

Research Article

Certain Classes of Harmonic Multivalent Functions Based on Hadamard Product

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Received 25 February 2009; Accepted 12 September 2009

Recommended by Yong Zhou

We define and investigate two special subclasses of the class of complex-valued harmonic multivalent functions based on Hadamard product.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain \mathbb{C} if both u and v are real harmonic in \mathbb{C} . In any simply connected domain $D \subseteq \mathbb{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the co-analytic part of f . Note that $f = h + \bar{g}$ reduces to h if the coanalytic part g is zero.

For $p \geq 1$, denote by $H(p)$ the set of all multivalent harmonic functions $f = h + \bar{g}$ defined in the open unit disc U , where h and g defined by

$$h(z) = z^p + \sum_{n=p+t}^{\infty} a_n z^n, \quad g(z) = \sum_{n=p+t-1}^{\infty} b_n z^n, \quad |b_{p+t-1}| < 1, \quad t \in \mathbb{N} := \{1, 2, \dots\} \quad (1.1)$$

are analytic functions in U .

Let F be a fixed multivalent harmonic function given by

$$F(z) = H(z) + \overline{G(z)} = z^p + \sum_{n=p+t}^{\infty} |A_n| z^n + \overline{\sum_{n=p+t-1}^{\infty} |B_n| z^n}, \quad |B_{p+t-1}| < 1, \quad t \in \mathbb{N}. \quad (1.2)$$

A function $f \in H(p)$ is said to be in the class $H_F(p, t, \alpha, k)$ if

$$\operatorname{Re} \left(\frac{z(f * F)'(z)}{z'(f * F)(z)} \right) \geq k \left| \frac{z(f * F)'(z)}{z'(f * F)(z)} - p \right| + p\alpha, \quad (1.3)$$

where $f * F$ is a harmonic convolution of f and F . Note that $z' = (\partial/\partial\theta)(r e^{i\theta})$, $f'(z) = (\partial/\partial\theta)f(r e^{i\theta})$. Using the fact

$$\operatorname{Re} w > k|w - p| + p\alpha \iff \operatorname{Re} \left((1 + ke^{i\theta})w - kpe^{i\theta} \right) \geq p\alpha, \quad (1.4)$$

it follows that $f \in H_F(p, t, \alpha, k)$ if and only if

$$\operatorname{Re} \left(\frac{(1 + ke^{i\theta})z(f * F)'(z)}{z'(f * F)(z)} - kpe^{i\theta} \right) \geq p\alpha, \quad 0 \leq \alpha < 1. \quad (1.5)$$

A function f in $H_F(p, t, \alpha, k)$ is called k -uniformly multivalent harmonic starlike function associated with a fixed multivalent harmonic function F . The set $H_F(p, t, \alpha, k)$ is a comprehensive family that contains several previously studied subclasses of $H(p)$; for example, if we let

$$F(z) = I(z) := \left(\frac{z^p}{1-z} \right) + \overline{\left(\frac{z^p}{1-z} \right)} = z^p + \sum_{n=p+1}^{\infty} z^n + \overline{\sum_{n=p}^{\infty} z^n}, \quad (1.6)$$

then

$$H_I(p, \alpha, 0) \equiv S_H^*(p, \alpha) := \left\{ f \in H(p) : \operatorname{Re} \left(\frac{z f'(z)}{z' f(z)} \right) \geq p\alpha \right\}, \quad (1.7)$$

(see [1, 2]);

$$H_I(p, 1, 0, 0) \equiv S_H^*(p, 0) \equiv S_H^*(p), \quad (1.8)$$

(see [3]);

$$H_I(1, 1, \alpha, 0) \equiv S_H^*(1, \alpha) \equiv S_H^*(\alpha), \quad (1.9)$$

(see [4]);

$$H_I(1, 1, 0, 0) \equiv S_H^*(0) \equiv S_H^*, \quad (1.10)$$

(see [5, 6]);

$$H_I(1, 1, \alpha, k) \equiv G_H(k, \alpha) := \left\{ f \in H(1) : \operatorname{Re} \left(\frac{(1 + ke^{i\theta})zf'(z)}{z'f(z)} - ke^{i\theta} \right) \geq \alpha \right\}, \quad (1.11)$$

(see [7]);

$$H_I(p, 1, \alpha, 1) \equiv M_H(p, \alpha) := \left\{ f \in H(p) : \operatorname{Re} \left(\left(1 + e^{i\theta}\right) \frac{zf'(z)}{z'f(z)} - pe^{i\theta} \right) \geq p\alpha \right\}, \quad (1.12)$$

(see [8]).

