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Research Article

Boundedness of the Maximal, Potential and Singular Operators in the Generalized Morrey Spaces

Vagif S. Guliyev[1, 2]

[1] *Department of Mathematics, Ahi Evran University, Kirsehir, Turkey*
[2] *Institute of Mathematics and Mechanics, Baku, Azerbaijan*

Correspondence should be addressed to Vagif S. Guliyev, vagif@guliyev.com

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We consider generalized Morrey spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ with a general function $\omega(x,r)$ defining the Morrey-type norm. We find the conditions on the pair (ω_1,ω_2) which ensures the boundedness of the maximal operator and Calderón-Zygmund singular integral operators from one generalized Morrey space $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to another $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$, $1 < p < \infty$, and from the space $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to the weak space $W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)$. We also prove a Sobolev-Adams type $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n) \to \mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$-theorem for the potential operators I_α. In all the cases the conditions for the boundedness are given it terms of Zygmund-type integral inequalities on (ω_1,ω_2), which do not assume any assumption on monotonicity of ω_1,ω_2 in r. As applications, we establish the boundedness of some Schrödinger type operators on generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class. As an another application, we prove the boundedness of various operators on generalized Morrey spaces which are estimated by Riesz potentials.

1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,r)$ denote its complement.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The maximal operator M, fractional maximal operator M_α, and the Riesz potential I_α are defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|\,dy,$$

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+(\alpha/n)} \int_{B(x,t)} |f(y)|\,dy, \quad 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n, \tag{1.1}$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

Let T be a singular integral Calderon-Zygmund operator, briefly a Calderon-Zygmund operator, that is, a linear operator bounded from $L_2(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $Tf \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \quad \text{a.e. on } \operatorname{supp} f. \tag{1.2}$$

Here $K(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates; there exist $c_1 > 0$ and $0 < \varepsilon \le 1$ such that

$$|K(x,y)| \le c_1 |x-y|^{-n} \tag{1.3}$$

for all $x, y \in \mathbb{R}^n$, $x \ne y$, and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le c_1 \left(\frac{|x-x'|}{|x-y|} \right)^{\varepsilon} |x-y|^{-n}, \tag{1.4}$$

whenever $2|x-x'| \le |x-y|$. Such operators were introduced in [1].

The operators $M \equiv M_0$, M_α, I_α, and T play an important role in real and harmonic analysis and applications (see, e.g., [2, 3]).

Generalized Morrey spaces of such a kind were studied in [4–20]. In the present work, we study the boundedness of maximal operator M and Calderón-Zygmund singular integral operators T from one generalized Morrey space $\mathcal{M}_{p,\omega_1}$ to another $\mathcal{M}_{p,\omega_2}$, $1 < p < \infty$, and from the space $\mathcal{M}_{1,\omega_1}$ to the weak space $W\mathcal{M}_{1,\omega_2}$. Also we study the boundedness of fractional maximal operator M_α and Riesz potential operators M_α from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{q,\omega_2}$, $1 < p < q < \infty$, and from the space $\mathcal{M}_{1,\omega_1}$ to the weak space $W\mathcal{M}_{1,\omega_2}$, $1 < q < \infty$.

As applications, we establish the boundedness of some Schödinger type operators on generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class. As an another application, we prove the boundedness of various operators on generalized Morrey spaces which are estimated by Riesz potentials.

2. Morrey Spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role; see [21, 22].

Introduced by Morrey [23] in 1938, they are defined by the norm

$$\|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x,r>0} r^{-\lambda/p} \|f\|_{L_p(B(x,r))}, \tag{2.1}$$

where $0 \leq \lambda < n, 1 \leq p < \infty$.

We also denote by $W\mathcal{M}_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in(\mathbb{R}^n), r>0} r^{-\lambda/p} \|f\|_{WL_p(B(x,r))} < \infty, \tag{2.2}$$

where WL_p denotes the weak L_p-space.

Chiarenza and Frasca [24] studied the boundedness of the maximal operator M in these spaces. Their results can be summarized as follows.

Theorem 2.1. *Let $1 \leq p < \infty$ and $0 \leq \lambda < n$. Then for $p > 1$ the operator M is bounded in $\mathcal{M}_{p,\lambda}$ and for $p = 1$ M is bounded from $\mathcal{M}_{1,\lambda}$ to $W\mathcal{M}_{1,\lambda}$.*

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n(1/p - 1/q)$ and for $p = 1 < q < \infty$, I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n(1 - 1/q)$. S. Spanne (published by Peetre [25]) and Adams [26] studied boundedness of the Riesz potential in Morrey spaces. Their results can be summarized as follows.

Theorem 2.2 (Spanne, but published by Peetre [25]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n-\alpha p$. Set $1/p - 1/q = \alpha/n$ and $\lambda/p = \mu/q$. Then there exists a constant $C > 0$ independent of f such*

$$\|I_\alpha f\|_{\mathcal{M}_{q,\mu}} \leq C\|f\|_{\mathcal{M}_{p,\lambda}} \tag{2.3}$$

for every $f \in \mathcal{M}_{p,\lambda}$.

Theorem 2.3 (Adams [26]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, and $1/p - 1/q = \alpha/(n - \lambda)$. Then there exists a constant $C > 0$ independent of f such*

$$\|I_\alpha f\|_{\mathcal{M}_{q,\lambda}} \leq C\|f\|_{\mathcal{M}_{p,\lambda}} \tag{2.4}$$

for every $f \in \mathcal{M}_{p,\lambda}$.

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \leq v_n^{\alpha/n-1} I_\alpha(|f|)(x), \tag{2.5}$$

hence Theorems 2.2 and 2.3 also imply boundedness of the fractional maximal operator M_α, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

The classical result for Calderon-Zygmund operators states that if $1 < p < \infty$ then T is bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, and if $p = 1$ then T is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ (see, e.g., [2]).

