac{x_{i_j}}{1+x_{i_j}}\right)^{1/r} \leq \left[r\frac{A_n(x)}{A_n(1+x)}\right]^{n!/(r\cdot r!(n-r)!)};$$
$$\text{(ii)} \quad \prod_{1\leq i_1<i_2<\cdots<i_r\leq n}\left(\sum_{j=1}^{r}\frac{1}{1+x_{i_j}}\right)^{1/r} \geq \left[r\frac{1}{A_n(1+x)}\right]^{n!/(r\cdot r!(n-r)!)}. \tag{4.2}$$

Proof. We clearly see that

$$(A_n(x), A_n(x), \ldots, A_n(x)) \prec (x_1, x_2, \ldots, x_n) = x. \tag{4.3}$$

Therefore, Theorem 4.2(i) follows from (4.3) and Theorem 3.1 together with (1.7), and Theorem 4.2(ii) follows from (4.3) and Theorem 3.4 together with (1.7). □

If we take $r = 1$ in Theorem 4.2(i) and $r = n$ in Theorem 4.2, respectively, then we have the following corollary.

Corollary 4.3. *If* $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ *and* $G_n(x) = (\prod_{i=1}^{n} x_i)^{1/n}$, *then*

$$\text{(i)} \quad \frac{G_n(x)}{G_n(1+x)} \leq \frac{A_n(x)}{A_n(1+x)};$$
$$\text{(ii)} \quad A_n\left(\frac{x}{1+x}\right) \leq \frac{A_n(x)}{A_n(1+x)};$$
$$\text{(iii)} \quad A_n\left(\frac{1}{1+x}\right) \geq \frac{1}{A_n(1+x)}. \tag{4.4}$$

Remark 4.4. If we take $\sum_{i=1}^{n} x_i = 1$ in Corollary 4.3(i), then we obtain the Weierstrass inequality: (see [20, page 260])

$$\prod_{i=1}^{n}\left(\frac{1}{x_i}+1\right) \geq (n+1)^n. \tag{4.5}$$

Theorem 4.5. *If $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $r \in \{1, 2, \ldots, n\}$, then*

$$\text{(i)} \quad \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\sum_{j=1}^{r} \frac{x_{i_j}}{1 + x_{i_j}} \right)^{1/r} \ge \left[r \frac{H_n(x)}{1 + H_n(x)} \right]^{n!/(r \cdot r!(n-r)!)};$$
$$\text{(ii)} \quad \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\sum_{j=1}^{r} \frac{1}{1 + x_{i_j}} \right)^{1/r} \le \left[r \frac{1}{1 + H_n(x)} \right]^{n!/(r \cdot r!(n-r)!)}. \tag{4.6}$$

Proof. We clearly see that

$$\left(\frac{1}{H_n(x)}, \frac{1}{H_n(x)}, \ldots, \frac{1}{H_n(x)} \right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right) = \frac{1}{x}. \tag{4.7}$$

Therefore, Theorem 4.5(i) follows from (4.7) and Theorem 3.4 together with (1.7), and Theorem 4.5(ii) follows from (4.7) and Theorem 3.1 together with (1.7). □

If we take $r = 1$ and $r = n$ in Theorem 4.5, respectively, then we get the following corollary.

Corollary 4.6. *If $x = (x_1, x_2, \ldots, x_n) \in R_+^n$, then*

$$\text{(i)} \quad \frac{G_n(x)}{G_n(1+x)} \ge \frac{H_n(x)}{1 + H_n(x)};$$
$$\text{(ii)} \quad G_n(1+x) \ge 1 + H_n(x);$$
$$\text{(iii)} \quad A_n\left(\frac{x}{1+x} \right) \ge \frac{H_n(x)}{1 + H_n(x)};$$
$$\text{(iv)} \quad A_n\left(\frac{1}{1+x} \right) \le \frac{1}{1 + H_n(x)}. \tag{4.8}$$

