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Research Article

On a Hilbert-Type Operator with a Class of Homogeneous Kernels

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By using the way of weight coefficient and the theory of operators, we define a Hilbert-type operator with a class of homogeneous kernels and obtain its norm. As applications, an extended basic theorem on Hilbert-type inequalities with the decreasing homogeneous kernels of $-\lambda$-degree is established, and some particular cases are considered.

1. Introduction

In 1908, Weyl published the well-known Hilbert's inequality as the following. If $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are real sequences, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then [1]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right)^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. In 1925, Hardy gave an extension of (1.1) by introducing one pair of conjugate exponents $(p,q)(1/p+1/q=1)$ as [2]. If $p>1$, $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.2}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. We named (1.2) Hardy-Hilbert's inequality. In 1934, Hardy et al. [3] gave some applications of (1.1)-(1.2) and a basic theorem with the general kernel (see [3, Theorem 318]).

Theorem A. *Suppose that $p > 1,\ 1/p + 1/q = 1,\ k(x,y)$ is a homogeneous function of -1-degree, and $k = \int_0^\infty k(u,1)u^{-1/p}\,du$ is a positive number. If both $k(u,1)u^{-1/p}$ and $k(1,u)u^{-1/q}$ are strictly decreasing functions for $u > 0$, $a_n, b_n \geq 0$, $0 < \|a\|_p = (\sum_{n=1}^\infty a_n^p)^{1/p} < \infty$, and $0 < \|b\|_q = (\sum_{n=1}^\infty b_n^q)^{1/q} < \infty$, then one has the following equivalent inequalities:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty k(m,n)a_m b_n < k\|a\|_p\|b\|_q, \tag{1.3}$$

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty k(m,n)a_m\right)^p < k^p\|a\|_p^p, \tag{1.4}$$

where the constant factors k and k^p are the best possible.

Note. Hardy did not prove this theorem in [3]. In particular, we find some classical Hilbert-type inequalities as,

(i) for $k(x,y) = 1/(x+y)$ in (1.3), it reduces (1.2),

(ii) for $k(x,y) = 1/\max\{x,y\}$ in (1.3), it reduces to (see [3, Theorem 341])

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < pq\left(\sum_{n=1}^\infty a_n^p\right)^{1/p}\left(\sum_{n=1}^\infty b_n^q\right)^{1/q}, \tag{1.5}$$

(iii) for $k(x,y) = \ln(x/y)/(x-y)$ in (1.3), it reduces to (see [3, Theorem 342])

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln(m/n)a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\pi/p)}\right]^2\left(\sum_{n=1}^\infty a_n^p\right)^{1/p}\left(\sum_{n=1}^\infty b_n^q\right)^{1/q}. \tag{1.6}$$

Hardy also gave some multiple extensions of (1.3) (see [3, Theorem 322]). About introducing one pair of nonconjugate exponents (p,q) in (1.1), Hardy et al. [3] gave that if $p,q > 1,\ 1/p + 1/q \geq 1,\ 0 < \lambda = 2 - (1/p + 1/q) \leq 1$, then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} \leq K(p,q)\left(\sum_{n=1}^\infty a_n^p\right)^{1/p}\left(\sum_{n=1}^\infty b_n^q\right)^{1/q}. \tag{1.7}$$

In 1951, Bonsall [4] considered (1.7) in the case of general kernel; in 1991, Mitrinović et al. [5] summarized the above results.

In 2001, Yang [6] gave an extension of (1.1) as for $0 < \lambda \leq 4$,

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\sum_{n=1}^\infty n^{1-\lambda}a_n^2 \sum_{n=1}^\infty n^{1-\lambda}b_n^2\right)^{1/2}, \tag{1.8}$$

where the constant $B(\lambda/2, \lambda/2,)$ is the best possible ($B(u, v)$ is the Beta function). For $\lambda = 1$, (1.8) reduces to (1.1). And Yang [7] also gave an extension of (1.2) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}} < \frac{\pi}{\lambda\sin(\pi/p)}\left\{\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q\right\}^{1/q}, \tag{1.9}$$

where the constant factor $\pi/\lambda\sin(\pi/p)$ $(0 < \lambda \leq 2)$ is the best possible.

