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Research Article

Iterative Schemes for Generalized Equilibrium Problem and Two Maximal Monotone Operators

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The purpose of this paper is to introduce and study two new hybrid proximal-point algorithms for finding a common element of the set of solutions to a generalized equilibrium problem and the sets of zeros of two maximal monotone operators in a uniformly smooth and uniformly convex Banach space. We established strong and weak convergence theorems for these two modified hybrid proximal-point algorithms, respectively.

1. Introduction

Let X be a real Banach space and X^* its dual space. The normalized duality mapping $J : X \to 2^{X^*}$ is defined as

$$J(x) := \left\{ x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in X, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. Recall that if X is a smooth Banach space then J is singlevalued. Throughout this paper, we will still denote by J the single-valued normalized duality mapping. Let C be a nonempty closed convex subset of X, f a bifunction from $C\times C$ to R, and $A : C \to X^*$ a nonlinear mapping. The generalized equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by EP. Problem (1.2) and similar problems have been extensively studied; see, for example, [1–11]. Whenever $A = 0$, problem (1.2) reduces to the equilibrium problem of finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $EP(f)$. Whenever $f = 0$, problem (1.2) reduces to the variational inequality problem of finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

Whenever $X = H$ a Hilbert space, problem (1.2) was very recently introduced and considered by Kamimura and Takahashi [12]. Problem (1.2) is very general in the sense that it includes, as spacial cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and others; see, for example, [13, 14]. A mapping $S : C \to X$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. Iterative schemes for finding common elements of EP and fixed points set of nonexpansive mappings have been studied recently; see, for example, [12, 15–17] and the references therein.

On the other hand, a classical method of solving $0 \in Tx$ in a Hilbert space H is the proximal point algorithm which generates, for any starting point $x_0 \in H$, a sequence $\{x_n\}$ in H by the iterative scheme

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where $\{r_n\}$ is a sequence in $(0, \infty)$, $J_r = (I + rT)^{-1}$ for each $r > 0$ is the resolvent operator for T, and I is the identity operator on H. This algorithm was first introduced by Martinet [14] and generally studied by Rockafellar [18] in the framework of a Hilbert space H. Later many authors studied (1.5) and its variants in a Hilbert space H or in a Banach space X; see, for example, [13, 19–23] and the references therein.

Let X be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let f be a bifunction from $C \times C$ to R satisfying the following conditions (A1)–(A4) which were imposed in [24]:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Let $T : X \to 2^{X^*}$ be a maximal monotone operator such that

(A5) $T^{-1}0 \cap EP(f) \neq \emptyset$.

The purpose of this paper is to introduce and study two new iterative algorithms for finding a common element of the set EP of solutions for the generalized equilibrium problem (1.2) and the set $T^{-1}0 \cap \widetilde{T}^{-1}0$ for maximal monotone operators $T, \widetilde{T}$ in a uniformly smooth and uniformly convex Banach space X. First, motivated by Kamimura and Takahashi

[12, Theorem 3.1], Ceng et al. [16, Theorem 3.1], and Zhang [17, Theorem 3.1], we introduce a sequence $\{x_n\}$ that, under some appropriate conditions, is strongly convergent to $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$ in Section 3. Second, inspired by Kamimura and Takahashi [12, Theorem 3.1], Ceng et al. [16, Theorem 4.1], and Zhang [17, Theorem 3.1], we define a sequence weakly convergent to an element $z \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$, where $z = \lim_{n\to\infty}\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_n$ in Section 4. Our results represent a generalization of known results in the literature, including Takahashi and Zembayashi [15], Kamimura and Takahashi [12], Li and Song [22], Ceng and Yao [25], and Ceng et al. [16]. In particular, compared with Theorems 3.1 and 4.1 in [16], our results (i.e., Theorems 3.2 and 4.2 in this paper) extend the problem of finding an element of $T^{-1}0 \cap EP(f)$ to the one of finding an element of $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$. Meantime, the algorithms in this paper are very different from those in [16] (because of considering the complexity involving the problem of finding an element of $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$).

2. Preliminaries

In the sequel, we denote the strong convergence, weak convergence and weak* convergence of a sequence $\{x_n\}$ to a point $x \in X$ by $x_n \to x$, $x_n \rightharpoonup x$ and $x_n \overset{*}{\rightharpoonup} x$, respectively.

A Banach space X is said to be strictly convex, if $\|x+y\|/2 < 1$ for all $x, y \in U = \{z \in X : \|z\| = 1\}$ with $x \neq y$. X is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that $\|x+y\|/2 \le 1-\delta$ for all $x, y \in U$ with $\|x-y\| \ge \epsilon$. Recall that each uniformly convex Banach space has the Kadec-Klee property, that is,

$$\left.\begin{array}{c} x_n \rightharpoonup x \\ \|x_n\| \longrightarrow \|x\| \end{array}\right\} \Longrightarrow x_n \longrightarrow x \tag{2.1}$$

The proof of the main results of Sections 3 and 4 will be based on the following assumption.

Assumption A. Let X be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let f be a bifunction from $C \times C$ to R satisfying the same conditions (A1)–(A4) as in Section 1. Let $T, \widetilde{T} : X \to 2^{X^*}$ be two maximal monotone operators such that

(A5)′ $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP \neq \emptyset$.

Recall that if C is a nonempty closed convex subset of a Hilbert space H, then the metric projection $P_C : H \to C$ of H onto C is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. In this connection, Alber [26] recently introduced a generalized projection operator Π_C in a Banach space X which is an analogue of the metric projection in Hilbert spaces. Consider the functional defined as in [26] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{2.2}$$

It is clear that in a Hilbert space H, (2.2) reduces to $\phi(x, y) = \|x-y\|^2$, $\forall x, y \in H$.