Theorem 4.7. *If $x = (x_1, x_2, \ldots, x_n) \in [1, \infty)^n$ and $r \in \{1, 2, \ldots, n\}$, then*

$$\prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left(\sum_{j=1}^{r} \frac{x_{i_j}}{1 + x_{i_j}} \right)^{1/r} \le \left[r \frac{G_n(x)}{1 + G_n(x)} \right]^{n!/(r \cdot r!(n-r)!)}. \tag{4.9}$$

Proof. We clearly see that

$$\log(G_n(x), G_n(x), \ldots, G_n(x)) \prec \log(x_1, x_2, \ldots, x_n). \tag{4.10}$$

Therefore, Theorem 4.7 follows from (4.10), Theorem 3.2, and (1.7). □

If we take $r = 1$ and $r = n$ in Theorem 4.7, respectively, then we get the following corollary.

Corollary 4.8. *If* $x = (x_1, x_2, \ldots, x_n) \in [1, \infty)^n$, *then*

$$\text{(i)}\quad A_n\left(\frac{x}{1+x}\right) \leq \frac{G_n(x)}{1+G_n(x)};$$
$$\text{(ii)}\quad G_n(1+x) \geq 1 + G_n(x). \tag{4.11}$$

Remark 4.9. From Remark 3.3 and (4.10) together with (1.7) we clearly see that inequality in Corollary 4.8(i) is reversed for $x \in (0,1]^n$ and inequality in Corollary 4.8(ii) is true for $x \in R_+^n$.

Theorem 4.10. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, *then*

$$\text{(i)}\quad \prod_{1\leq i_1<i_2\cdots<i_r\leq n}\left(\sum_{j=1}^{r}\frac{1+x_{i_j}}{2+x_{i_j}}\right)^{1/r} \geq \left[\frac{1+\sum_{i=1}^{n}x_i}{2+\sum_{i=1}^{n}x_i}+\frac{r-1}{2}\right]^{(n-1)!/r!(n-r)!} \times \left(\frac{r}{2}\right)^{(n-1)!/r\cdot r!(n-r-1)!}$$
$$\textit{for } 1 \leq r \leq n-1;$$

$$\text{(ii)}\quad \sum_{i=1}^{n}\frac{1+x_i}{2+x_i} \geq \frac{1+\sum_{i=1}^{n}x_i}{2+\sum_{i=1}^{n}x_i}+\frac{n-1}{2};$$

$$\text{(iii)}\quad \prod_{1\leq i_1<i_2\cdots<i_r\leq n}\left(\sum_{j=1}^{r}\frac{1}{2+x_{i_j}}\right)^{1/r} \leq \left(\frac{1}{2+\sum_{i=1}^{n}x_i}+\frac{r-1}{2}\right)^{(n-1)!/(r-1)!(n-r)!} \times \left(\frac{r}{2}\right)^{(n-1)!/r\cdot r!(n-r-1)!}$$
$$\textit{for } 1 \leq r \leq n-1;$$

$$\text{(iv)}\quad \sum_{i=1}^{n}\frac{1}{2+x_i} \leq \frac{1}{2+\sum_{i=1}^{n}x_i}+\frac{n-1}{2}. \tag{4.12}$$

Proof. Theorem 4.10 follows from Theorems 3.1, 3.4, and (1.7) together with the fact that

$$(1+x_1, 1+x_2, \ldots, 1+x_n) \prec \left(1+\sum_{i=1}^{n}x_i, 1, 1, \ldots, 1\right). \tag{4.13}$$