In 2004, Yang [8] published the dual form of (1.2) as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty} n^{p-2} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{q-2} b_n^q\right)^{1/q}, \tag{1.10}$$

where $\pi/\sin(\pi/p)$ is the best possible. For $p = q = 2$, both (1.10) and (1.2) reduce to (1.1). It means that there are more than two different best extensions of (1.1). In 2005, Yang [9] gave an extension of (1.8)–(1.10) with two pairs of conjugate exponents (p, q), (r, s) $(p, r > 1)$, and two parameters $\alpha, \lambda > 0$ $(\alpha\lambda \leq \min\{r, s\})$ as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\alpha})^{\lambda}} < k_{\alpha\lambda}(r)\left\{\sum_{n=1}^{\infty} n^{p(1-\alpha\lambda/r)-1} a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q(1-\alpha\lambda/s)-1} b_n^q\right\}^{1/q}, \tag{1.11}$$

where the constant factor $k_{\alpha\lambda}(r) = (1/\alpha)B(\lambda/r, \lambda/s)$ is the best possible; Krnić and Pečarić [10] also considered (1.11) in the general homogeneous kernel, but the best possible property of the constant factor was not proved by [10].

Note. For $A = B = \alpha = \beta = 1$ in [10, inequality (37)], it reduces to the equivalent result of (3.1) in this paper.

In 2006-2007, some authors also studied the operator expressing of (1.3) and (1.4).

Suppose that $k(x, y) (\geq 0)$ is a symmetric function with $k(y, x) = k(x, y)$, and $k_0(p) := \int_0^{\infty} k(x, y)(x/y)^{1/r}\, dy$ $(r = p, q;\ x > 0)$ is a positive number independent of x. Define an operator $T : l^r \to l^r$ $(r = p, q)$ as follows. For $a_m \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, there exists only $Ta = c = \{c_n\}_{n=1}^{\infty} \in l^p$, satisfying

$$(Ta)(n) = c_n := \sum_{m=1}^{\infty} k(m, n) a_m \quad (n \in \mathbf{N}). \tag{1.12}$$

Then the formal inner product of Ta and b are defined as follows:

$$(Ta, b) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(m, n) a_m b_n. \tag{1.13}$$

In 2007, Yang [11] proved that if for $\varepsilon \geq 0$ small enough, $k(x,y)(x/y)^{(1+\varepsilon)/r}$ is strictly decreasing for $y > 0$, the integral $\int_0^\infty k(x,y)(x/y)^{(1+\varepsilon)/r}\,dy = k_\varepsilon(p)$ is also a positive number independent of $x > 0$, $k_\varepsilon(p) = k_0(p) + o(1)$ $(\varepsilon \to 0^+)$, and

$$\sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \int_0^1 k(m,t)\left(\frac{m}{t}\right)^{(1+\varepsilon)/r} dt = O(1) \quad (\varepsilon \longrightarrow 0^+;\ r = p, q), \tag{1.14}$$

then $\|T\|_p = k_0(p)$; in this case, if $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p > 0$, $\|b\|_q > 0$, then we have two equivalent inequalities as

$$(Ta, b) < \|T\|_p \|a\|_p \|b\|_q; \qquad \|Ta\|_p < \|T\|_p \|a\|_p, \tag{1.15}$$

where the constant factor $\|T\|_p$ is the best possible. In particular, for $k(x,y)$ being -1-degree homogeneous, inequalities (1.15) reduce to (1.3)-(1.4) (in the symmetric kernel). Yang [12] also considered (1.15) in the real space l^2.

In this paper, by using the way of weight coefficient and the theory of operators, we define a new Hilbert-type operator and obtain its norm. As applications, an extended basic theorem on Hilbert-type inequalities with the decreasing homogeneous kernel of $-\lambda$-degree is established; some particular cases are considered.