The generalized projection $\Pi_C : X \to C$ is a mapping that assigns to an arbitrary point $x \in X$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.3}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [27]). In a Hilbert space, $\Pi_C = P_C$. From [26], in a smooth strictly convex and reflexive Banach space X, we have

$$\left(\|y\| - \|x\|\right)^2 \le \phi(y, x) \le \left(\|y\| + \|x\|\right)^2, \quad \forall x, y \in X. \tag{2.4}$$

Moreover, by the property of subdifferential of convex functions, we easily get the following inequality:

$$\phi(x, y) \le \phi\Big(x, J^{-1}(Jy + Jz)\Big) - 2\langle y - x, Jz\rangle, \quad \forall x, y, z \in X. \tag{2.5}$$

Let S be a mapping from C into itself. A point p in C is called an asymptotically fixed point of S if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Sx_n - x_n \to 0$ [28]. The set of asymptotically fixed points of S will be denoted by $\widehat{F}(S)$. A mapping C from S into itself is called relatively nonexpansive if $\widehat{F}(S) = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$, for all $x \in C$ and $p \in F(S)$ [15].

Observe that, if X is a reflexive strictly convex and smooth Banach space, then for any $x, y \in X$, $\phi(x, y) = 0$ if and only if $x = y$. To this end, it is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. Actually, from (2.4), we have $\|x\| = \|y\|$ which implies that $\langle x, Jy\rangle = \|x\|^2 = \|y\|^2$. From the definition of J, we have $Jx = Jy$ and therefore, $x = y$; see [29] for more details.

We need the following lemmas for the proof of our main results.

Lemma 2.1 (Kamimura and Takahashi [12]). *Let X be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2 (Alber [26], Kamimura and Takahashi [12]). *Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space X. Let $x \in X$ and let $z \in C$. Then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz\rangle \le 0, \quad \forall y \in C. \tag{2.6}$$

Lemma 2.3 (Alber [26], Kamimura and Takahashi [12]). *Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space X. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y), \quad \forall x \in C,\ y \in X. \tag{2.7}$$

Lemma 2.4 (Rockafellar [18]). *Let X be a reflexive strictly convex and smooth Banach space and let $T : X \to 2^{X^*}$ be a multivalued operator. Then there hold the following hold:*

(i) *$T^{-1}0$ is closed and convex if T is maximal monotone such that $T^{-1}0 \neq \emptyset$;*

(ii) *T is maximal monotone if and only if T is monotone with $R(J + rT) = X^*$ for all $r > 0$.*

Lemma 2.5 (Xu [30]). *Let X be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|), \tag{2.8}$$

for all $x, y \in B_r$ and $t \in [0,1]$, where $B_r = \{z \in X : \|z\| \le r\}$.

Lemma 2.6 (Kamimura and Takahashi [12]). *Let X be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \le \phi(x,y), \quad \forall x, y \in B_r. \tag{2.9}$$

The following result is due to Blum and Oettli [24].

Lemma 2.7 (Blum and Oettli [24]). *Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space X, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$ and $x \in X$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C. \tag{2.10}$$

Motivated by Combettes and Hirstoaga [31] in a Hilbert space, Takahashi and Zembayashi [15] established the following lemma.

Lemma 2.8 (Takahashi and Zembayashi [15]). *Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space X, and let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4). For $r > 0$ and $x \in X$, define a mapping $T_r : X \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0,\ \forall y \in C \right\} \tag{2.11}$$

for all $x \in X$. Then, the following hold:

(i) *T_r is singlevalued;*

(ii) *T_r is a firmly nonexpansive-type mapping, that is, for all $x, y \in X$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.12}$$

(iii) $F(T_r) = \widehat{F}(T_r) = EP(f)$;

(iv) $EP(f)$ *is closed and convex.*

Using Lemma 2.8, one has the following result.

Lemma 2.9 (Takahashi and Zembayashi [15]). *Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space X, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$. Then, for $x \in X$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.13}$$

Utilizing Lemmas 2.7, 2.8 and 2.9 as previously mentioned, Zhang [17] derived the following result.

Proposition 2.10 (Zhang [21, Lemma]). *Let X be a smooth strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of X. Let $A : C \to X^*$ be an α -inverse-strongly monotone mapping, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$. Then the following hold:*

for $x \in X$, there exists $u \in C$ such that

$$f(u, y) + \langle Au, y - u \rangle + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C, \tag{2.14}$$

if X is additionally uniformly smooth and $K_r : X \to C$ is defined as

$$K_r(x) = \left\{ u \in C : f(u, y) + \langle Au, y - u \rangle + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0,\ \forall y \in C \right\}, \quad \forall x \in X, \tag{2.15}$$

then the mapping K_r has the following properties:

(i) K_r *is singlevalued,*

(ii) K_r *is a firmly nonexpansive-type mapping, that is,*

$$\langle K_r x - K_r y, JK_r x - JK_r y \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle, \quad \forall x, y \in X, \tag{2.16}$$

(iii) $F(K_r) = \widehat{F}(K_r) = EP$,

(iv) EP *is a closed convex subset of* C,

(v) $\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x)$, *for all* $p \in F(K_r)$.