Finally, denote by $TH(p)$ the subclass of functions $f(z) = h(z) + \overline{g(z)}$ in $H(p)$ where

$$h(z) = z^p - \sum_{n=p+t}^{\infty} |a_n|z^n, \quad g(z) = \sum_{n=p+t-1}^{\infty} |b_n|z^n. \quad (1.13)$$

Let $H_{\overline{F}}(p, t, \alpha, k) := TH(p) \cap H_F(p, t, \alpha, k)$.

In this paper, we investigate coefficient conditions, extreme points, and distortion bounds for functions in the family $H_{\overline{F}}(p, t, \alpha, k)$. We observe that the results so obtained for this main family can be viewed as extensions and generalizations for various subclasses of $H(p)$ and $H(1)$.

2. Main Results

Theorem 2.1. *Let $f = h + \overline{g}$ be such that h and g are given by (1.1). Then $f \in H_{\overline{F}}(p, t, \alpha, k)$ if the inequality*

$$\sum_{n=p+t}^{\infty} \left[\frac{n(1+k) - p(k+\alpha)}{(p(1-\alpha)+1) - |p(1-\alpha)-1|} \right] |a_n A_n| + \sum_{n=p+t-1}^{\infty} \left[\frac{n(1+k) + p(k+\alpha)}{(p(1-\alpha)+1) - |p(1-\alpha)-1|} \right] |b_n B_n| \leq \frac{1}{2} \quad (2.1)$$

is satisfied for some k ($k \geq 0$), p ($p \geq 1$), α ($0 \leq \alpha < 1$), and t ($t \geq 1$).

Proof. In view of (1.5), we need to prove that $\operatorname{Re}(w) > 0$, where

$$w = \frac{(ke^{i\theta} + 1) \left[z(h * H)'(z) - \overline{z(g * G)'(z)} \right] - p(ke^{i\theta} + \alpha) \left[(h * H)(z) + \overline{(g * G)(z)} \right]}{(h * H)(z) + \overline{(g * G)(z)}} := \frac{A(z)}{B(z)}. \quad (2.2)$$

Using the fact that $\operatorname{Re}(w) > 0 \Leftrightarrow |1 + w| \geq |1 - w|$, it suffices to show that

$$|A(z) + B(z)| - |A(z) - B(z)| \geq 0. \quad (2.3)$$

Therefore, we obtain

$$\begin{aligned}
& |A(z) + B(z)| - |A(z) - B(z)| \\
& \geq ([p(1 - \alpha) + 1] - |p(1 - \alpha) - 1|)|z|^p - \sum_{n=p+t}^{\infty} [n(1 + k) - p(k + \alpha) + 1]|a_n A_n||z|^n \\
& \quad - \sum_{n=p+t-1}^{\infty} [n(1 + k) + p(k + \alpha) - 1]|b_n B_n||z|^n - \sum_{n=p+t}^{\infty} [n(1 + k) - p(k + \alpha) - 1]|a_n A_n||z|^n \\
& \quad - \sum_{n=p+t-1}^{\infty} [n(1 + k) + p(k + \alpha) + 1]|b_n B_n||z|^n \\
& = ([p(1 - \alpha) + 1] - |p(1 - \alpha) - 1|)|z|^p \\
& \quad - \sum_{n=p+t}^{\infty} 2[n(1 + k) - p(k + \alpha)]|a_n A_n||z|^n - \sum_{n=p+t-1}^{\infty} 2[n(1 + k) + p(k + \alpha)]|b_n B_n||z|^n \\
& = ([p(1 - \alpha) + 1] - |p(1 - \alpha) - 1|)|z|^p \\
& \quad \times \left\{ 1 - \sum_{n=p+t}^{\infty} \frac{2[n(1 + k) - p(k + \alpha)]}{[p(1 - \alpha) + 1] - |p(1 - \alpha) - 1|} |a_n A_n| \right. \\
& \quad \left. - \sum_{n=p+t-1}^{\infty} \frac{2[n(1 + k) + p(k + \alpha)]}{[p(1 - \alpha) + 1] - |p(1 - \alpha) - 1|} |b_n B_n| \right\} \geq 0.
\end{aligned} \tag{2.4}$$