2. On a New Hilbert-Type Operator and the Norm

If $k_\lambda(x,y)$ is a measurable function, satisfying for $\lambda, u, x, y > 0$, $k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x,y)$, then we call $k_\lambda(x,y)$ the homogeneous function of $-\lambda$-degree.

For $k_\lambda(x,y) \geq 0$, setting $x = uy$, we find $k_\lambda(x,y)(1/x^{1-\lambda/r}) = (1/y^{1+\lambda/s}) k_\lambda(u,1) u^{\lambda/r-1}$. Hence, the following two words are equivalent: (a) $k_\lambda(u,1)u^{\lambda/r-1}$ is decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$; (b) for any $y > 0$, $k_\lambda(x,y)(1/x^{1-\lambda/r})$ is decreasing in $x \in (0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$. The following two words are also equivalent: (a)$'$ $k_\lambda(1,u)u^{\lambda/s-1}$ is decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$; (b)$'$ for any $x > 0$, $k_\lambda(x,y)(1/y^{1-\lambda/s})$ is decreasing in $y \in (0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$.

Lemma 2.1. *If $f(x) (\geq 0)$ is decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$, and $I_0 := \int_0^\infty f(x)dx < \infty$, then*

$$I_1 := \int_1^\infty f(x)dx \leq \sum_{n=1}^{\infty} f(n) < I_0. \tag{2.1}$$

Proof. By the assumption, we find $\int_n^{n+1} f(x)dx \leq f(n) \leq \int_{n-1}^{n} f(x)dx$ $(n \in \mathbf{N})$, and there exists $(n_0 - 1, n_0] \subset (0,\infty)$, such that $f(n_0) < \int_{n_0-1}^{n_0} f(x)dx$. Hence,

$$I_1 = \sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx \leq \sum_{n=1}^{\infty} f(n) < \sum_{n=1}^{\infty} \int_{n-1}^{n} f(x)dx = I_0. \tag{2.2}$$

□

Lemma 2.2. *If* $r > 1$, $1/r + 1/s = 1$, $\lambda > 0$, $k_\lambda(x,y)(\geq 0)$ *is a homogeneous function of* $-\lambda$*-degree, and* $k_\lambda(r) := \int_0^\infty k_\lambda(u,1)u^{\lambda/r-1}\,du$ *is a positive number, then* (i) $\int_0^\infty k_\lambda(1,u)u^{\lambda/s-1}\,du = k_\lambda(r)$; (ii) *for* $x, y \in (0,\infty)$, *setting the weight functions as*

$$\omega_\lambda(r,y) := \int_0^\infty k_\lambda(x,y)\frac{y^{\lambda/s}}{x^{1-\lambda/r}}dx, \qquad \varpi_\lambda(s,x) := \int_0^\infty k_\lambda(x,y)\frac{x^{\lambda/r}}{y^{1-\lambda/s}}dy, \tag{2.3}$$

then $\omega_\lambda(r,y) = \varpi_\lambda(s,x) = k_\lambda(r)$.

Proof. (i) Setting $v = 1/u$, by the assumption, we obtain $\int_0^\infty k_\lambda(1,u)u^{\lambda/s-1}\,du = \int_0^\infty k_\lambda(v,1)v^{\lambda/r-1}\,dv = k_\lambda(r)$. (ii) Setting $x = yu$ and $y = xu$ in the integrals $\omega_\lambda(r,y)$ and $\varpi_\lambda(s,x)$, respectively, in view of (i), we still find that $\omega_\lambda(r,y) = \varpi_\lambda(s,x) = k_\lambda(r)$. □

For $p > 1$, $1/p + 1/q = 1$, we set $\phi(x) = x^{p(1-\lambda/r)-1}$, $\psi(x) = x^{q(1-\lambda/s)-1}$, and $\psi^{1-p}(x) = x^{p\lambda/s-1}$, $x \in (0,\infty)$. Define the real space as $l_\phi^p := \{a = \{a_n\}_{n=1}^\infty;\ \|a\|_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{1/p} < \infty\}$, and then we may also define the spaces l_ψ^q and $l_{\psi^{1-p}}^p$.

