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Research Article

Bounds of Eigenvalues of $K_{3,3}$-Minor Free Graphs

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The spectral radius $\rho(G)$ of a graph G is the largest eigenvalue of its adjacency matrix. Let $\lambda(G)$ be the smallest eigenvalue of G. In this paper, we have described the $K_{3,3}$-minor free graphs and showed that (A) let G be a simple graph with order $n \geq 7$. If G has no $K_{3,3}$-minor, then $\rho(G) \leq 1+\sqrt{3n-8}$. (B) Let G be a simple connected graph with order $n \geq 3$. If G has no $K_{3,3}$-minor, then $\lambda(G) \geq -\sqrt{2n-4}$, where equality holds if and only if G is isomorphic to $K_{2,n-2}$.

1. Introduction

In this paper, all graphs are finite undirected graphs without loops and multiple edges. Let G be a graph with $n = n(G)$ vertices, $m = m(G)$ edges, and minimum degree δ or $\delta(G)$. The spectral radius $\rho(G)$ of G is the largest eigenvalue of its adjacency matrix. Let $\lambda(G)$ be the smallest eigenvalue of G. The join $G\nabla H$ is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. A graph H is said to be a minor of G if H can be obtained from G by deleting edges, contracting edges, and deleting isolated vertices. A graph G is H-minor free if G has no H-minor.

Brualdi and Hoffman [1] showed that the spectral radius satisfies $\rho(G) \leq k-1$, where $m = k(k-1)/2$, with equality if and only if G is isomorphic to the disjoint union of the complete graph K_k and isolated vertices. Stanley [2] improved the above result. Hong et al. [3] showed that if G is a simple connected graph then $\rho \leq (\delta - 1 + \sqrt{(\delta+1)^2 + 4(2m - n\delta)})/2$ with equality if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$. Hong [4] showed that if G is a K_5-minor free graph then (1) $\rho(G) \leq 1 + \sqrt{3n-8}$, where equality holds if and only if G is isomorphic to $K_3\nabla(n-3)K_1$; (2) $\lambda(G) \geq -\sqrt{3n-9}$, where equality holds if and only if G is isomorphic to $K_{3,n-3}(n \geq 5)$.

In this paper, we have described the $K_{3,3}$-minor free graphs and obtained that

(a) let G be a simple graph with order $n \geq 7$. If G has no $K_{3,3}$-minor, then $\rho(G) \leq 1 + \sqrt{3n-8}$;

(b) let G be a simple connected graph with order $n \geq 3$. If G has no $K_{3,3}$-minor, then $\lambda(G) \geq -\sqrt{2n-4}$, where equality holds if and only if G is isomorphic to $K_{2,n-2}$.

2. $K_{3,3}$-Minor Free Graphs

The intersection $G \cap H$ of G and H is the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. Suppose G is a connected graph and S be a minimal separating vertex set of G. Then we can write $G = G_1 \cup G_2$, where G_1 and G_2 are connected and $G_1 \cap G_2 = G(S)$. Now suppose further that $G(S)$ is a complete graph. We say that G is a k-sum of G_1 and G_2, denoted by $G \equiv G_1 \oplus G_2$, if $|S| = k$. In particular, let $G_1 \oplus_2 G_2$ denote a 2–sum of G_1 and G_2. Moreover, if G_1 or G_2(say G_1) has a separating vertex set which induces a complete graph, then we can write $G_1 = G_3 \cup G_4$ such that G_3 and G_4 are connected and $G_3 \cap G_4$ is a complete subgraph of G. We proceed like this until none of the resulting subgraphs $G_1, G_2, \cdots, G_t$ has a complete separating subgraph. The graphs $G_1, G_2, \cdots, G_t$ are called the simplical summands of G. It is easy to show that the subgraphs $G_1, G_2, \cdots, G_t$ are independent of the order in which the decomposition is carried out (see [5]).

Theorem 2.1 (see [6], D. W. Hall; K. Wagner). *A graph has no $K_{3,3}$-minor if and only if it can be obtained by 0-, 1-, 2-summing starting from planar graphs and K_5.*

A graph G is said to be a edge-maximal H-minor free graph if G has no H-minor and G' has at least an H-minor, where G' is obtained from G by joining any two nonadjacent vertices of G. A graph G is called a maximal planar graph if the planarity will be not held by joining any two nonadjacent vertices of G.