By hypothesis, last expression is nonnegative. Thus the proof is complete. \square

The coefficient bounds (2.1) is sharp for the function

$$\begin{aligned}
f(z) &= z^p + \sum_{n=p+t}^{\infty} \frac{[p(1 - \alpha) + 1] - |p(1 - \alpha) - 1|}{2[n(1 + k) - p(k + \alpha)]} X_n z^n \\
& \quad + \sum_{n=p+t-1}^{\infty} \frac{[p(1 - \alpha) + 1] - |p(1 - \alpha) - 1|}{2[n(1 + k) + p(k + \alpha)] B_n} \overline{Y_n z^n},
\end{aligned} \tag{2.5}$$

where $\sum_{n=p+t}^{\infty} |X_n| + \sum_{n=p+t-1}^{\infty} |Y_n| = 1$.

Corollary 2.2. For $p \geq (1/1 - \alpha)$, $0 \leq \alpha < 1$, if the inequality

$$\sum_{n=p+t}^{\infty} [n(1 + k) - p(k + \alpha)]|a_n A_n| + \sum_{n=p+t-1}^{\infty} [n(1 + k) + p(k + \alpha)]|b_n B_n| \leq 1 \tag{2.6}$$

holds, then $f \in H_F(p, t, \alpha, k)$.

Corollary 2.3. For $0 \leq \alpha < 1$ and $1 \leq p \leq (1/1 - \alpha)$, if the inequality

$$\sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] |b_n B_n| \leq p(1-\alpha) \tag{2.7}$$

holds, then $f \in H_F(p, t, \alpha, k)$.

Theorem 2.4. Let $f = h + \bar{g}$ be such that h and g are given by (1.13). Also, suppose that $k \geq 0$ and $0 \leq \alpha < 1$. Then

(i) for $1 \leq p \leq (1/1 - \alpha)$, $f \in H_{\bar{F}}(p, t, \alpha, k)$ if and only if

$$\sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] |b_n B_n| \leq p(1-\alpha); \tag{2.8}$$

(ii) for $p(1 - \alpha) \geq 1$, $f \in H_{\bar{F}}(p, t, \alpha, k)$ if and only if

$$\sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] |b_n B_n| \leq 1. \tag{2.9}$$

Proof. According to Corollaries 2.2 and 2.3, we must show that if the condition (2.9) does not hold, then $f \notin H_{\bar{F}}(p, t, \alpha, k)$, that is, we must have

$$\operatorname{Re} \left(\frac{(1 + ke^{i\theta}) \left[z(h * H)'(z) - \overline{z(g * G)'(z)} \right] - p(ke^{i\theta} + \alpha) \left[(h * H)(z) + \overline{(g * G)(z)} \right]}{(h * H)(z) + \overline{(g * G)(z)}} \right) \geq 0. \tag{2.10}$$