Lemma 2.3. *As the assumption of Lemma 2.2, for* $a_m \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_\phi^p$, *setting* $c_n = \sum_{m=1}^\infty k_\lambda(m,n)a_m$, *if* $k_\lambda(u,1)u^{\lambda/r-1}$ *and* $k_\lambda(1,u)u^{\lambda/s-1}$ *are decreasing in* $(0,\infty)$ *and strictly decreasing in a subinterval of* $(0,\infty)$, *then* $c = \{c_n\}_{n=1}^\infty \in l_{\psi^{1-p}}^p$.

Proof. By Hölder's inequality [13] and Lemmas 2.1-2.2, we obtain

$$\begin{aligned}
c_n^p &= \left\{\sum_{m=1}^\infty k_\lambda(m,n)\left[\frac{m^{(1-\lambda/r)/q}}{n^{(1-\lambda/s)/p}}a_m\right]\left[\frac{n^{(1-\lambda/s)/p}}{m^{(1-\lambda/r)/q}}\right]\right\}^p \\
&\leq \left[\sum_{m=1}^\infty k_\lambda(m,n)\frac{m^{(1-\lambda/r)p/q}}{n^{1-\lambda/s}}a_m^p\right]\left[\sum_{m=1}^\infty k_\lambda(m,n)\frac{n^{(1-\lambda/s)q/p}}{m^{1-\lambda/r}}\right]^{p-1} \\
&\leq \omega_\lambda^{p-1}(r,n)n^{1-p\lambda/s}\sum_{m=1}^\infty k_\lambda(m,n)\frac{m^{(1-\lambda/r)p/q}}{n^{1-\lambda/s}}a_m^p \\
&= k_\lambda^{p-1}(r)n^{1-p\lambda/s}\sum_{m=1}^\infty k_\lambda(m,n)\frac{m^{(1-\lambda/r)p/q}}{n^{1-\lambda/s}}a_m^p, \\
\|c\|_{p,\psi^{1-p}} &= \left\{\sum_{n=1}^\infty n^{p\lambda/s-1}c_n^p\right\}^{1/p} = \left\{\sum_{n=1}^\infty n^{p\lambda/s-1}\left[\sum_{m=1}^\infty k_\lambda(m,n)a_m\right]^p\right\}^{1/p} \\
&\leq k_\lambda^{1/q}(r)\left\{\sum_{n=1}^\infty\sum_{m=1}^\infty k_\lambda(m,n)\frac{m^{(1-\lambda/r)p/q}}{n^{1-\lambda/s}}a_m^p\right\}^{1/p} \\
&= k_\lambda^{1/q}(r)\left\{\sum_{m=1}^\infty\left[\sum_{n=1}^\infty k_\lambda(m,n)\frac{m^{\lambda/r}}{n^{1-\lambda/s}}\right]m^{p(1-\lambda/r)-1}a_m^p\right\}^{1/p} \\
&< k_\lambda^{1/q}(r)\left\{\sum_{m=1}^\infty \varpi_\lambda(s,m)m^{p(1-\lambda/r)-1}a_m^p\right\}^{1/p} = k_\lambda(r)\|a\|_{p,\phi} < \infty.
\end{aligned} \tag{2.4}$$

Therefore, $c = \{c_n\}_{n=1}^\infty \in l_{\psi^{1-p}}^p$. □

For $a_m \geq 0,\ a = \{a_m\}_{m=1}^{\infty} \in l_{\phi}^{p}$, define a Hilbert-type operator $T : l_{\phi}^{p} \to l_{\psi^{1-p}}^{p}$ as $Ta = c$, satisfying $c = \{c_n\}_{n=1}^{\infty}$,

$$(Ta)(n) := c_n = \sum_{m=1}^{\infty} k_{\lambda}(m,n) a_m \quad (n \in \mathbf{N}). \tag{2.5}$$

In view of Lemma 2.3, $c \in l_{\psi^{1-p}}^{p}$ and then T exists. If there exists $M > 0$, such that for any $a \in l_{\phi}^{p}$, $\|Ta\|_{p,\psi^{1-p}} \leq M\|a\|_{p,\phi}$, then T is bounded and $\|T\| = \sup_{\|a\|_{p,\phi}=1} \|Ta\|_{p,\psi^{1-p}} \leq M$. Hence by (2.4), we find $\|T\| \leq k_{\lambda}(r)$ and T is bounded.