Fazio and Ragusa [27] studied the boundedness of the Calderón-Zygmund singular integral operators in Morrey spaces, and their results imply the following statement for Calderón-Zygmund operators T.

Theorem 2.4. *Let $1 \leq p < \infty$, $0 < \lambda < n$. Then for $1 < p < \infty$ Calderón-Zygmund singular integral operator T is bounded in $\mathcal{M}_{p,\lambda}$ and for $p = 1$ T is bounded from $\mathcal{M}_{1,\lambda}$ to $W\mathcal{M}_{1,\lambda}$.*

Note that in the case of the classical Calderón-Zygmund singular integral operators Theorem 2.4 was proved by Peetre [25]. If $\lambda = 0$, the statement of Theorem 2.4 reduces to the aforementioned result for $L_p(\mathbb{R}^n)$.

3. Generalized Morrey Spaces

Everywhere in the sequel the functions $\omega(x,r), \omega_1(x,r)$ and $\omega_2(x,r)$, used in the body of the paper are nonnegative measurable function on $\mathbb{R}^n \times (0,\infty)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 3.1. Let $1 \leq p < \infty$. The generalized Morrey space $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ is defined of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ by the finite norm

$$\|f\|_{\mathcal{M}_{p,\omega}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-n/p}}{\omega(x,r)} \|f\|_{L_p(B(x,r))}. \tag{3.1}$$

According to this definition, we recover the space $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ under the choice $\omega(x,r) = r^{(\lambda-n)/p}$:

$$\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \mathcal{M}_{p,\omega}(\mathbb{R}^n)\big|_{\omega(x,r)=r^{(\lambda-n)/p}}. \tag{3.2}$$

In [4, 5, 17, 18] there were obtained sufficient conditions on weights ω_1 and ω_2 for the boundedness of the singular operator T from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$. In [18] the following condition was imposed on $w(x,r)$:

$$c^{-1}\omega(x,r) \leq \omega(x,t) \leq c\, \omega(x,r), \tag{3.3}$$

whenever $r \leq t \leq 2r$, where $c(\geq 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition

$$\int_r^\infty \omega(x,t)^p \frac{dt}{t} \leq C\, \omega(x,r)^p, \tag{3.4}$$

for the maximal or singular operator and the condition

$$\int_r^\infty t^{\alpha p}\omega(x,t)^p \frac{dt}{t} \leq C\, r^{\alpha p}\omega(x,r)^p \tag{3.5}$$

for potential and fractional maximal operators, where $C(>0)$ does not depend on r and $x \in \mathbb{R}^n$.

Note that integral conditions of type (3.4) after the paper [28] of 1956 are often referred to as Bary-Stechkin or Zygmund-Bary-Stechkin conditions; see also [29]. The classes of almost monotonic functions satisfying such integral conditions were later studied in a number of papers, see [30–32] and references therein, where the characterization of integral inequalities of such a kind was given in terms of certain lower and upper indices known as Matuszewska-Orlicz indices. Note that in the cited papers the integral inequalities were studied as $r \to 0$. Such inequalities are also of interest when they allow to impose different conditions as $r \to 0$ and $r \to \infty$; such a case was dealt with in [33, 34].

In [18] the following statements were proved.

Theorem 3.2 ([18]). *Let $1 \le p < \infty$ and $\omega(x,r)$ satisfy conditions* (3.3)-(3.4). *Then for $p > 1$ the operators M and T are bounded in $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ and for $p = 1$ M and T are bounded from $\mathcal{M}_{1,\omega}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\omega}(\mathbb{R}^n)$.*

Theorem 3.3 ([18]). *Let $1 \le p < \infty, 0 < \alpha < (n/p)$, $1/q = 1/p - \alpha/n$ and $\omega(x,t)$ satisfy conditions* (3.3) *and* (3.5). *Then for $p > 1$ the operators M_α and I_α are bounded from $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega}(\mathbb{R}^n)$ and for $p = 1$ M_α and I_α are bounded from $\mathcal{M}_{1,\omega}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\omega}(\mathbb{R}^n)$.*

4. The Maximal Operator in the Spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$

Theorem 4.1. *Let $1 \le p < \infty$ and $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$. Then for $p > 1$*

$$\|Mf\|_{L_p(B(x,t))} \le Ct^{n/p} \int_t^\infty r^{-n/p-1} \|f\|_{L_p(B(x,r))} dr, \tag{4.1}$$

and for $p = 1$

$$\|Mf\|_{WL_1(B(x,t))} \le Ct^{n} \int_t^\infty r^{-n-1} \|f\|_{L_1(B(x,r))} dr, \tag{4.2}$$

where C does not depend on f, $x \in \mathbb{R}^n$ and $t > 0$.

Proof. Let $1 < p < \infty$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\complement B(x,2t)}(y), \quad t > 0, \tag{4.3}$$

and have

$$\|Mf\|_{L_p(B(x,t))} \le \|Mf_1\|_{L_p(B(x,t))} + \|Mf_2\|_{L_p(B(x,t))}. \tag{4.4}$$

By boundedness of the operator M in $L_p(\mathbb{R}^n)$, $1 < p < \infty$ we obtain

$$\|Mf_1\|_{L_p(B(x,t))} \le \|Mf_1\|_{L_p(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(B(x,2t))}, \tag{4.5}$$

where C does not depend on f. From (4.5) we have

$$\begin{aligned}\|Mf_1\|_{L_p(B(x,t))} &\leq Ct^{n/p}\int_{2t}^{\infty} r^{-n/p-1}\|f\|_{L_p(B(x,r))}dr \\ &\leq Ct^{n/p}\int_{t}^{\infty} r^{-n/p-1}\|f\|_{L_p(B(x,r))}dr\end{aligned} \tag{4.6}$$

easily obtained from the fact that $\|f\|_{L_p(B(x,2t))}$ is nondecreasing in t, so that $\|f\|_{L_p(B(x,2t))}$ on the right-hand side of (4.5) is dominated by the right-hand side of (4.6).