□

Theorem 4.11. *Let* $\mathcal{A} = A_1A_2\cdots A_{n+1}$ *be an* n*-dimensional simplex in* R^n *and let* P *be an arbitrary point in the interior of* $\mathcal{A}$. *If* B_i *is the intersection point of straight line* A_iP *and hyperplane* $\sum_i = A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}, i = 1, 2, \ldots, n+1$, *then for* $r \in \{1, 2, \ldots, n+1\}$ *one has*

$$\text{(i)}\quad \prod_{1\leq i_1<i_2\cdots<i_r\leq n+1}\left(\sum_{j=1}^{r}\frac{PB_{i_j}}{A_{i_j}B_{i_j}+PB_{i_j}}\right)^{1/r} \leq \left(\frac{r}{n+2}\right)^{(n+1)!/r\cdot r!(n-r+1)!};$$

$$\text{(ii)} \quad \prod_{1\le i_1<i_2\cdots<i_r\le n+1}\left(\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{A_{i_j}B_{i_j}+PB_{i_j}}\right)^{1/r} \ge \left[r\left(\frac{n+1}{n+2}\right)\right]^{(n+1)!/r\cdot r!(n-r+1)!};$$

$$\text{(iii)} \quad \prod_{1\le i_1<i_2\cdots<i_r\le n+1}\left(\sum_{j=1}^{r}\frac{PA_{i_j}}{A_{i_j}B_{i_j}+PA_{i_j}}\right)^{1/r} \le \left[r\left(\frac{n}{2n+1}\right)\right]^{(n+1)!/r\cdot r!(n-r+1)!};$$

$$\text{(iv)} \quad \prod_{1\le i_1<i_2\cdots<i_r\le n+1}\left(\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{A_{i_j}B_{i_j}+PA_{i_j}}\right)^{1/r} \ge \left[r\left(\frac{n+1}{2n+1}\right)\right]^{(n+1)!/r\cdot r!(n-r+1)!}. \tag{4.14}$$

Proof. It is easy to see that $\sum_{i=1}^{n+1}(PB_i/A_iB_i) = 1$ and $\sum_{i=1}^{n+1}(PA_i/A_iB_i) = n$, these identical equations imply

$$\begin{gathered}\left(\frac{1}{n+1},\frac{1}{n+1},\dots,\frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2},\dots,\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right),\\ \left(\frac{n}{n+1},\frac{n}{n+1},\dots,\frac{n}{n+1}\right) \prec \left(\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2},\dots,\frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right).\end{gathered} \tag{4.15}$$

Therefore, Theorem 4.11 follows from (4.15), Theorems 3.1, 3.4, and (1.7). □

Remark 4.12. Mitrinovic' et al. [21, pages 473–479] established a series of inequalities for PA_i/A_iB_i and PB_i/A_iB_i, $i = 1,2,\dots,n+1$. Obvious, our inequalities in Theorem 4.11 are different from theirs.

Theorem 4.13. *Suppose that $A \in M_n(C)$ $(n \ge 2)$ is a complex matrix, $\lambda_1,\lambda_2,\dots,\lambda_n$, and $\sigma_1,\sigma_2,\dots,\sigma_n$ are the eigenvalues and singular values of A, respectively. If A is a positive definite Hermitian matrix and $r \in \{1,2,\dots,n\}$, then*

$$\text{(i)} \quad \prod_{1\le i_1<i_2\cdots<i_r\le n}\left(\sum_{j=1}^{r}\frac{\lambda_{i_j}}{1+\lambda_{i_j}}\right)^{1/r} \le \left[r\left(\frac{\mathrm{tr}A}{n+\mathrm{tr}A}\right)\right]^{n!/r\cdot r!(n-r)!};$$

$$\text{(ii)} \quad \prod_{1\le i_1<i_2\cdots<i_r\le n}\left(\sum_{j=1}^{r}\frac{1}{1+\lambda_{i_j}}\right)^{1/r} \ge \left[r\left(\frac{n}{n+\mathrm{tr}A}\right)\right]^{n!/r\cdot r!(n-r)!};$$

$$\text{(iii)} \quad \prod_{1\le i_1<i_2\cdots<i_r\le n}\left(\sum_{j=1}^{r}\frac{1+\lambda_{i_j}}{2+\lambda_{i_j}}\right)^{1/r} \le \left[r\left(\frac{\sqrt[n]{\det(I+A)}}{1+\sqrt[n]{\det(I+A)}}\right)\right]^{n!/r\cdot r!(n-r)!};$$

$$\text{(iv)} \quad \prod_{1\le i_1<i_2\cdots<i_r\le n}\left(\sum_{j=1}^{r}\frac{1}{\mathrm{tr}A+\lambda_{i_j}}\right)^{1/r} \le \left[r\left(\frac{1}{\mathrm{tr}A+\sqrt[n]{\det A}}\right)\right]^{n!/r\cdot r!(n-r)!};$$