Theorem 2.4. *As the assumption of Lemma 2.3, it follows $\|T\| = k_{\lambda}(r)$.*

Proof. For $a_m, b_n \geq 0,\ a = \{a_m\}_{m=1}^{\infty} \in l_{\phi}^{p},\ b = \{b_n\}_{n=1}^{\infty} \in l_{\psi}^{q},\ \|a\|_{p,\phi} > 0,\ \|b\|_{q,\psi} > 0$, by Hölder's inequality [12], we find

$$\begin{aligned}
(Ta, b) &= \sum_{n=1}^{\infty} \left[n^{\lambda/s-1/p} \sum_{m=1}^{\infty} k_{\lambda}(m,n) a_m \right] \left[n^{-\lambda/s+1/p} b_n \right] \\
&\leq \left\{ \sum_{n=1}^{\infty} n^{p\lambda/s-1} \left[\sum_{m=1}^{\infty} k_{\lambda}(m,n) a_m \right]^p \right\}^{1/p} \|b\|_{q,\psi}.
\end{aligned} \tag{2.6}$$

Then by (2.4), we obtain

$$(Ta, b) < k_{\lambda}(r) \|a\|_{p,\phi} \|b\|_{q,\psi}. \tag{2.7}$$

For $0 < \varepsilon < \min\{p\lambda/r,\ q\lambda/s\}$, setting $\tilde{a} = \{\tilde{a}_n\}_{n=1}^{\infty}$, $\tilde{b} = \{\tilde{b}_n\}_{n=1}^{\infty}$ as $\tilde{a}_n = n^{\lambda/r-\varepsilon/p-1}$, $\tilde{b}_n = n^{\lambda/s-\varepsilon/q-1}$, for $n \in \mathbf{N}$, if there exists a constant $0 < k \leq k_{\lambda}(r)$, such that (2.7) is still valid when we replace $k_{\lambda}(r)$ by k, then by Lemma 2.1,

$$\varepsilon(T\tilde{a}, \tilde{b}) < \varepsilon k \|\tilde{a}\|_{p,\phi} \|\tilde{b}\|_{q,\psi} = \varepsilon k \left(1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\varepsilon}} \right) < \varepsilon k \left(1 + \int_{1}^{\infty} \frac{1}{y^{1+\varepsilon}} dy \right) = k(\varepsilon + 1), \tag{2.8}$$

$$\begin{aligned}
\varepsilon(T\tilde{a}, \tilde{b}) &= \varepsilon \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} k_{\lambda}(m,n) m^{\lambda/r-1} m^{-\varepsilon/p} \right] n^{\lambda/s-\varepsilon/q-1} \\
&\geq \varepsilon \sum_{n=1}^{\infty} \left[\int_{1}^{\infty} k_{\lambda}(x,n) x^{\lambda/r-\varepsilon/p-1}\, dx \right] n^{\lambda/s-\varepsilon/q-1} \\
&= \varepsilon \int_{1}^{\infty} \left[\sum_{n=1}^{\infty} k_{\lambda}(x,n) n^{\lambda/s-\varepsilon/q-1} \right] x^{\lambda/r-\varepsilon/p-1}\, dx \\
&\geq \varepsilon \int_{1}^{\infty} \left[\int_{1}^{\infty} k_{\lambda}(x,y) y^{\lambda/s-\varepsilon/q-1} x^{\lambda/r-\varepsilon/p-1}\, dy \right] dx.
\end{aligned} \tag{2.9}$$