Choosing the values of $z = r$ on positive real axis where $0 \leq z = r < 1$, and using $\operatorname{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the inequality (2.10) reduces to

$$\begin{aligned} &= \operatorname{Re} \left\{ \frac{(1 + ke^{i\theta}) \left[pz^p - \sum_{n=p+t}^{\infty} n |a_n A_n| z^n - \sum_{n=p+t-1}^{\infty} n |b_n B_n| z^n \right] - \mathfrak{A}}{p \left[z^p - \sum_{n=p+t}^{\infty} |a_n A_n| z^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| z^n \right]} \right\} \\ &\geq \left\{ \frac{(1+k) \left[pr^p - \sum_{n=p+t}^{\infty} n |a_n A_n| r^n - \sum_{n=p+t-1}^{\infty} n |b_n B_n| r^n \right] - \mathfrak{B}}{p \left[r^p - \sum_{n=p+t}^{\infty} |a_n A_n| r^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| r^n \right]} \right\} \tag{2.11} \\ &= \left\{ \frac{p(1-\alpha)r^p - \sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] |a_n A_n| r^n - \mathfrak{C}}{p \left[r^p - \sum_{n=p+t}^{\infty} |a_n A_n| r^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| r^n \right]} \right\}, \end{aligned}$$

where \mathfrak{A} denotes $(k e^{i\theta} + \alpha)p[z^p - \sum_{n=p+t}^{\infty} |a_n A_n| z^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| z^n]$, \mathfrak{B} denotes $(k + \alpha)p[r^p - \sum_{n=p+t}^{\infty} |a_n A_n| r^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| r^n]$, and \mathfrak{C} denotes $\sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] |b_n B_n| r^n$.

Letting $r \rightarrow 1^-$, we obtain

$$\frac{p(1-\alpha) - \left\{ \sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] |a_n A_n| + \sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] |b_n B_n| \right\}}{1 - \sum_{n=p+t}^{\infty} |a_n A_n| + \sum_{n=p+t-1}^{\infty} |b_n B_n|} \geq 0. \quad (2.12)$$

If the condition (2.10) does not hold, then the numerator in (2.12) is negative for r sufficiently close to 1. Hence there exists $z_0 = r_0$ in $(0, 1)$ for which (2.12) is negative. Therefore, it follows that $f \notin H_{\overline{F}}(p, t, \alpha, k)$ and so the proof is complete. \square

Theorem 2.5. *If $f \in H_{\overline{F}}(p, t, \alpha, k)$, then for $|z| = r < 1$, $|A_{p+t}| \leq |A_n| \leq |B_n|$, and $A_{p+t} \neq 0$,*

$$|f(z)| \leq \begin{cases} \left((1 + |b_{p+t-1}|) r^{p+t-1} \right. \\ \left. + \left(\frac{p(1-\alpha)}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} \right. \right. \\ \left. \left. - \frac{[(p+t-1)(1+k) + p(k+\alpha)]}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} |b_{p+t-1} B_{p+t-1}| \right) r^{p+1}; \quad p(1-\alpha) \leq 1 \right. \\ \left. (1 + |b_{p+t-1}|) r^{p+t-1} \right. \\ \left. + \left(\frac{1}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} \right. \right. \\ \left. \left. - \frac{[(p+t-1)(1+k) + p(k+\alpha)]}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} |b_{p+t-1} B_{p+t-1}| \right) r^{p+1}; \quad p(1-\alpha) \geq 1, \right. \end{cases}$$

$$|f(z)| \geq \begin{cases} \left((1 - |b_{p+t-1}|) r^{p+t-1} \right. \\ \left. - \left(\frac{p(1-\alpha)}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} \right. \right. \\ \left. \left. - \frac{[(p+t-1)(1+k) + p(k+\alpha)]}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} |b_{p+t-1} B_{p+t-1}| \right) r^{p+1}; \quad p(1-\alpha) \leq 1 \right. \\ \left. (1 - |b_{p+t-1}|) r^{p+t-1} \right. \\ \left. - \left(\frac{1}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} \right. \right. \\ \left. \left. - \frac{[(p+t-1)(1+k) + p(k+\alpha)]}{[(p+t)(1+k) - p(k+\alpha)] |A_{p+t}|} |b_{p+t-1} B_{p+t-1}| \right) r^{p+1}; \quad p(1-\alpha) \geq 1. \right. \end{cases} \quad (2.13)$$

These bounds are sharp.