Let $T, \widetilde{T} : X \to 2^{X^*}$ be two maximal monotone operators in a smooth Banach space X. We denote the resolvent operators of T and $\widetilde{T}$ by $J_r = (J + rT)^{-1}J$ and $\widetilde{J}_r = (J + r\widetilde{T})J$ for each $r > 0$, respectively. Then $J_r : X \to D(T)$ and $\widetilde{J}_r : X \to D(\widetilde{T})$ are two single-valued mappings. Also, $T^{-1}0 = F(J_r)$ and $\widetilde{T}^{-1}0 = F(\widetilde{J}_r)$ for each $r > 0$, where $F(J_r)$ and $F(\widetilde{J}_r)$ are the sets of fixed points of J_r and $\widetilde{J}_r$, respectively. For each $r > 0$, the Yosida approximations of T and $\widetilde{T}$ are defined by $A_r = (J - JJ_r)/r$ and $\widetilde{A}_r = (J - J\widetilde{J}_r)/r$, respectively. It is known that

$$A_r x \in T(J_r x), \quad \widetilde{A}_r x \in \widetilde{T}\left(\widetilde{J}_r x\right), \quad \forall r > 0,\ x \in X. \tag{2.17}$$

Lemma 2.11 (Kohsaka and Takahashi [13]). *Let X be a reflexive strictly convex and smooth Banach space and let $T : X \to 2^{X^*}$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$. Then*

$$\phi(z, J_r x) + \phi(J_r x, x) \leq \phi(z, x), \quad \forall r > 0,\ z \in T^{-1}0,\ x \in X. \tag{2.18}$$

Lemma 2.12 (Tan and Xu [32]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality: $a_{n+1} \leq a_n + b_n$ for all $n \geq 0$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem and the set $T^{-1}0 \cap \widetilde{T}^{-1}0$ for two maximal monotone operators T and $\widetilde{T}$.

Lemma 3.1. *Let X be a reflexive strictly convex and smooth Banach space and let $T : X \to 2^{X^*}$ be a maximal monotone operator. Then for each $r \in (0, \infty)$, the following holds:*

$$\langle Ju - Jv, J_r u - J_r v \rangle \geq \langle JJ_r u - JJ_r v, J_r u - J_r v \rangle, \quad \forall u, v \in X, \tag{3.1}$$

where $J_r = (J + rT)^{-1} J$ and J is the duality mapping on X. In particular, whenever $X = H$ a real Hilbert space, J_r is a nonexpansive mapping on H.

Proof. Since for each $u, v \in X$

$$J_r u = (J + rT)^{-1} Ju, \qquad J_r v = (J + rT)^{-1} Jv, \tag{3.2}$$

we have that

$$\frac{1}{r} \cdot (Ju - JJ_r u) \in TJ_r u, \qquad \frac{1}{r} \cdot (Jv - JJ_r v) \in TJ_r v. \tag{3.3}$$

Thus, from the monotonicity of T it follows that

$$\left\langle \frac{1}{r} \cdot (Ju - JJ_r u) - \frac{1}{r} \cdot (Jv - JJ_r v), J_r u - J_r v \right\rangle \geq 0, \tag{3.4}$$

and hence

$$\langle Ju - Jv, J_r u - J_r v \rangle \geq \langle JJ_r u - JJ_r v, J_r u - J_r v \rangle. \tag{3.5}$$

□

Theorem 3.2. *Suppose that Assumption A is fulfilled and let $x_0 \in X$ be chosen arbitrarily. Consider the sequence*

$$x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \tag{3.6}$$

where

$$
\begin{aligned}
H_n &= \Big\{ z \in C : \phi(z, K_{r_n} y_n) \le \Big[\alpha_n + \widetilde{\beta}_n - \alpha_n \widetilde{\beta}_n - \widetilde{\alpha}_n \widetilde{\beta}_n + \alpha_n \widetilde{\alpha}_n \widetilde{\beta}_n\Big] \phi(z, x_0) \\
&\qquad + \Big[(1-\alpha_n)(1-\widetilde{\alpha}_n)\Big(1-\widetilde{\beta}_n\Big) + \widetilde{\alpha}_n (1-\alpha_n)\Big] \phi(z, x_n) \Big\}, \\
W_n &= \{ z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0 \}, \\
\widetilde{x}_n &= J^{-1}(\alpha_n J x_0 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J J_{r_n} x_n)), \\
y_n &= J^{-1}\Big(\widetilde{\alpha}_n J \widetilde{x}_n + (1-\widetilde{\alpha}_n) J \widetilde{J}_{r_n} J^{-1}\Big(\widetilde{\beta}_n J x_0 + \Big(1-\widetilde{\beta}_n\Big) J \widetilde{x}_n\Big)\Big),
\end{aligned}
\tag{3.7}
$$

K_r is defined by (2.15)*, $\{\alpha_n\}, \{\beta_n\}, \{\widetilde{\alpha}_n\}, \{\widetilde{\beta}_n\} \subset [0,1]$ satisfy*

$$
\lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \widetilde{\beta}_n = 0, \qquad \liminf_{n\to\infty} \beta_n (1-\beta_n) > 0, \qquad \liminf_{n\to\infty} \widetilde{\alpha}_n (1-\widetilde{\alpha}_n) > 0, \tag{3.8}
$$

and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP} x_0$ provided $\|\widetilde{J}_{r_n} v_n - \widetilde{J}_{r_n} \widetilde{x}_n\| \to 0$ for any sequence $\{v_n\} \subset X$ with $\|v_n - \widetilde{x}_n\| \to 0$, where $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}$ is the generalized projection of X onto $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$.

Remark 3.3. In Theorem 3.2, if $X = H$ a real Hilbert space, then $\{\widetilde{J}_{r_n}\}$ is a sequence of nonexpansive mappings on H. This implies that as $n \to \infty$,

$$
\Big\| \widetilde{J}_{r_n} v_n - \widetilde{J}_{r_n} \widetilde{x}_n \Big\| \le \| v_n - \widetilde{x}_n \| \longrightarrow 0. \tag{3.9}
$$

In this case, we can remove the requirement that $\|\widetilde{J}_{r_n} v_n - \widetilde{J}_{r_n} \widetilde{x}_n\| \to 0$ for any sequence $\{v_n\} \subset X$ with $\|v_n - \widetilde{x}_n\| \to 0$.