In view of (2.8) and (2.9), setting $u = x/y$, by Fubini's theorem [13], it follows

$$\begin{aligned}
k(\varepsilon+1) &> \varepsilon\int_1^{\infty} x^{-1-\varepsilon}\left(\int_0^x k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du\right)dx \\
&= \int_0^1 k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du + \varepsilon\int_1^{\infty} x^{-1-\varepsilon}\left(\int_1^x k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du\right)dx \\
&= \int_0^1 k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du + \varepsilon\int_1^{\infty}\left(\int_u^{\infty} x^{-1-\varepsilon}\,dx\right)k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du \\
&= \int_0^1 k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du + \int_1^{\infty} k_\lambda(u,1)u^{\lambda/r-\varepsilon/p-1}\,du.
\end{aligned} \tag{2.10}$$

Setting $\varepsilon \to 0^+$ in the above inequality, by Fatou's lemma [14], we find

$$\begin{aligned}
k &\geq \varliminf_{\varepsilon\to 0^+}\left[\int_0^1 k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du + \int_1^{\infty} k_\lambda(u,1)u^{\lambda/r-\varepsilon/p-1}\,du\right] \\
&\geq \int_0^1 \varliminf_{\varepsilon\to 0^+} k_\lambda(u,1)u^{\lambda/r+\varepsilon/q-1}\,du + \int_1^{\infty} \varliminf_{\varepsilon\to 0^+} k_\lambda(u,1)u^{\lambda/r-\varepsilon/p-1}\,du \\
&= \int_0^1 k_\lambda(u,1)u^{\lambda/r-1}\,du + \int_1^{\infty} k_\lambda(u,1)u^{\lambda/r-1}\,du = k_\lambda(r).
\end{aligned} \tag{2.11}$$

Hence $k = k_\lambda(r)$ is the best value of (2.7). We conform that $k_\lambda(r)$ is the best value of (2.4). Otherwise, we can get a contradiction by (2.6) that the constant factor in (2.7) is not the best possible. It follows that $\|T\| = k_\lambda(r)$. □

3. An Extended Basic Theorem on Hilbert-Type Inequalities

Still setting $\phi(x) = x^{p(1-\lambda/r)-1}$, $\psi(x) = x^{q(1-\lambda/s)-1}$, $\psi^{1-p}(x) = x^{p\lambda/s-1}$, $x \in (0,\infty)$, and $l_\phi^p = \{a = \{a_n\}_{n=1}^{\infty};\ \|a\|_{p,\phi} := \{\sum_{n=1}^{\infty}\phi(n)|a_n|^p\}^{1/p} < \infty\}$, we have the following theorem.

Theorem 3.1. *Suppose that $p, r > 1$, $1/p + 1/q = 1$, $1/r + 1/s = 1$, $\lambda > 0$, $k_\lambda(x,y)(\geq 0)$ is a homogeneous function of $-\lambda$-degree, $k_\lambda(r) = \int_0^{\infty} k_\lambda(u,1)u^{\lambda/r-1}\,du$ is a positive number, both $k_\lambda(u,1)u^{\lambda/r-1}$ and $k_\lambda(1,u)u^{\lambda/s-1}$ are decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$. If $a_n, b_n \geq 0$, $a = \{a_n\}_{n=1}^{\infty} \in l_\phi^p$, $b = \{b_n\}_{n=1}^{\infty} \in l_\psi^q$, $\|a\|_{p,\phi} > 0$, $\|b\|_{q,\psi} > 0$, then one has the equivalent inequalities as*

$$(Ta,b) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(m,n)a_m b_n < k_\lambda(r)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{3.1}$$

$$\|Ta\|_{p,\psi^{1-p}}^p = \sum_{n=1}^{\infty} n^{p\lambda/s-1}\left(\sum_{m=1}^{\infty} k_\lambda(m,n)a_m\right)^p < k_\lambda^p(r)\|a\|_{p,\phi}^p, \tag{3.2}$$

where the constant factors $k_\lambda(r)$ and $k_\lambda^p(r)$ are the best possible.