Corollary 2.2. *If G is an edge maximal $K_{3,3}$-minor free graph then it can be obtained by 2-summing starting from K_5 and edge maximal planar graphs.*

Proof. This follows from Theorem 2.1. □

Lemma 2.3. *If G_1 and G_2 are two maximal planar graphs with order $n_1 \geq 3$ and $n_2 \geq 3$, respectively, then $G_1 \oplus_2 G_2$ is not a maximal planar graph.*

Proof. We denote a planar embedding of G_i by G_i still. Since G_i is a maximal planar graph, every face boundary in G_i is a 3-cycle. Hence the outside face boundary in $G_1 \oplus_2 G_2$ is a 4-cycle, this implies that the graph $G_1 \oplus_2 G_2$ is not maximal planar.

Further, we have the following results. □

Theorem 2.4. *If G is an edge-maximal $K_{3,3}$-minor free graph with $n \geq 3$ vertices then $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, where $t = (n - n_0)/3$, G_0 is a maximal planar graph with order $2 \leq n_0 \leq n$. In particular,*

(1) *when* $n_0 = 2$, $\quad G \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, *where* $t = (n-2)/3$;

(2) *when* $n_0 = 3$, $\quad G \cong K_3 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, *where* $t = (n-3)/3$;

(3) *when* $n_0 = 4$, $\quad G \cong K_4 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, *where* $t = (n-4)/3$;

(4) *when* $n_0 = n$, $\quad G \cong G_0$ *is a maximal planar graph.*

Proof. Suppose that the graphs $G_1, G_2, \cdots, G_t (t \geq 1)$ are the simplical summands of G, namely $G \cong G_1 \oplus_2 G_2 \oplus_2 \cdots \oplus_2 G_t$. By Corollary 2.2, G_i is either a maximal planar graph or a K_5. By Lemma 2.3, there is at most a maximal planar graph in $G_i, 1 \leq i \leq t$. Hence we have $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$, where $t = (n - n_0)/3$, G_0 is a maximal planar graph with order $2 \leq n_0 \leq n$. □

Lemma 2.5 (see [7]). *Let G be a simple planar bipartite graph with $n \geq 3$ vertices and m edges. Then $m \leq 2n - 4$.*

Theorem 2.6. *Let G be a simple connected bipartite graph with $n \geq 3$ vertices and m edges. If G has no $K_{3,3}$-minor, then $m \leq 2n - 4$.*

Proof. Let H be a simple connected edge-maximal $K_{3,3}$-minor free graph with $n(H) = n(G)$ vertices and $m(H)$ edges. Suppose that the graphs $H_1, H_2, \cdots, H_t (t \geq 1)$ are the simplical summands of H. Then H_i is either a maximal planar graph or the graph K_5 by Corollary 2.2. Further, without loss generality, we may assume that G is a spanning subgraph of H. Let the graph G_i be the intersection of G and $H_i (1 \leq i \leq t)$. Then $n(G_i) = n(H_i)$ for $1 \leq i \leq t$. If $H_i \cong K_5$ then G_i is a subgraph of $K_{2,3}$, implies that $m(G_i) \leq 6 = 2n(G_i) - 4$. If H_i is a maximal planar graph then G_i is a simple planar bipartite graph, implies that $m(G_i) \leq 2n(G_i) - 4$ by Lemma 2.5. Next we prove this result by induction on t. For $t = 1$, $m = m(G) = m(G_1) \leq 2n(G_1) - 4 = 2n(G) - 4$. Now we assume it is true for $t = k$ and prove it for $t = k + 1$. Let $H' = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ and $G' = G \cap H'$. Then $m(G') \leq 2n(G') - 4$ by the induction hypothesis. $H = H' \oplus_2 H_{k+1}$. Hence $m(G) \leq m(G') + m(G_{k+1}) \leq 2(n(G') + n(G_{k+1}) - 2) - 4 = 2n(G) - 4$. □

3. Bounds of Eigenvalues of $K_{3,3}$-Minor Free Graphs

Lemma 3.1 (see [3]). *If G is a simple connected graph then $\rho \leq (\delta - 1 + \sqrt{(\delta+1)^2 + 4(2m - n\delta)})/2$ with equality if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.*

Lemma 3.2. *Let G be a simple connected graph with n vertices and m edges. If $\delta(G) \geq k$,then $\rho \leq (k - 1 + \sqrt{(k+1)^2 + 4(2m - kn)})/2$, where equality holds if and only if $\delta(G) = k$ and G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.*