Proof of Theorem 3.2. For the sake of simplicity, we define

$$
u_n := K_{r_n} y_n, \qquad z_n := J^{-1}(\beta_n J x_n + (1-\beta_n) J J_{r_n} x_n), \qquad \widetilde{z}_n := \widetilde{J}_{r_n} J^{-1}\Big(\widetilde{\beta}_n J x_0 + \Big(1-\widetilde{\beta}_n\Big) J \widetilde{x}_n\Big), \tag{3.10}
$$

so that

$$
\widetilde{x}_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n), \qquad y_n = J^{-1}(\widetilde{\alpha}_n J \widetilde{x}_n + (1-\widetilde{\alpha}_n) J \widetilde{z}_n). \tag{3.11}
$$

We divide the proof into several steps.

Step 1. We claim that $H_n \cap W_n$ is closed and convex for each $n \ge 0$.

Indeed, it is obvious that H_n is closed and W_n is closed and convex for each $n \ge 0$. Let us show that H_n is convex. For $z_1, z_2 \in H_n$ and $t \in (0,1)$, put $z = tz_1 + (1-t)z_2$. It is sufficient to show that $z \in H_n$. We first write $\gamma_n = \alpha_n + \widetilde{\beta}_n - \alpha_n\widetilde{\beta}_n - \widetilde{\alpha}_n\widetilde{\beta}_n + \alpha_n\widetilde{\alpha}_n\widetilde{\beta}_n$ for each $n \geq 0$. Next, we prove that

$$\phi(z, u_n) \leq \gamma_n\phi(z, x_0) + (1-\gamma_n)\phi(z, x_n) \tag{3.12}$$

is equivalent to

$$2\gamma_n\langle z, Jx_0\rangle + 2(1-\gamma_n)\langle z, Jx_n\rangle - 2\langle z, Ju_n\rangle \leq \gamma_n\|x_0\|^2 + (1-\gamma_n)\|x_n\|^2 - \|u_n\|^2. \tag{3.13}$$

Indeed, from (2.4) we deduce that the following equations hold:

$$\begin{aligned}
\phi(z, x_0) &= \|z\|^2 - 2\langle z, Jx_0\rangle + \|x_0\|^2,\\
\phi(z, x_n) &= \|z\|^2 - 2\langle z, Jx_n\rangle + \|x_n\|^2,\\
\phi(z, u_n) &= \|z\|^2 - 2\langle z, Ju_n\rangle + \|u_n\|^2,
\end{aligned} \tag{3.14}$$

which combined with (3.12) yield that (3.12) is equivalent to (3.13). Thus we have

$$\begin{aligned}
&2\gamma_n\langle z, Jx_0\rangle + 2(1-\gamma_n)\langle z, Jx_n\rangle - 2\langle z, Ju_n\rangle\\
&\quad= 2\gamma_n\langle tz_1 + (1-t)z_2, Jx_0\rangle\\
&\qquad + 2(1-\gamma_n)\langle tz_1 + (1-t)z_2, Jx_n\rangle\\
&\qquad - 2\langle tz_1 + (1-t)z_2, Ju_n\rangle\\
&\quad= 2t\gamma_n\langle z_1, Jx_0\rangle + 2(1-t)\gamma_n\langle z_2, Jx_0\rangle + 2(1-\gamma_n)t\langle z_1, Jx_n\rangle\\
&\qquad + 2(1-\gamma_n)(1-t)\langle z_2, Jx_n\rangle - 2t\langle z_1, Ju_n\rangle - 2(1-t)\langle z_2, Ju_n\rangle\\
&\quad\leq \gamma_n\|x_0\|^2 + (1-\gamma_n)\|x_n\|^2 - \|u_n\|^2.
\end{aligned} \tag{3.15}$$

This implies that $z \in H_n$. Therefore, H_n is closed and convex.

Step 2. We claim that $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$ for each $n \geq 0$ and that $\{x_n\}$ is well defined. Indeed, take $w \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$ arbitrarily. Note that $u_n = K_{r_n}y_n$ is equivalent to

$$u_n \in C \quad \textit{such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C. \tag{3.16}$$

Then from Lemma 2.11 we obtain

$$\begin{aligned}
\phi(w,z_n) &= \phi\Big(w, J^{-1}(\beta_n J x_n + (1-\beta_n) J J_{r_n} x_n)\Big) \\
&= \|w\|^2 - 2\langle w, \beta_n J x_n + (1-\beta_n) J J_{r_n} x_n\rangle + \big\|\beta_n J x_n + (1-\beta_n) J J_{r_n} x_n\big\|^2 \\
&\le \|w\|^2 - 2\beta_n\langle w, Jx_n\rangle - 2(1-\beta_n)\langle w, J J_{r_n} x_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n} x_n\|^2 \\
&= \beta_n\phi(w,x_n) + (1-\beta_n)\phi(w, J_{r_n} x_n) \\
&\le \beta_n\phi(w,x_n) + (1-\beta_n)\phi(w,x_n) = \phi(w,x_n),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\phi(w,\tilde{x}_n) &= \phi\Big(w, J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n)\Big) \\
&= \|w\|^2 - 2\langle w, \alpha_n J x_0 + (1-\alpha_n) J z_n\rangle + \|\alpha_n J x_0 + (1-\alpha_n) J z_n\|^2 \\
&\le \|w\|^2 - 2\alpha_n\langle w, Jx_0\rangle - 2(1-\alpha_n)\langle w, J z_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n\phi(w,x_0) + (1-\alpha_n)\phi(w,z_n) \\
&\le \alpha_n\phi(w,x_0) + (1-\alpha_n)\phi(w,x_n).
\end{aligned} \tag{3.18}$$