To estimate Mf_2, we first prove the following auxiliary inequality:

$$\int_{\complement B(x,t)} |x-y|^{-n}|f(y)|dy \leq C\int_{t}^{\infty} s^{-(n/p)-1}\|f\|_{L_p(B(x,s))}ds, \quad 0<t<\infty. \tag{4.7}$$

To this end, we choose $\beta > n/p$ and proceed as follows:

$$\begin{aligned}\int_{\complement B(x,t)} |x-y|^{-n}|f(y)|dy &\leq \beta\int_{\complement B(x,t)} |x-y|^{-n+\beta}|f(y)|dy\int_{|x-y|}^{\infty} s^{-\beta-1}ds \\ &= \beta\int_{t}^{\infty} s^{-\beta-1}ds\int_{\{y\in\mathbb{R}^n: t\leq|x-y|\leq s\}} |x-y|^{-n+\beta}|f(y)|dy \\ &\leq C\int_{t}^{\infty} s^{-\beta-1}\|f\|_{L_p(B(x,s))}\left\||x-y|^{-n+\beta}\right\|_{L_{p'}(B(x,s))}ds.\end{aligned} \tag{4.8}$$

For $z \in B(x,t)$ we get

$$\begin{aligned}Mf_2(z) &= \sup_{r>0}|B(z,r)|^{-1}\int_{B(z,r)} |f_2(y)|dy \\ &\leq C\sup_{r\geq 2t}\int_{(\complement B(x,2t))\cap B(z,r)} |y-z|^{-n}|f(y)|dy \\ &\leq C\sup_{r\geq 2t}\int_{(\complement B(x,2t))\cap B(z,r)} |x-y|^{-n}|f(y)|dy \\ &\leq C\int_{\complement B(x,2t)} |x-y|^{-n}|f(y)|dy.\end{aligned} \tag{4.9}$$

Then by (4.7)

$$\begin{aligned}Mf_2(z) &\leq C\int_{2t}^{\infty} s^{-n/p-1}\|f\|_{L_p(B(x,s))}ds \\ &\leq C\int_{t}^{\infty} s^{-n/p-1}\|f\|_{L_p(B(x,s))}ds,\end{aligned} \tag{4.10}$$

where C does not depend on x, r. Thus, the function $Mf_2(z)$, with fixed x and t, is dominated by the expression not depending on z. Then

$$\|Mf_2\|_{L_p(B(x,t))} \leq C\int_t^{\infty} s^{-n/p-1}\|f\|_{L_p(B(x,s))}ds\ \|1\|_{L_p(B(x,t))}. \tag{4.11}$$

Since $\|1\|_{L_p(B(x,t))} = Ct^{n/p}$, we then obtain (4.1) from (4.6) and (4.11).

Let $p = 1$. It is obvious that for any ball $B = B(x, r)$

$$\|Mf\|_{WL_1(B(x,t))} \leq \|Mf_1\|_{WL_1(B(x,t))} + \|Mf_2\|_{WL_1(B(x,t))}. \tag{4.12}$$

By boundedness of the operator M from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ we have

$$\|Mf_1\|_{WL_1(B(x,t))} \leq C\|f\|_{L_1(B(x,2t))}, \tag{4.13}$$

where C does not depend on x, t.

Note that inequality (4.11) also true in the case $p = 1$. Then by (4.11), we get inequality (4.2). □

Theorem 4.2. *Let $1 \leq p < \infty$ and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition*

$$\int_t^{\infty}\omega_1(x,r)\frac{dr}{r} \leq C\ \omega_2(x,t), \tag{4.14}$$

where C does not depend on x and t. Then for $p > 1$ the maximal operator M is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$ and for $p = 1$ M is bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$ and $f \in \mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$. By Theorem 4.1 we obtain

$$\begin{aligned}\|Mf\|_{\mathcal{M}_{p,\omega_2}} &= \sup_{x\in\mathbb{R}^n,\ t>0}\omega_2^{-1}(x,t)t^{-n/p}\|Mf\|_{L_p(B(x,t))}\\ &\leq C\sup_{x\in\mathbb{R}^n,\ t>0}\omega_2^{-1}(x,t)\int_t^{\infty}r^{-n/p-1}\|f\|_{L_p(B(x,r))}dr.\end{aligned} \tag{4.15}$$

Hence

$$\begin{aligned}\|Mf\|_{\mathcal{M}_{p,\omega_2}} &\leq C\|f\|_{\mathcal{M}_{p,\omega_1}}\sup_{x\in\mathbb{R}^n,\ t>0}\frac{1}{\omega_2(x,t)}\int_t^{\infty}\omega_1(x,r)\frac{dr}{r}\\ &\leq C\|f\|_{\mathcal{M}_{p,\omega_1}}\end{aligned} \tag{4.16}$$

by (4.14), which completes the proof for $1 < p < \infty$.