Proof. Because when $n - 1 \leq m \leq n(n-1)/2$ and $2m \geq xn$, $f(x) = (x - 1 + \sqrt{(x+1)^2 + 4(2m - nx)})/2$ is a decreasing function of x for $1 \leq x \leq n - 1$, this follows from Lemma 3.1. □

Lemma 3.3. *Let G_0 be a maximal planar graph with order n_0, and let G be a graph with n vertices and m edges.*

(1) *If $G \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 5$, where $t = (n-2)/3$, then $m = 3n - 5, \delta(G) = 4$.*

(2) *If $G \cong K_3 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 6$, where $t = (n-3)/3$, then $m = 3n - 6$, $\delta(G) = 2$.*

(3) *If $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq n_0 \geq 4$, where $t = (n - n_0)/3$, then $m = 3n - 6$, $\delta(G) \geq 3$.*

Proof. Applying the properties of the maximal planar graphs, this follows by calculating. □

Lemma 3.4. *Let G_0 be a maximal planar graph with order n_0, and let G be a graph with n vertices.*

(1) *If $G \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 5$, where $t = n - 2/3$, then $\rho(G) \leq (3 + \sqrt{8n-15})/2$.*

(2) *If $G \cong K_3 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq 6$, where $t = n - 3/3$, then $\rho(G) < (3 + \sqrt{8n+1})/2$.*

(3) *If $G \cong G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}$ and $n \geq n_0 \geq 4$, where $t = n - n_0/3$, then $\rho(G) \leq 1 + \sqrt{3n-8}$.*

Proof. It follows that (1) and (3) are true by Lemma 3.2 and 5(1)(3). Next we prove that (2) is true too.

Let G^* be a graph obtained from G by expanding K_3 (in the simplcal summands of G) to K_5, such that G^* can be obtained by 2-summing K_5, namely, $G^* \cong \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t+1}$.

This implies that $\rho(G^*) \leq (3 + \sqrt{8n^* - 15})/2$ by (1). Also we have $n^* = n(G^*) = n(G) + 2 = n + 2$, so $\rho(G) < \rho(G^*) \leq (3 + \sqrt{8n+1})/2$. □

Theorem 3.5. *Let G be a simple graph with order $n \geq 7$. If G has no $K_{3,3}$-minor, then $\rho(G) \leq 1 + \sqrt{3n-8}$.*

Proof. Since when adding an edge in G the spectral radius $\rho(G)$ is strict increasing, we consider the edge-maximal $K_{3,3}$-minor free graph only. Next we may assume that G is an edge-maximal $K_{3,3}$-minor free graph.

By Theorem 2.4 and Lemma 3.4, when $n \geq 4$, $\rho(G) \leq \max\{(1 + \sqrt{3n-8}), (3 + (\sqrt{8n-15})/2),\ 3 + (\sqrt{8n+1}/2)\}$.

When $n \geq 14$, $1 + \sqrt{3n-8} > max\{3 + (\sqrt{8n-15})/2,\ (3 + \sqrt{8n+1})/2\}$.

When $7 \leq n \leq 13$, we have $\rho(G) \leq \rho(G_0 \oplus_2 \underbrace{K_5 \oplus_2 \cdots \oplus_2 K_5}_{t}) \leq 1 + \sqrt{3n-8}$ by calculating directly, where $t = (n - n_0)/3$, G_0 is a maximal planar graph with order $2 \leq n_0 \leq n$ (see Theorem 2.4).

Therefore when $n \geq 7$, $\rho(G) \leq 1 + \sqrt{3n-8}$. □

Remark 3.6. In Theorem 3.5, the equality holds only if $n = 8$, for the others, the upper bounds of $\rho(G)$ are not sharp. We conjecture that the best bound of $\rho(G)$ is $(3 + \sqrt{8n-15})/2$ still.

Lemma 3.7 (see [7]). *If G is a simple connected graph with n vertices, then there exists a connected bipartite subgraph H of G such that $\lambda(G) \geq \lambda(H)$ with equality holding if and only if $G \cong H$.*

Lemma 3.8 (see [7]). *If G is a connected bipartite graph with n vertices and m edges, then $\lambda(G) \geq -\sqrt{m}$, where equality holds if and only if G is a complete bipartite graph.*

Theorem 3.9. *Let G be a simple connected graph with $n \geq 3$ vertices. If G has no $K_{3,3}$-minor, then $\lambda(G) \geq -\sqrt{2n-4}$, where equality holds if and only if G is isomorphic to $K_{2,n-2}$.*

Proof. This follows from Lemmas 3.7, 3.8 and Theorem 2.6. □

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