Moreover, we have

$$\begin{aligned}
\phi(w,\tilde{z}_n) &= \phi\Big(w, \tilde{J}_{r_n} J^{-1}\Big(\tilde{\beta}_n J x_0 + \Big(1-\tilde{\beta}_n\Big) J\tilde{x}_n\Big)\Big) \\
&\le \phi\Big(w, J^{-1}\Big(\tilde{\beta}_n J x_0 + \Big(1-\tilde{\beta}_n\Big)\tilde{x}_n\Big)\Big) \\
&= \|w\|^2 - 2\Big\langle w, \tilde{\beta}_n J x_0 + \Big(1-\tilde{\beta}_n\Big) J\tilde{x}_n\Big\rangle + \Big\|\tilde{\beta}_n J x_0 + \Big(1-\tilde{\beta}_n\Big) J\tilde{x}_n\Big\|^2 \\
&\le \|w\|^2 - 2\tilde{\beta}_n\langle w, Jx_0\rangle - 2\Big(1-\tilde{\beta}_n\Big)\langle w, J\tilde{x}_n\rangle + \tilde{\beta}_n\|x_0\|^2 + \Big(1-\tilde{\beta}_n\Big)\|\tilde{x}_n\|^2 \\
&= \tilde{\beta}_n\phi(w,x_0) + \Big(1-\tilde{\beta}_n\Big)\phi(w,\tilde{x}_n) \\
&\le \tilde{\beta}_n\phi(w,x_0) + \Big(1-\tilde{\beta}_n\Big)\left[\alpha_n\phi(w,x_0) + (1-\alpha_n)\phi(w,x_n)\right] \\
&= \Big[\tilde{\beta}_n + \Big(1-\tilde{\beta}_n\Big)\alpha_n\Big]\phi(w,x_0) + \Big(1-\tilde{\beta}_n\Big)(1-\alpha_n)\phi(w,x_n), \\
\phi(w,y_n) &= \phi\Big(w, J^{-1}(\tilde{\alpha}_n J\tilde{x}_n + (1-\tilde{\alpha}_n) J\tilde{z}_n)\Big) \\
&\le \|w\|^2 - 2\tilde{\alpha}_n\langle w, J\tilde{x}_n\rangle - 2(1-\tilde{\alpha}_n)\langle w, J\tilde{z}_n\rangle + \tilde{\alpha}_n\|\tilde{x}_n\|^2 + (1-\tilde{\alpha}_n)\|\tilde{z}_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\alpha}_n \phi(w, \tilde{x}_n) + (1 - \tilde{\alpha}_n)\phi(w, \tilde{z}_n) \\
&\le \tilde{\alpha}_n \left[\alpha_n \phi(w, x_0) + (1 - \alpha_n)\phi(w, x_n)\right] \\
&\quad + (1 - \tilde{\alpha}_n)\left\{\left[\tilde{\beta}_n + \left(1 - \tilde{\beta}_n\right)\alpha_n\right]\phi(w, x_0) + \left(1 - \tilde{\beta}_n\right)(1 - \alpha_n)\phi(w, x_n)\right\} \\
&= \left[\alpha_n + \tilde{\beta}_n - \alpha_n\tilde{\beta}_n - \tilde{\alpha}_n\tilde{\beta}_n + \alpha_n\tilde{\alpha}_n\tilde{\beta}_n\right]\phi(w, x_0) \\
&\quad + \left[(1 - \alpha_n)(1 - \tilde{\alpha}_n)\left(1 - \tilde{\beta}_n\right) + \tilde{\alpha}_n(1 - \alpha_n)\right]\phi(w, x_n) \\
&= \gamma_n \phi(w, x_0) + (1 - \gamma_n)\phi(w, x_n),
\end{aligned} \tag{3.19}
$$

where $\gamma_n = \alpha_n + \tilde{\beta}_n - \alpha_n\tilde{\beta}_n - \tilde{\alpha}_n\tilde{\beta}_n + \alpha_n\tilde{\alpha}_n\tilde{\beta}_n$. So $w \in H_n$ for all $n \ge 0$. Now, let us show that

$$
T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset W_n \quad \forall n \ge 0. \tag{3.20}
$$

We prove this by induction. For $n = 0$, we have $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset C = W_0$. Assume that $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset W_n$. Since x_{n+1} is the projection of x_0 onto $H_n \cap W_n$, by Lemma 2.2 we have

$$
\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall z \in H_n \cap W_n. \tag{3.21}
$$

As $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP$. This, together with the definition of W_{n+1} implies that $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset W_{n+1}$. Hence (3.20) holds for all $n \ge 0$. So, $T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$ for all $n \ge 0$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We claim that $\{x_n\}$ is bounded and that $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$.

Indeed, it follows from the definition of W_n that $x_n = \Pi_{W_n} x_0$. Since $x_n = \Pi_{W_n} x_0$ and $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in W_n$, so $\phi(x_n, x_0) \le \phi(x_{n+1}, x_0)$ for all $n \ge 0$; that is, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from $x_n = \Pi_{W_n} x_0$ and Lemma 2.3 that

$$
\phi(x_n, x_0) = \phi(\Pi_{W_n} x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0) \tag{3.22}
$$

for each $p \in T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP \subset W_n$ for each $n \ge 0$. Therefore, $\{\phi(x_n, x_0)\}$ is bounded which implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Since

$$
(\|x_n\| - \|x_0\|)^2 \le \phi(x_n, x_0) \le (\|x_n\| + \|x_0\|)^2, \quad \forall n \ge 0, \tag{3.23}
$$

so $\{x_n\}$ is bounded. From Lemma 2.3, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{W_n} x_0) \le \phi(x_{n+1}, x_0) - \phi(\Pi_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0),\end{aligned} \tag{3.24}$$

for each $n \ge 0$. This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.25}$$

Step 4. We claim that $\lim_{n\to\infty}\|x_n - u_n\| = 0$, $\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0$, and $\lim_{n\to\infty}\|\tilde{x}_n - \tilde{J}_{r_n}\tilde{x}_n\| = 0$.