Proof. In view of (2.7) and (2.4), we have (3.1) and (3.2). Based on Theorem 2.4, it follows that the constant factors in (3.1) and (3.2) are the best possible.

If (3.2) is valid, then by (2.6), we have (3.1). Suppose that (3.1) is valid. By (2.4), $\|Ta\|^p_{p,\psi^{1-p}} < \infty$. If $\|Ta\|^p_{p,\psi^{1-p}} = 0$, then (3.2) is naturally valid; if $\|Ta\|^p_{p,\psi^{1-p}} > 0$, setting $b_n = n^{p\lambda/s-1}(\sum_{m=1}^{\infty} k_\lambda(m,n)a_m)^{p-1}$, then $0 < \|b\|^q_{q,\psi} = \|Ta\|^p_{p,\psi^{1-p}} < \infty$. By (3.1), we obtain

$$\|b\|^q_{q,\psi} = \|Ta\|^p_{p,\psi^{1-p}} = (Ta,b) < k_\lambda(r)\|a\|_{p,\phi}\|b\|_{q,\psi}$$
$$\|b\|^{q-1}_{q,\psi} = \|Ta\|_{p,\psi^{1-p}} < k_\lambda(r)\|a\|_{p,\phi}, \tag{3.3}$$

and we have (3.2). Hence (3.1) and (3.2) are equivalent. □

Remark 3.2. (a) For $\lambda = 1,\ s = p,\ r = q$, (3.1) and (3.2) reduce, respectively, to (1.6) and (1.7). Hence, Theorem 3.1 is an extension of Theorem A.

(b) Replacing the condition "$k_\lambda(u,1)u^{\lambda/r-1}$ and $k_\lambda(1,u)u^{\lambda/s-1}$ are decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$" by "for $0 < \lambda \leq \min\{r,s\}$, $k_\lambda(u,1)$ and $k_\lambda(1,u)$ are decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$," the theorem is still valid. Then in particular,

(i) for $k_{\alpha\lambda}(x,y) = 1/(x^\alpha + y^\alpha)^\lambda$ $(\alpha,\lambda > 0,\ \alpha\lambda \leq \min\{r,s\})$ in (3.1), we find

$$k_{\alpha\lambda}(r) = \int_0^\infty \frac{u^{\alpha\lambda/r-1}}{(u^\alpha+1)^\lambda}du = \frac{1}{\alpha}\int_0^\infty \frac{v^{\lambda/r-1}}{(v+1)^\lambda}dv = \frac{1}{\alpha}B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right), \tag{3.4}$$

and then it deduces to (1.11);

(ii) for $k_\lambda(x,y) = (1/\max\{x^\lambda,y^\lambda\})$ $(0 < \lambda \leq \min\{r,s\})$ in (3.1), we find

$$k_\lambda(r) = \int_0^\infty \frac{1}{\max\{u^\lambda,1\}}u^{\lambda/r-1}\,du = \frac{rs}{\lambda}, \tag{3.5}$$

and then it deduces to the best extension of (1.5) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\})^\lambda} < \frac{rs}{\lambda}\|a\|_{p,\phi}\|b\|_{q,\psi}; \tag{3.6}$$

(iii) for $k_\lambda(x,y) = (\ln(x/y)/(x^\lambda - y^\lambda))$ $(0 < \lambda \leq \min\{r,s\})$ in (3.1), we find [3]

$$k_\lambda(r) = \int_0^\infty \frac{\ln u}{u^\lambda - 1}u^{\lambda/r-1}\,du = \left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^2, \tag{3.7}$$

and $(\ln u/(u^\lambda - 1))' < 0$, and then it deduces to the best extension of (1.6) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m^\lambda - n^\lambda} < \left[\frac{\pi}{\lambda\sin(\pi/r)}\right]^2\|a\|_{p,\phi}\|b\|_{q,\psi}. \tag{3.8}$$

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