Let $p = 1$ and $f \in \mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$. By Theorem 4.1 we obtain

$$\begin{aligned} \|Mf\|_{W\mathcal{M}_{1,\omega_2}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \omega_2^{-1}(x,t)t^{-n}\|Mf\|_{WL_1(B(x,t))} \\ &\leq C \sup_{x\in\mathbb{R}^n,\, t>0} \omega_2^{-1}(x,t)\int_t^\infty r^{-n-1}\|f\|_{L_1(B(x,r))}dr. \end{aligned} \tag{4.17}$$

Hence

$$\begin{aligned} \|Mf\|_{W\mathcal{M}_{1,\omega_2}} &\leq C\|f\|_{\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\omega_2(x,t)}\int_t^\infty \omega_1(x,r)\frac{dr}{r} \\ &\leq C\|f\|_{\mathcal{M}_{1,\omega_1}} \end{aligned} \tag{4.18}$$

by (4.14), which completes the proof for $p = 1$. □

Remark 4.3. Note that Theorems 4.1 and 4.2 were proved in [4] (see also [5]). Theorem 4.2 do not impose the pointwise doubling conditions (3.3) and (3.4). In the case $\omega_1(x,r) = \omega_2(x,r) = \omega(x,r)$, Theorem 4.2 is containing the results of Theorem 3.2.

5. Riesz Potential Operator in the Spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$

5.1. Spanne Type Result

Theorem 5.1. *Let $1 \leq p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, and $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$. Then for $p > 1$*

$$\|I_\alpha f\|_{L_q(B(x,t))} \leq Ct^{n/q}\int_t^\infty r^{-n/q-1}\|f\|_{L_p(B(x,r))}dr, \tag{5.1}$$

and for $p = 1$

$$\|I_\alpha f\|_{WL_q(B(x,t))} \leq Ct^{n/q}\int_t^\infty r^{-n/q-1}\|f\|_{L_1(B(x,r))}dr, \tag{5.2}$$

where C does not depend on f, $x \in \mathbb{R}^n$ and $t > 0$.

Proof. As in the proof of Theorem 4.1, we represent function f in form (4.3) and have

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x). \tag{5.3}$$

Let $1 < p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$. By boundedness of the operator I_α from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ we obtain

$$\begin{aligned} \|I_\alpha f_1\|_{L_q(B(x,t))} &\leq \|I_\alpha f_1\|_{L_q(\mathbb{R}^n)} \\ &\leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(B(x,2t))}. \end{aligned} \tag{5.4}$$

Then

$$\|I_\alpha f_1\|_{L_q(B(x,t))} \le C\|f\|_{L_p(B(x,2t))}, \tag{5.5}$$

where the constant C is independent of f.

Taking into account that

$$\|f\|_{L_p(B(x,2t))} \le Ct^{n/q}\int_{2t}^{\infty} r^{-n/q-1}\|f\|_{L_p(B(x,r))}dr, \tag{5.6}$$

we get

$$\|I_\alpha f_1\|_{L_q(B(x,t))} \le Ct^{n/q}\int_{2t}^{\infty} r^{-n/q-1}\|f\|_{L_p(B(x,r))}dr. \tag{5.7}$$

When $|x-z| \le t$, $|z-y| \ge 2t$, we have $(1/2)|z-y| \le |x-y| \le (3/2)|z-y|$, and therefore

$$\begin{aligned}\|I_\alpha f_2\|_{L_q(B(x,t))} &\le \left\| \int_{\complement_{B(x,2t)}} |z-y|^{\alpha-n} f(y)dy \right\|_{L_q(B(x,t))} \\ &\le C\int_{\complement_{B(x,2t)}} |x-y|^{\alpha-n}|f(y)|dy \|\chi_{B(x,t)}\|_{L_q(\mathbb{R}^n)}.\end{aligned} \tag{5.8}$$

We choose $\beta > n/q$ and obtain

$$\begin{aligned}\int_{\complement_{B(x,2t)}} |x-y|^{\alpha-n}|f(y)|dy &= \beta\int_{\complement_{B(x,2t)}} |x-y|^{\alpha-n+\beta}|f(y)|\left(\int_{|x-y|}^{\infty} s^{-\beta-1}ds\right)dy \\ &= \beta\int_{2t}^{\infty} s^{-\beta-1}\left(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}} |x-y|^{\alpha-n+\beta}|f(y)|dy\right)ds \\ &\le C\int_{2t}^{\infty} s^{-\beta-1}\|f\|_{L_p(B(x,s))}\left\||x-y|^{\alpha-n+\beta}\right\|_{L_{p'}(B(x,s))}ds \\ &\le C\int_{2t}^{\infty} s^{\alpha-n/p-1}\|f\|_{L_p(B(x,s))}ds.\end{aligned} \tag{5.9}$$

Therefore

$$\|I_\alpha f_2\|_{L_q(B(x,t))} \le Ct^{n/q}\int_{2t}^{\infty} s^{-n/q-1}\|f\|_{L_p(B(x,s))}ds, \tag{5.10}$$

which together with (5.7) yields (5.1).

Let $p = 1$. It is obvious that for any ball $B = B(x,r)$

$$\|I_\alpha f\|_{WL_1(B(x,t))} \le \|I_\alpha f_1\|_{WL_1(B(x,t))} + \|I_\alpha f_2\|_{WL_1(B(x,t))}. \tag{5.11}$$

By boundedness of the operator I_α from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ we have

$$\|I_\alpha f_1\|_{WL_1(B(x,t))} \le C\|f\|_{L_q(B(x,2t))}, \tag{5.12}$$

where C does not depend on x, t.