Proof. Suppose $p(1 - \alpha) \leq 1$. Let $f \in H_{\bar{F}}(p, t, \alpha, k)$ and $|A_{p+t}| \leq |B_n|$. In view of (1.13), we get

$$\begin{aligned}
 |f(z)| &= \left| z^p + |b_{p+t-1}|(\bar{z})^{p+t-1} - \sum_{n=p+t}^{\infty} (|a_n|z^n - |b_n|\bar{z}^n) \right| \\
 &\leq (1 + |b_{p+t-1}|)r^{p+t-1} + \sum_{n=p+t}^{\infty} (|a_n| + |b_n|)r^{p+1} \\
 &\leq (1 + |b_{p+t-1}|)r^{p+t-1} + \frac{p(1 - \alpha)}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} \\
 &\quad \times \sum_{n=p+t}^{\infty} \frac{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|}{p(1 - \alpha)} (|a_n| + |b_n|)r^{p+1} \\
 &\leq (1 + |b_{p+t-1}|)r^{p+t-1} + \frac{p(1 - \alpha)}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} \\
 &\quad \times \left(\sum_{n=p+t}^{\infty} \frac{[n(1+k) - p(k+\alpha)]|A_{p+t}|}{p(1 - \alpha)} |a_n| + \frac{[n(1+k) + p(k+\alpha)]|A_{p+t}|}{p(1 - \alpha)} |b_n| \right) r^{p+1} \\
 &\leq (1 + |b_{p+t-1}|)r^{p+t-1} + \frac{p(1 - \alpha)}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} \\
 &\quad \times \left(\sum_{n=p+t}^{\infty} \frac{[n(1+k) - p(k+\alpha)]}{p(1 - \alpha)} |A_n a_n| + \frac{[n(1+k) + p(k+\alpha)]}{p(1 - \alpha)} |B_n b_n| \right) r^{p+1}.
 \end{aligned} \tag{2.14}$$

Using Theorem 2.4(i), we obtain

$$\begin{aligned}
 |f(z)| &\leq (1 + |b_{p+t-1}|)r^{p+t-1} + \frac{p(1 - \alpha)}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} \\
 &\quad \times \left(1 - \frac{[(p+t-1)(1+k) + p(k+\alpha)]}{p(1 - \alpha)} |b_{p+t-1} B_{p+t-1}| \right) r^{p+1} \\
 &= (1 + |b_{p+t-1}|)r^{p+t-1} \\
 &\quad + \left(\frac{p(1 - \alpha)}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} \right. \\
 &\quad \left. - \frac{[(p+t-1)(1+k) + p(k+\alpha)]}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} |b_{p+t-1} B_{p+t-1}| \right) r^{p+1}.
 \end{aligned} \tag{2.15}$$

The proofs of other cases are similar and so are omitted. \square

Corollary 2.6. *If $f \in H_{\overline{F}}(p, t, \alpha, k)$, then*

$$w : |w(z)| < \left\{ \begin{array}{l} 1 - \frac{p(1-\alpha)}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} \\ - \frac{[(p+t)(1+k) - p(k+\alpha) - [(p+t-1)(1+k) + p(k+\alpha)]|B_{p+t-1}|]|b_{p+t-1}|}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|}; \\ p(1-\alpha) \leq 1 \\ \\ 1 - \frac{1}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|} \\ - \frac{[(p+t)(1+k) - p(k+\alpha) - [(p+t-1)(1+k) + p(k+\alpha)]|B_{p+t-1}|]|b_{p+t-1}|}{[(p+t)(1+k) - p(k+\alpha)]|A_{p+t}|}; \\ p(1-\alpha) \geq 1 \end{array} \right\} \\ \subset f(U). \quad (2.16)$$