(v) $$\prod_{1\le i_1<i_2\cdots<i_r\le n}\left(\sum_{j=1}^{r}\frac{1}{\lambda_{i_j}+\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}\right)^{1/r} \ge \prod_{1\le i_1<i_2\cdots<i_r\ge n}\left(\sum_{j=1}^{r}\frac{1}{\sigma_{i_j}+\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}\right)^{1/r}. \tag{4.16}$$

Proof. (i)–(ii) We clearly see that $\lambda_i > 0$ $(i = 1,2,\dots,n)$ and $\sum_{i=1}^{n}\lambda_i = \operatorname{tr}A$, then we have

$$\left(\frac{\operatorname{tr}A}{n},\frac{\operatorname{tr}A}{n},\dots,\frac{\operatorname{tr}A}{n}\right) \prec (\lambda_1,\lambda_2,\dots,\lambda_n). \tag{4.17}$$

Therefore, Theorem 4.13(i) and (ii) follows from (4.17), Theorems 3.1, 3.4, and (1.7).

(iii) It is easy to see that $1+\lambda_1, 1+\lambda_2,\dots,1+\lambda_n$ are the eigenvalues of matrix $I+A$ and $\prod_{i=1}^{n}(1+\lambda_i) = \det(I+A)$, then we get

$$\begin{gathered}\log\left(\sqrt[n]{\det(I+A)},\sqrt[n]{\det(I+A)},\dots,\sqrt[n]{\det(I+A)}\right) \prec \log(1+\lambda_1,1+\lambda_2,\dots,1+\lambda_n)\\ 1+\lambda_i \ge 1,\quad i=1,2,\dots,n.\end{gathered} \tag{4.18}$$

Therefore, Theorem 4.13(iii) follows from (4.18), Theorem 3.2, and (1.7).

(iv) It is not difficult to verify that

$$\begin{gathered}\log\left(\frac{\operatorname{tr}A}{\sqrt[n]{\det A}},\frac{\operatorname{tr}A}{\sqrt[n]{\det A}},\dots,\frac{\operatorname{tr}A}{\sqrt[n]{\det A}}\right) \prec \log\left(\frac{\operatorname{tr}A}{\lambda_1},\frac{\operatorname{tr}A}{\lambda_2},\dots,\frac{\operatorname{tr}A}{\lambda_n}\right),\\ \frac{\operatorname{tr}A}{\lambda_i} \ge 1,\quad i=1,2,\dots,n.\end{gathered} \tag{4.19}$$

Therefore, Theorem 4.13(iv) follows from (4.19), and Theorem 3.2 together with (1.7).

(v) A result due to Weyl [22] gives

$$\log(\lambda_1,\lambda_2,\dots,\lambda_n) \prec \log(\sigma_1,\sigma_2,\dots,\sigma_n). \tag{4.20}$$

From (4.20), we clearly see that

$$\begin{gathered}\log\left(\frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\lambda_1},\frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\lambda_2},\dots,\frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\lambda_n}\right)\\ \prec \log\left(\frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\sigma_1},\frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\sigma_2},\dots,\frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\sigma_n}\right),\\ \frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\lambda_i},\ \frac{\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\sigma_i}{\sigma_i} \ge 1,\quad i=1,2,\dots,n.\end{gathered} \tag{4.21}$$

Therefore, Theorem 4.13(v) follows from (4.21), Theorem 3.2, and (1.7). □

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