Note that inequality (5.10) also true in the case $p = 1$. Then by (5.10), we get inequality (5.2). □

Theorem 5.2. *Let $1 \le p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$ and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int_r^\infty t^\alpha \omega_1(x,t)\frac{dt}{t} \le C\,\omega_2(x,r), \tag{5.13}$$

where C does not depend on x and r. Then for $p > 1$ the operators M_α and I_α are bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$ and for $p = 1$ M_α and I_α are bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$ and $f \in \mathcal{M}_{p,\omega}(\mathbb{R}^n)$. By Theorem 5.1 we obtain

$$\begin{aligned}\|I_\alpha f\|_{\mathcal{M}_{q,\omega_2}} &\le C \sup_{x\in\mathbb{R}^n,\, t>0}\frac{1}{\omega_2(x,t)}\int_t^\infty r^{-n/q-1}\|f\|_{L_p(B(x,r))}dr \\ &\le C\|f\|_{\mathcal{M}_{p,\omega_1}}\sup_{x\in\mathbb{R}^n,\, t>0}\frac{1}{\omega_2(x,t)}\int_t^\infty r^\alpha\omega_1(x,r)\frac{dr}{r}\end{aligned} \tag{5.14}$$

by (5.13), which completes the proof for $1 < p < \infty$.

Let $p = 1$ and $f \in \mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$. By Theorem 5.1 we obtain

$$\begin{aligned}\|I_\alpha f\|_{W\mathcal{M}_{q,\omega_2}} &= \sup_{x\in\mathbb{R}^n,\, t>0}\omega_2^{-1}(x,t)t^{-n/q}\|I_\alpha f\|_{WL_q(B(x,t))} \\ &\le C\sup_{x\in\mathbb{R}^n,\, t>0}\omega_2^{-1}(x,t)\int_t^\infty r^{-(n/q)-1}\|f\|_{L_1(B(x,r))}dr.\end{aligned} \tag{5.15}$$

Hence

$$\begin{aligned}\|I_\alpha f\|_{W\mathcal{M}_{q,\omega_2}} &\le C\|f\|_{\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, t>0}\frac{1}{\omega_2(x,t)}\int_t^\infty r^\alpha\omega_1(x,r)\frac{dr}{r} \\ &\le C\|f\|_{\mathcal{M}_{1,\omega_1}}\end{aligned} \tag{5.16}$$

by (5.13), which completes the proof for $p = 1$. □

Remark 5.3. Note that Theorems 5.1 and 5.2 were proved in [4] (see also [5]). Theorem 5.2 do not impose the pointwise doubling condition, (3.3) and (3.5). In the case $\omega_1(x,r) = \omega_2(x,r) = \omega(x,r)$, Theorem 5.2 is containing the results of Theorem 3.3.

5.2. Adams Type Result

Theorem 5.4. *Let $1 \le p < \infty$, $0 < \alpha < n/p$, and $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\left|I_\alpha f(x)\right| \le Ct^\alpha \; Mf(x) + C\int_t^\infty r^{\alpha-n/p-1}\|f\|_{L_p(B(x,r))}dr, \tag{5.17}$$

where C does not depend on f, x, and t.

Proof. As in the proof of Theorem 4.1, we represent function f in form (4.3) and have

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x). \tag{5.18}$$

For $I_\alpha f_1(x)$, following Hedberg's trick (see for instance [2], page 354), we obtain $|I_\alpha f_1(x)| \le C_1 t^\alpha M f(x)$. For $I_\alpha f_2(x)$ we have

$$\begin{aligned}
\left|I_\alpha f_2(x)\right| &\le \int_{\complement B(x,2t)} |x-y|^{\alpha-n}|f(y)|dy \\
&\le C\int_{\complement B(x,2t)} |f(y)|dy\int_{|x-y|}^\infty r^{\alpha-n-1}dr \\
&\le C\int_{2t}^\infty\left(\int_{2t<|x-y|<r}|f(y)|dy\right)r^{\alpha-n-1}dr \\
&\le C\int_t^\infty r^{\alpha-n/p-1}\|f\|_{L_p(B(x,r))}dr,
\end{aligned} \tag{5.19}$$

which proves (5.17). □

Theorem 5.5. *Let $1 \le p < \infty$, $0 < \alpha < n/p$ and let $\omega(x,t)$ satisfy condition* (4.14) *and the conditions*

$$t^\alpha\omega(x,t) + \int_t^\infty r^\alpha \; \omega(x,r)\frac{dr}{r} \le C\omega(x,t)^{p/q}, \tag{5.20}$$

where $q \ge p$ and C does not depend on $x \in \mathbb{R}^n$ and $t > 0$. Suppose also that for almost every $x \in \mathbb{R}^n$, the function $w(x,r)$ fulfills the condition

$$\text{there exist an } a = a(x) > 0 \text{ such that } \omega(x,\cdot) : [0,\infty] \longrightarrow [a,\infty) \text{ is surjective.} \tag{5.21}$$

Then for $p > 1$ the operators M_α and I_α are bounded from $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega^{p/q}}(\mathbb{R}^n)$ and for $p = 1$ the operators M_α and I_α are bounded from $\mathcal{M}_{1,\omega}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\omega^{1/q}}(\mathbb{R}^n)$.

Proof. In view of the well-known pointwise estimate $M_\alpha f(x) \le C(I_\alpha|f|)(x)$, it suffices to treat only the case of the operator I_α.