Theorem 2.7. *Suppose $A_n \neq 0$ ($n = p+t, n = p+t+1, \dots$) and $B_n \neq 0$ ($n = p+t-1, n = p+t, \dots$). Then $f \in \text{clco } H_{\overline{F}}(p, t, \alpha, k)$ if and only if*

$$f(z) = \sum_{n=p+t-1}^{\infty} (X_n h_n(z) + Y_n g_n(z)), \quad z \in U, \quad (2.17)$$

where

$$h_{p+t-1}(z) = z^p$$

$$h_n(z) = \begin{cases} z^p - \frac{p(1-\alpha)}{[n(1+k) - p(k+\alpha)]|A_n|} z^n; & (n = p+t, p+t+1, \dots), p(1-\alpha) \leq 1, \\ z^p - \frac{1}{[n(1+k) - p(k+\alpha)]|A_n|} z^n; & (n = p+t, p+t+1, \dots), p(1-\alpha) \geq 1, \end{cases}$$

$$g_n(z) = \begin{cases} z^p + \frac{p(1-\alpha)}{[n(1+k) + p(k+\alpha)]|B_n|} \overline{z^n}; & (n = p+t-1, p+t, \dots), p(1-\alpha) \leq 1 \\ z^p + \frac{1}{[n(1+k) + p(k+\alpha)]|B_n|} \overline{z^n}; & (n = p+t-1, p+t, \dots), p(1-\alpha) \geq 1 \end{cases}$$

$$X_{p+t-1} \equiv X_p = 1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right), \quad X_n \geq 0, Y_n \geq 0. \quad (2.18)$$

In particular, the extreme points of $H_{\overline{F}}(p, t, \alpha, k)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose $p(1 - \alpha) \leq 1$. For functions of the form (2.17), we can write

$$f(z) = z^p - \sum_{n=p+t}^{\infty} \frac{p(1-\alpha)}{[n(1+k) - p(k+\alpha)]|A_n|} X_n z^n + \sum_{n=p+t-1}^{\infty} \frac{p(1-\alpha)}{[n(1+k) + p(k+\alpha)]|B_n|} Y_n \bar{z}^n. \quad (2.19)$$

On the other hand, for $0 \leq X_p \leq 1$, we obtain

$$\begin{aligned} & \sum_{n=p+t}^{\infty} \frac{[n(1+k) - p(k+\alpha)]|A_n|}{p(1-\alpha)} \left(\frac{p(1-\alpha)}{[n(1+k) - p(k+\alpha)]|A_n|} X_n \right) \\ & + \sum_{n=p+t-1}^{\infty} \frac{[n(1+k) + p(k+\alpha)]|B_n|}{p(1-\alpha)} \left(\frac{p(1-\alpha)}{[n(1+k) + p(k+\alpha)]|B_n|} Y_n \right) \quad (2.20) \\ & = \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right) = 1 - X_p \leq 1. \end{aligned}$$

Thus $f \in H_{\bar{F}}(p, t, \alpha, k)$, by Theorem 2.4.

Conversely, suppose that $f \in H_{\bar{F}}(p, t, \alpha, k)$. Then, it follows Theorem 2.4 that

$$|a_n| \leq \frac{p(1-\alpha)}{[n(1+k) - p(k+\alpha)]|A_n|}, \quad |b_n| \leq \frac{p(1-\alpha)}{[n(1+k) + p(k+\alpha)]|B_n|}. \quad (2.21)$$

Setting

$$X_n = \frac{[n(1+k) - p(k+\alpha)]|a_n A_n|}{p(1-\alpha)}, \quad Y_n = \frac{[n(1+k) + p(k+\alpha)]|b_n B_n|}{p(1-\alpha)}, \quad (2.22)$$

and defining

$$X_p = 1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right), \quad (2.23)$$