Indeed, from $x_{n+1} = \Pi_{H_n\cap W_n} x_0 \in H_n$, we have

$$\begin{aligned}\phi(x_{n+1}, u_n) \le{}& \left[\alpha_n + \tilde{\beta}_n - \alpha_n\tilde{\beta}_n - \tilde{\alpha}_n\tilde{\beta}_n + \alpha_n\tilde{\alpha}_n\tilde{\beta}_n\right]\phi(x_{n+1}, x_0) \\ &+ \left[(1-\alpha_n)(1-\tilde{\alpha}_n)\left(1-\tilde{\beta}_n\right) + \tilde{\alpha}_n(1-\alpha_n)\right]\phi(x_{n+1}, x_n)\end{aligned} \tag{3.26}$$

for all $n \ge 0$. Therefore, from $\alpha_n \to 0$, $\tilde{\beta}_n \to 0$ and $\phi(x_{n+1}, x_n) \to 0$, it follows that $\lim_{n\to\infty}\phi(x_{n+1}, u_n) = 0$. Since $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = \lim_{n\to\infty}\phi(x_{n+1}, u_n) = 0$ and X is uniformly convex and smooth, we have from Lemma 2.1 that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_{n+1} - u_n\| = 0, \tag{3.27}$$

and therefore, $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded subsets of X and $x_n - u_n \to 0$, then $\lim_{n\to\infty}\|Jx_n - Ju_n\| = 0$.

Let us set $\Omega := T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP$. Then, according to Lemma 2.4 and Proposition 2.10, we know that Ω is a nonempty closed convex subset of X such that $\Omega \subset C$. Fix $u \in \Omega$ arbitrarily. As in the proof of Step 2 we can show that $\phi(u, z_n) \le \phi(u, x_n)$, $\phi(u, \tilde{x}_n) \le \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, x_n)$, $\phi(u, \tilde{z}_n) \le \tilde{\beta}_n\phi(u, x_0) + (1-\tilde{\beta}_n)\phi(u, \tilde{x}_n)$, $\phi(u, y_n) \le \tilde{\alpha}_n\phi(u, \tilde{x}_n) + (1-\tilde{\alpha}_n)\phi(u, \tilde{z}_n)$, and $\phi(u, u_n) \le \tilde{\alpha}_n\phi(u, \tilde{x}_n) + (1-\tilde{\alpha}_n)\phi(u, \tilde{z}_n)$. Hence it follows from the boundedness of $\{x_n\}$ that $\{z_n\}, \{\tilde{x}_n\}, \{\tilde{z}_n\}, \{y_n\}$, and $\{u_n\}$ are also bounded. Let $r = \sup\{\|x_n\|, \|\tilde{x}_n\|, \|J_{r_n}x_n\|, \|\tilde{z}_n\| : n \ge 0\}$. Since X is a uniformly smooth Banach space, we know that X^* is a uniformly convex Banach space. Therefore, by Lemma 2.5 there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that

$$\left\|\alpha x^* + (1-\alpha)y^*\right\|^2 \le \alpha\|x^*\|^2 + (1-\alpha)\left\|y^*\right\|^2 - \alpha(1-\alpha)g\left(\left\|x^* - y^*\right\|\right), \tag{3.28}$$

for $x^*, y^* \in B_r^*$ and $\alpha \in [0,1]$. So, we have that

$$\begin{aligned}
\phi(u, z_n) &= \phi\Big(u, J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n)\Big) \\
&= \|u\|^2 - 2\langle u, \beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\rangle + \left\|\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\right\|^2 \\
&\leq \|u\|^2 - 2\beta_n\langle u, Jx_n\rangle - 2(1-\beta_n)\langle u, JJ_{r_n}x_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n}x_n\|^2 - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&= \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, J_{r_n}x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&\leq \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&= \phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|),
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
\phi(u, \widetilde{z}_n) &= \phi\Big(u, \widetilde{J}_{r_n}J^{-1}\Big(\widetilde{\beta}_n Jx_0 + \Big(1-\widetilde{\beta}_n\Big)J\widetilde{x}_n\Big)\Big) \\
&\leq \phi\Big(u, J^{-1}\Big(\widetilde{\beta}_n Jx_0 + \Big(1-\widetilde{\beta}_n\Big)J\widetilde{x}_n\Big)\Big) \\
&\leq \|u\|^2 - 2\Big\langle u, \widetilde{\beta}_n Jx_0 + \Big(1-\widetilde{\beta}_n\Big)J\widetilde{x}_n\Big\rangle + \widetilde{\beta}_n\|x_0\|^2 + \Big(1-\widetilde{\beta}_n\Big)\|\widetilde{x}_n\|^2 \\
&= \widetilde{\beta}_n\phi(u, x_0) + \Big(1-\widetilde{\beta}_n\Big)\phi(u, \widetilde{x}_n) \\
&\leq \widetilde{\beta}_n\phi(u, x_0) + \Big(1-\widetilde{\beta}_n\Big)\left[\phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|)\right] \\
&= \widetilde{\beta}_n\phi(u, x_0) + \Big(1-\widetilde{\beta}_n\Big)\phi(u, x_n) - \Big(1-\widetilde{\beta}_n\Big)\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|),
\end{aligned} \tag{3.30}$$