Let $1 \le p < \infty$ and $f \in \mathcal{M}_{p,\omega}(\mathbb{R}^n)$. By Theorem 5.4 we get

$$\left|I_\alpha f(x)\right| \le Cr^\alpha \; Mf(x) + C\|f\|_{\mathcal{M}_{p,\omega}} \; \int_r^\infty t^\alpha \omega(x,t)\frac{dt}{t}. \tag{5.22}$$

From (5.20) we have $r^\alpha \omega(x,r\;) \le C\omega(x,r)^{p/q}$. Making also use of condition (5.20), we obtain

$$\left|I_\alpha f(x)\right| \le C\omega(x,r)^{p/q-1} \; Mf(x) + C\omega(x,r)^{p/q} \;\; \|f\|_{\mathcal{M}_{p,\omega}}. \tag{5.23}$$

Since $\omega(x,r)$ is surjective, we can choose $r > 0$ so that $\omega(x,r) = Mf(x) \;\; \|f\|^{-1}_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)}$, assuming that f is not identical 0. Hence, for every $x \in \mathbb{R}^n$, we have

$$\left|I_\alpha f(x)\right| \le C(Mf(x))^{p/q} \;\; \|f\|^{1-p/q}_{\mathcal{M}_{p,\omega}}. \tag{5.24}$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator M in $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ provided by Theorem 4.2 in virtue of condition (4.14)

$$\begin{aligned}
\|I_\alpha f\|_{\mathcal{M}_{q,\omega^{p/q}}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \omega(x,t)^{-p/q} t^{-n/q} \|I_\alpha f\|_{L_q(B(x,t))} \\
&\le C \;\; \|f\|^{1-p/q}_{\mathcal{M}_{p,\omega}} \sup_{x\in\mathbb{R}^n,\, t>0} \omega(x,t)^{-p/q} t^{-n/q} \|Mf\|^{p/q}_{L_p(B(x,t))} \\
&\le C \;\; \|f\|_{\mathcal{M}_{p,\omega}},
\end{aligned} \tag{5.25}$$

if $1 < p < q < \infty$ and

$$\begin{aligned}
\|I_\alpha f\|_{W\mathcal{M}_{q,\omega^{1/q}}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \omega(x,t)^{-1/q} t^{-n/q} \|I_\alpha f\|_{WL_q(B(x,t))} \\
&\le C \;\; \|f\|^{1-(1/q)}_{\mathcal{M}_{1,\omega}} \sup_{x\in\mathbb{R}^n,\, t>0} \omega(x,t)^{-1/q} t^{-n/q} \|Mf\|^{1/q}_{WL_1(B(x,t))} \\
&\le C \;\; \|f\|_{\mathcal{M}_{1,\omega}},
\end{aligned} \tag{5.26}$$

if $p = 1 < q < \infty$. □

6. Singular Operators in the Spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$

Theorem 6.1. *Let $1 \le p < \infty$ and $f \in L^{\text{loc}}_p(\mathbb{R}^n)$. Then for $p > 1$*

$$\|Tf\|_{L_p(B(x,t))} \le Ct^{n/p} \int_t^\infty r^{-n/p-1} \|f\|_{L_p(B(x,r))} dr, \tag{6.1}$$

and for $p = 1$

$$\|Tf\|_{WL_1(B(x,t))} \le Ct^n \int_t^\infty r^{-n-1} \|f\|_{L_1(B(x,r))} dr, \tag{6.2}$$

where C *does not depend on* f, $x \in \mathbb{R}^n$ *and* $t > 0$.

Proof. Let $1 < p < \infty$. We represent function f as in (4.3) and have

$$\|Tf\|_{L_p(B(x,t))} \le \|Tf_1\|_{L_p(B(x,t))} + \|Tf_2\|_{L_p(B(x,t))}. \tag{6.3}$$

By boundedness of the operator T in $L_p(\mathbb{R}^n)$, $1 < p < \infty$ we obtain $\|Tf_1\|_{L_p(B(x,t))} \le \|Tf_1\|_{L_p(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)}$, so that

$$\|Tf_1\|_{L_p(B(x,t))} \le C\|f\|_{L_p(B(x,2t))}. \tag{6.4}$$

Taking into account the inequality

$$\|f\|_{L_p(B(x,t))} \le Ct^{n/p} \int_{2t}^\infty r^{-n/p-1} \|f\|_{L_p(B(x,r))} dr, \tag{6.5}$$

we get

$$\|Tf_1\|_{L_p(B(x,t))} \le Ct^{n/p} \int_{2t}^\infty r^{-n/p-1} \|f\|_{L_p(B(x,r))} dr. \tag{6.6}$$

To estimate $\|Tf_2\|_{L_p(B(x,t))}$, we observe that

$$|Tf_2(z)| \le C \int_{\complement B(x,2t)} \frac{|f(y)|dy}{|y-z|^n}, \tag{6.7}$$

where $z \in B(x,t)$ and the inequalities $|x - z| \le t$, $|z - y| \ge 2t$ imply $(1/2)|z - y| \le |x - y| \le (3/2)|z - y|$, and therefore

$$\|Tf_2\|_{L_p(B(x,t))} \le C \int_{\complement B(x,2t)} |x-y|^{-n} |f(y)| dy \|\chi_{B(x,t)}\|_{L_p(\mathbb{R}^n)}. \tag{6.8}$$

Hence by inequality (4.7), we get

$$\|Tf_2\|_{L_p(B(x,t))} \le Ct^{n/p} \int_{2t}^\infty r^{-n/p-1} \|f\|_{L_p(B(x,r))} dr. \tag{6.9}$$

From (6.6) and (6.9) we arrive at (6.1).

Let $p = 1$. It is obvious that for any ball $B(x,r)$

$$\|Tf\|_{WL_1(B(x,t))} \le \|Tf_1\|_{WL_1(B(x,t))} + \|Tf_2\|_{WL_1(B(x,t))}. \tag{6.10}$$

By boundedness of the operator T from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ we have

$$\|Tf_1\|_{WL_1(B(x,t))} \le C\|f\|_{L_1(B(x,2t))}, \tag{6.11}$$

where C does not depend on x, t.