where $X_p \geq 0$, we obtain

$$\begin{aligned}
 f(z) &= z^p - \sum_{n=p+t}^{\infty} |a_n| z^n + \sum_{n=p+t-1}^{\infty} |b_n| (\bar{z})^n \\
 &= z^p - \sum_{n=p+t}^{\infty} \frac{p(1-\alpha)X_n}{[n(1+k) - p(k+\alpha)]|A_n|} z^n + \sum_{n=p+t-1}^{\infty} \frac{p(1-\alpha)Y_n}{[n(1+k) + p(k+\alpha)]|B_n|} (\bar{z})^n \\
 &= z^p - \sum_{n=p+t}^{\infty} (z^p - h_n(z))X_n - \sum_{n=p+t-1}^{\infty} (z^p - g_n(z))Y_n \tag{2.24} \\
 &= \left(1 - \left(\sum_{n=p+t}^{\infty} X_n + \sum_{n=p+t-1}^{\infty} Y_n \right) \right) z^p + \sum_{n=p+t}^{\infty} h_n(z)X_n + \sum_{n=p+t-1}^{\infty} g_n(z)Y_n \\
 &= X_p z^p + \sum_{n=p+t}^{\infty} X_n h_n(z) + \sum_{n=p+t-1}^{\infty} Y_n g_n(z).
 \end{aligned}$$

Thus f can be expressed as (2.17). The proof for the case $p(1-\alpha) \geq 1$ is similar and hence is omitted. \square

Theorem 2.8. *The class $H_{\bar{F}}(p, t, \alpha, k)$ is closed under convex combinations.*

Proof. For $j = 1, 2, \dots$, let the functions f_j given by

$$f_j(z) = z^p - \sum_{n=p+t}^{\infty} |a_{j_n}| z^n + \sum_{n=p+t-1}^{\infty} |b_{j_n}| (\bar{z})^n \tag{2.25}$$

are in $H_{\bar{F}}(p, t, \alpha, k)$. Also suppose the given fixed harmonic functions are given by

$$F_j(z) = z^p + \sum_{n=p+t}^{\infty} |A_{j_n}| z^n + \sum_{n=p+t-1}^{\infty} |B_{j_n}| (\bar{z})^n. \tag{2.26}$$

For $\sum_{j=1}^{\infty} \mu_j = 1$, $0 \leq \mu_j \leq 1$ the convex combinations of f_j can be expressed as

$$\sum_{j=1}^{\infty} \mu_j f_j(z) = z^p - \sum_{n=p+t}^{\infty} \left(\sum_{j=1}^{\infty} \mu_j |a_{j_n}| \right) z^n + \sum_{n=p+t-1}^{\infty} \left(\sum_{j=1}^{\infty} \mu_j |b_{j_n}| \right) (\bar{z})^n. \tag{2.27}$$

Since

$$\begin{aligned} & \sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] |a_{j_n} A_{j_n}| + \sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] |b_{j_n} B_{j_n}| \\ & \leq \begin{cases} p(1-\alpha) & \text{if } p(1-\alpha) \geq 1 \\ 1 & \text{if } p(1-\alpha) \leq 1, \end{cases} \end{aligned} \quad (2.28)$$

(2.27) yields

$$\begin{aligned} & \sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] \sum_{j=1}^{\infty} \mu_j |a_{j_n} A_{j_n}| + \sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] \sum_{j=1}^{\infty} \mu_j |b_{j_n} B_{j_n}| \\ & = \sum_{j=1}^{\infty} \mu_j \left\{ \sum_{n=p+t}^{\infty} [n(1+k) - p(k+\alpha)] |a_{j_n} A_{j_n}| + \sum_{n=p+t-1}^{\infty} [n(1+k) + p(k+\alpha)] |b_{j_n} B_{j_n}| \right\} \\ & \leq \begin{cases} p(1-\alpha) \sum_{j=1}^{\infty} \mu_j = p(1-\alpha) & \text{if } p(1-\alpha) \leq 1 \\ \sum_{j=1}^{\infty} \mu_j = 1 & \text{if } p(1-\alpha) \geq 1. \end{cases} \end{aligned} \quad (2.29)$$

Thus the coefficient estimate given by Theorem 2.4 holds. Therefore, we obtain

$$\sum_{j=1}^{\infty} \mu_j f_j(z) \in H_{\overline{F}}(p, t, \alpha, k). \quad (2.30)$$

□

Acknowledgment

This present investigation is supported with the Project no. DÜBAP-07-02-21 by Dicle University, The committee of the Scientific Research Projects.

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