and hence

$$\begin{aligned}
\phi(u, \widetilde{x}_n) &= \phi\Big(u, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n)\Big) \\
&= \|u\|^2 - 2\langle u, \alpha_n Jx_0 + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)Jz_n\|^2 \\
&\leq \|u\|^2 - 2\alpha_n\langle u, Jx_0\rangle - 2(1-\alpha_n)\langle u, Jz_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, z_n) \\
&\leq \alpha_n\phi(u, x_0) + (1-\alpha_n)\left[\phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|)\right] \\
&= \alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, x_n) - (1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\phi(u, u_n) &= \phi(u, K_{r_n}y_n) \leq \phi(u, y_n) \qquad \text{(using Proposition 2.10)} \\
&= \phi\Big(u, J^{-1}(\widetilde{\alpha}_n J\widetilde{x}_n + (1-\widetilde{\alpha}_n)J\widetilde{z}_n)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \|u\|^2 - 2\langle u, \widetilde{\alpha}_n J\widetilde{x}_n + (1-\widetilde{\alpha}_n)J\widetilde{z}_n\rangle + \|\widetilde{\alpha}_n J\widetilde{x}_n + (1-\widetilde{\alpha}_n)J\widetilde{z}_n\|^2 \\
&\leq \|u\|^2 - 2\widetilde{\alpha}_n\langle u, J\widetilde{x}_n\rangle - 2(1-\widetilde{\alpha}_n)\langle u, J\widetilde{z}_n\rangle + \widetilde{\alpha}_n\|\widetilde{x}_n\|^2 + (1-\widetilde{\alpha}_n)\|\widetilde{z}_n\|^2 \\
&\quad - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g(\|J\widetilde{x}_n - J\widetilde{z}_n\|) \\
&= \widetilde{\alpha}_n\phi(u,\widetilde{x}_n) + (1-\widetilde{\alpha}_n)\phi(u,\widetilde{z}_n) - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g(\|J\widetilde{x}_n - J\widetilde{z}_n\|) \\
&\leq \widetilde{\alpha}_n\left[\alpha_n\phi(u,x_0) + (1-\alpha_n)\phi(u,x_n) - (1-\alpha_n)\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|)\right] \\
&\quad + (1-\widetilde{\alpha}_n)\Big[\widetilde{\beta}_n\phi(u,x_0) + \left(1-\widetilde{\beta}_n\right)\phi(u,x_n) \\
&\quad -\left(1-\widetilde{\beta}_n\right)\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|)\Big] - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g(\|J\widetilde{x}_n - J\widetilde{z}_n\|) \\
&\leq \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + \left[\widetilde{\alpha}_n(1-\alpha_n) + (1-\widetilde{\alpha}_n)\left(1-\widetilde{\beta}_n\right)\right]\phi(u,x_n) \\
&\quad - \left[\widetilde{\alpha}_n(1-\alpha_n) + (1-\widetilde{\alpha}_n)\left(1-\widetilde{\beta}_n\right)\right]\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&\quad - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g(\|J\widetilde{x}_n - J\widetilde{z}_n\|) \\
&= \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + \left[1 - \alpha_n\widetilde{\alpha}_n - \widetilde{\beta}_n(1-\widetilde{\alpha}_n)\right]\phi(u,x_n) \\
&\quad - \left[1 - \alpha_n\widetilde{\alpha}_n - \widetilde{\beta}_n(1-\widetilde{\alpha}_n)\right]\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&\quad - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g(\|J\widetilde{x}_n - J\widetilde{z}_n\|) \\
&\leq \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + \phi(u,x_n) \\
&\quad - \left[1 - \alpha_n\widetilde{\alpha}_n - \widetilde{\beta}_n(1-\widetilde{\alpha}_n)\right]\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) \\
&\quad - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g(\|J\widetilde{x}_n - J\widetilde{z}_n\|),
\end{aligned}
\tag{3.32}
$$

for all $n \geq 0$. Consequently we have

$$
\begin{aligned}
&\left[1 - \alpha_n\widetilde{\alpha}_n - \widetilde{\beta}_n(1-\widetilde{\alpha}_n)\right]\beta_n(1-\beta_n)g(\|Jx_n - JJ_{r_n}x_n\|) + \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g(\|J\widetilde{x}_n - J\widetilde{z}_n\|) \\
&\quad \leq \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + \phi(u,x_n) - \phi(u,u_n) \\
&\quad = \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
&\quad \leq \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + \left|\|x_n\|^2 - \|u_n\|^2\right| + 2|\langle u, Jx_n - Ju_n\rangle| \\
&\quad \leq \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\| \\
&\quad \leq \left(\alpha_n + \widetilde{\beta}_n\right)\phi(u,x_0) + \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|.
\end{aligned}
\tag{3.33}
$$

Since $x_n - u_n \to 0$ and J is uniformly norm-to-norm continuous on bounded subsets of X, we obtain $Jx_n - Ju_n \to 0$. From $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\liminf_{n\to\infty}\widetilde{\alpha}_n(1-\widetilde{\alpha}_n) > 0$, and $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\widetilde{\beta}_n = 0$ we have

$$\lim_{n\to\infty} g(\|Jx_n - JJ_{r_n}x_n\|) = \lim_{n\to\infty} g(\|J\widetilde{x}_n - J\widetilde{z}_n\|) = 0. \tag{3.34}$$

Therefore, from the properties of g we get

$$\begin{aligned}
&\lim_{n\to\infty}\|Jx_n - JJ_{r_n}x_n\| = \lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0,\\
&\lim_{n\to\infty}\|J\widetilde{x}_n - J\widetilde{z}_n\| = \lim_{n\to\infty}\|\widetilde{x}_n - \widetilde{z}_n\| = 0,
\end{aligned} \tag{3.35}$$

recalling that J^{-1} is uniformly norm-to-norm continuous on bounded sunsets of X^*. Next let us show that

$$\lim_{n\to\infty}\left\|J\widetilde{x}_n - J\widetilde{J}_{r_n}\widetilde{x}_n\right\| = \lim_{n\to\infty}\left\|\widetilde{x}_n - \widetilde{J}_{r_n}\widetilde{x}_n\right\| = 0. \tag{3.36}$$