Note that inequality (6.9) also true in the case $p = 1$. Then by (4.11), we get inequality (6.2). □

Theorem 6.2. *Let $1 \le p < \infty$ and $\omega_1(x,t)$ and $\omega_2(x,r)$ fulfill condition* (4.14)*. Then for $p > 1$ the singular integral operator T is bounded from the space $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to the space $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$ and for $p = 1$ T is bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)$.*

Proof. Let $1 < p < \infty$ and $f \in \mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$. By Theorem 6.1 we obtain

$$\begin{aligned}\|Tf\|_{\mathcal{M}_{p,\omega_2}} &= \sup_{x\in\mathbb{R}^n,\ t>0} \omega_2^{-1}(x,t)t^{-n/p}\|Tf\|_{L_p(B(x,t))} \\ &\le C \sup_{x\in\mathbb{R}^n,\ t>0} \omega_2^{-1}(x,t)\int_t^\infty r^{-n/p-1}\|f\|_{L_p(B(x,r))}dr.\end{aligned} \tag{6.12}$$

Hence

$$\begin{aligned}\|Tf\|_{\mathcal{M}_{p,\omega_2}} &\le C\|f\|_{\mathcal{M}_{p,\omega_1}} \sup_{x\in\mathbb{R}^n,\ t>0} \frac{1}{\omega_2(x,t)}\int_t^\infty \omega_1(x,r)\frac{dr}{r} \\ &\le C\|f\|_{\mathcal{M}_{p,\omega_1}}\end{aligned} \tag{6.13}$$

by (4.14), which completes the proof for $1 < p < \infty$.

Let $p = 1$ and $f \in \mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$. By Theorem 6.1 we obtain

$$\begin{aligned}\|Tf\|_{W\mathcal{M}_{1,\omega_2}} &= \sup_{x\in\mathbb{R}^n,\ t>0} \omega_2^{-1}(x,t)t^{-n}\|Tf\|_{WL_1(B(x,t))} \\ &\le C \sup_{x\in\mathbb{R}^n,\ t>0} \omega_2^{-1}(x,t)\int_t^\infty r^{-n-1}\|f\|_{L_1(B(x,r))}dr.\end{aligned} \tag{6.14}$$

Hence

$$\begin{aligned}\|Tf\|_{W\mathcal{M}_{1,\omega_2}} &\leq C\|f\|_{\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\omega_2(x,t)} \int_t^{\infty} \omega_1(x,r)\frac{dr}{r} \\ &\leq C\|f\|_{\mathcal{M}_{1,\omega_1}}\end{aligned} \tag{6.15}$$

by (4.14), which completes the proof for $p = 1$. □

Remark 6.3. Note that Theorems 6.1 and 6.2 were proved in [4] (see also [5]). Theorem 6.2 does not impose the pointwise doubling conditions (3.3) and (3.4). In the case $\omega_1(x,r) = \omega_2(x,r) = \omega(x,r)$, Theorem 6.2 is containing the results of Theorem 3.2.

7. The Generalized Morrey Estimates for the Operators $V^{\gamma}(-\Delta + V)^{-\beta}$ and $V^{\gamma}\nabla(-\Delta + V)^{-\beta}$

In this section we consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$, where the nonnegative potential V belongs to the reverse Hölder class $B_{\infty}(\mathbb{R}^n)$ for some $q_1 \geq n$. The generalized Morrey $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ estimates for the operators $V^{\gamma}(-\Delta+V)^{-\beta}$ and $V^{\gamma}\nabla(-\Delta+V)^{-\beta}$ are obtained.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [35–37]). Shen [36] studied the Schrödinger operator $-\Delta + V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \geq n/2$ and he proved the L_p boundedness of the operators $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-1/2}$, and $\nabla(-\Delta + V)^{-1}$. Kurata and Sugano generalized Shens results to uniformly elliptic operators in [38]. Sugano [39] also extended some results of Shen to the operator $V^{\gamma}(-\Delta + V)^{-\beta}$, $0 \leq \gamma \leq \beta \leq 1$, and $V^{\gamma}\nabla(-\Delta + V)^{-\beta}$, $0 \leq \gamma \leq 1/2 \leq \beta \leq 1$ and $\beta - \gamma \geq 1/2$. Later, Lu [40] and Li [41] investigated the Schrödinger operators in a more general setting.

We investigate the generalized Morrey $\mathcal{M}_{p,\omega_1}$-$\mathcal{M}_{q,\omega_2}$ boundedness of the operators

$$\begin{aligned} T_1 &= V^{\gamma}(-\Delta + V)^{-\beta}, \quad 0 \leq \gamma \leq \beta \leq 1, \\ T_2 &= V^{\gamma}\nabla(-\Delta + V)^{-\beta}, \quad 0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1,\ \beta - \gamma \geq \frac{1}{2}. \end{aligned} \tag{7.1}$$

Note that the operators $V(-\Delta + V)^{-1}$ and $V^{1/2}\nabla(-\Delta + V)^{-1}$ in [41] are the special case of T_1 and T_2, respectively.

It is worth pointing out that we need to establish pointwise estimates for T_1, T_2 and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^n$ in [41]. And we prove the generalized Morrey estimates by using $\mathcal{M}_{p,\omega_1}-\mathcal{M}_{q,\omega_2}$ boundedness of the fractional maximal operators.

Let $V \geq 0$. We say $V \in B_\infty$, if there exists a constant $C > 0$ such that

$$\|V\|_{L_\infty(B)} \leq \frac{C}{|B|}\int_B V(x)dx \tag{7.2}$$

holds for every ball B in $\mathbb{R}^n$ (see [41]).

The following are two pointwise estimates for T_1 and T_2 which are proven in [37, Lemma 3.2] with the potential $V \in B_\infty$.

Theorem B. *Suppose $V \in B_\infty$ and $0 \leq \gamma \leq \beta \leq 1$. Then there exists a constant $C > 0$ such that*

$$\left|T_1 f(x)\right| \leq C M_\alpha f(x), \quad f \in C_0^\infty(\mathbb{R}^n), \tag{7.3}$$

where $\alpha = 2(\beta - \gamma)$.

Theorem C. *Suppose $V \in B_\infty$, $0 \leq \gamma \leq 1/2 \leq \beta \leq 1$ and $\beta - \gamma \geq 1/2$. Then there exists a constant $C > 0$ such that*

$$\left|T_2 f(x)\right| \leq C M_\alpha f(x), \quad f \in C_0^\infty(\mathbb{R}^n), \tag{7.4}$$

where $\alpha = 2(\beta - \gamma) - 1$.

The previous theorems will yield the generalized Morrey estimates for T_1 and T_2.

Corollary 7.1. *Assume that $V \in B_\infty$, and $0 \leq \gamma \leq \beta \leq 1$. Let $1 \leq p \leq q < \infty$, $2(\beta-\gamma) = n(1/p-1/q)$, and condition* (5.13) *be satisfied for $\alpha = 2(\beta - \gamma)$. Then for $p > 1$ the operator T_1 is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$ and for $p = 1$ T_1 is bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$.*

Corollary 7.2. *Assume that $V \in B_\infty$, $0 \leq \gamma \leq 1/2 \leq \beta \leq 1$, and $\beta - \gamma \geq 1/2$. Let $1 \leq p \leq q < \infty$, $2(\beta-\gamma)-1 = n(1/p-1/q)$, and condition* (5.13) *be satisfied for $\alpha = 2(\beta-\gamma)-1$. Then for $p > 1$ the operator T_2 is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$ and for $p = 1$ T_2 is bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$.*

8. Some Applications

The theorems of Section 2 can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$\left|p_t(x,y)\right| \leq \frac{c_1}{t^{n/2}} e^{-c_2(|x-y|^2/t)} \tag{8.1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x, y, and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}. \tag{8.2}$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. See, for example, [2, Chapter 5].

Theorem 8.1. *Let $0 < \alpha < n$, $1 \le p < q < \infty$, $\alpha/n = 1/p - 1/q$ and conditions* (5.13), (8.1) *are satisfied. Then for $p > 1$ the operator $L^{-\alpha/2}$ is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$ and for $p = 1$ $L^{-\alpha/2}$ is bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$.*

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (8.1), it follows that

$$\left| L^{-\alpha/2} f(x) \right| \le C I_\alpha |f|(x) \tag{8.3}$$

for all $x \in \mathbb{R}^n$, where $C > 0$ is independent of x (see [42]). Hence by Theorem 5.2 we have

$$\begin{aligned} \|L^{-\alpha/2} f\|_{\mathcal{M}_{q,\omega_2}} &\le C \| I_\alpha |f| \|_{\mathcal{M}_{q,\omega_2}} \le C \|f\|_{\mathcal{M}_{p,\omega_1}}, \quad \text{if } p > 1, \\ \|L^{-\alpha/2} f\|_{W\mathcal{M}_{q,\omega_2}} &\le C \| I_\alpha |f| \|_{W\mathcal{M}_{q,\omega_2}} \le C \|f\|_{\mathcal{M}_{1,\omega_1}}, \quad \text{if } p = 1, \end{aligned} \tag{8.4}$$

where the constant $C > 0$ is independent of f. □

Property (8.1) is satisfied for large classes of differential operators. We mention two of them.

(a) Consider a magnetic potential $\vec{a}$, that is, a real-valued vector potential $\vec{a} = (a_1, a_2, \ldots, a_n)$, and an electric potential V. We assume that for any $k = 1, 2, \ldots, n$, $a_k \in L_2^{\mathrm{loc}}$ and $0 \le V \in L_1^{\mathrm{loc}}$. The operator L, which is given by

$$L = -\left(\nabla - i\vec{a}\right)^2 + V(x), \tag{8.5}$$

is called the magnetic Schrödinger operator.

By the well-known diamagnetic inequality (see [43], Theorem 2.3) we have the following pointwise estimate. For any $t > 0$ and $f \in L_2$,

$$\left| e^{-tL} f \right| \le e^{-t\Delta} |f|, \tag{8.6}$$

which implies that the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies upper bound (8.1).

(b) Let $A = ((a_{ij}(x))_{1 \le i,j \le n}$ be an $n \times n$ matrix with complex-valued entries $a_{ij} \in L_\infty$ satisfying

$$\operatorname{Re} \sum_{i,j=1}^n a_{ij}(x) \zeta_i \zeta_j \ge \lambda |\zeta|^2 \tag{8.7}$$

for all $x \in \mathbb{R}^n, \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n$ and some $\lambda > 0$. Consider the divergence form operator

$$Lf \equiv -\operatorname{div}(A\nabla f), \tag{8.8}$$

which is interpreted in the usual weak sense via the appropriate sesquilinear form.

It is known that the Gaussian bound (8.1) for the kernel of e^{-tL} holds when A has real-valued entries (see, e.g., [44]), or when $n = 1, 2$ in the case of complex-valued entries (see [45, Chapter 1]).

Finally we note that under the appropriate assumptions (see [2, 46, Chapter 5]; [45, pages 58-59]) one can obtain results similar to Theorem 8.1 for a homogeneous elliptic operator L in L_2 of order $2m$ in the divergence form

$$Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left(a_{\alpha\beta} D^\beta f \right). \tag{8.9}$$

In this case estimate (8.1) should be replaced by

$$|p_t(x,y)| \leq \frac{c_3}{t^{n/2m}} e^{-c_4 \left(|x-y|/t^{1/(2m)}\right)^{2m/(2m-1)}} \tag{8.10}$$

for all $t > 0$ and all $x, y \in \mathbb{R}^n$, where $c_3, c_4 > 0$ are independent of x, y, and t.

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