Observe first that

$$\begin{aligned}
&\phi(u_n, x_n) - \phi(x_{n+1}, u_n)\\
&\quad= \|x_n\|^2 - \|x_{n+1}\|^2 - 2\langle u_n, Jx_n\rangle + 2\langle x_{n+1}, Ju_n\rangle\\
&\quad= (\|x_n\| - \|x_{n+1}\|)(\|x_n\| + \|x_{n+1}\|) + 2\langle x_{n+1} - u_n, Jx_n\rangle + 2\langle x_{n+1}, Ju_n - Jx_n\rangle\\
&\quad\le \|x_n - x_{n+1}\|(\|x_n\| + \|x_{n+1}\|) + 2\|x_{n+1} - u_n\|\|x_n\| + 2\|x_{n+1}\|\|Ju_n - Jx_n\|.
\end{aligned} \tag{3.37}$$

Since $\phi(x_{n+1}, u_n) \to 0$, $\|x_{n+1} - x_n\| \to 0$, $\|x_{n+1} - u_n\| \to 0$, $\|Ju_n - Jx_n\| \to 0$, and $\{x_n\}$ is bounded, so it follows that $\phi(u_n, x_n) \to 0$. Also, observe that

$$\begin{aligned}
\phi(u_n, J_{r_n}x_n) - \phi(u_n, x_n) &= \|J_{r_n}x_n\|^2 - \|x_n\|^2 + 2\langle u_n, Jx_n - JJ_{r_n}x_n\rangle\\
&= (\|J_{r_n}x_n\| - \|x_n\|)(\|J_{r_n}x_n\| + \|x_n\|) + 2\langle u_n, Jx_n - JJ_{r_n}x_n\rangle\\
&\le \|J_{r_n}x_n - x_n\|(\|J_{r_n}x_n\| + \|x_n\|) + 2\|u_{\mathrm{n}}\|\|Jx_n - JJ_{r_n}x_n\|.
\end{aligned} \tag{3.38}$$

Since $\phi(u_n, x_n) \to 0$, $\|J_{r_n}x_n - x_n\| \to 0$, $\|Jx_n - JJ_{r_n}x_n\| \to 0$ and the sequences $\{x_n\}, \{u_n\}, \{J_{r_n}x_n\}$ are bounded, so it follows that $\phi(u_n, J_{r_n}x_n) \to 0$. Meantime, observe that

$$\begin{aligned}
\phi(u_n, z_n) &= \phi\Big(u_n, J^{-1}(\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n)\Big) \\
&= \|u_n\|^2 - 2\langle u_n, \beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\rangle + \big\|\beta_n Jx_n + (1-\beta_n)JJ_{r_n}x_n\big\|^2 \\
&\le \|u_n\|^2 - 2\beta_n\langle u_n, Jx_n\rangle - 2(1-\beta_n)\langle u_n, JJ_{r_n}x_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|J_{r_n}x_n\|^2 \\
&= \beta_n\phi(u_n, x_n) + (1-\beta_n)\phi(u_n, J_{r_n}x_n) \\
&\le \phi(u_n, x_n) + \phi(u_n, J_{r_n}x_n),
\end{aligned} \tag{3.39}$$

and hence

$$\begin{aligned}
\phi(u_n, \tilde{x}_n) &= \phi\Big(u_n, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n)\Big) \\
&= \|u_n\|^2 - 2\langle u_n, \alpha_n Jx_0 + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)Jz_n\|^2 \\
&\le \|u_n\|^2 - 2\alpha_n\langle u_n, Jx_0\rangle - 2(1-\alpha_n)\langle u_n, Jz_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&= \alpha_n\phi(u_n, x_0) + (1-\alpha_n)\phi(u_n, z_n) \\
&\le \alpha_n\phi(u_n, x_0) + \phi(u_n, z_n) \\
&\le \alpha_n\phi(u_n, x_0) + \phi(u_n, x_n) + \phi(u_n, J_{r_n}x_n).
\end{aligned} \tag{3.40}$$

Since $\alpha_n \to 0$, $\phi(u_n, x_n) \to 0$ and $\phi(u_n, J_{r_n}x_n) \to 0$, it follows from the boundedness of $\{u_n\}$ that $\phi(u_n, \tilde{x}_n) \to 0$. Thus, in terms of Lemma 2.1, we have that $\|u_n - \tilde{x}_n\| \to 0$ and so $\|x_n - \tilde{x}_n\| \to 0$. Furthermore, since $\|\tilde{\beta}_n Jx_0 + (1-\tilde{\beta}_n)J\tilde{x}_n - J\tilde{x}_n\| = \tilde{\beta}_n\|Jx_0 - J\tilde{x}_n\| \to 0$, from the uniform norm-to-norm continuity of J^{-1} on bounded subsets of X^*, we obtain

$$\Big\|J^{-1}\Big(\tilde{\beta}_n Jx_0 + \Big(1-\tilde{\beta}_n\Big)J\tilde{x}_n\Big) - \tilde{x}_n\Big\| \longrightarrow 0. \tag{3.41}$$

Observe that

$$\begin{aligned}
\Big\|\tilde{x}_n - \tilde{J}_{r_n}\tilde{x}_n\Big\| &\le \|\tilde{x}_n - \tilde{z}_n\| + \Big\|\tilde{z}_n - \tilde{J}_{r_n}\tilde{x}_n\Big\| \\
&= \|\tilde{x}_n - \tilde{z}_n\| + \Big\|\tilde{J}_{r_n}J^{-1}\Big(\tilde{\beta}_n Jx_0 + \Big(1-\tilde{\beta}_n\Big)J\tilde{x}_n\Big) - \tilde{J}_{r_n}\tilde{x}_n\Big\|.
\end{aligned} \tag{3.42}$$

Thus, from (3.35) it follows that $\|\tilde{x}_n - \tilde{J}_{r_n}\tilde{x}_n\| \to 0$. Since J is uniformly norm-to-norm continuous on bounded subsets of X, it follows that $\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\| \to 0$.

Step 5. We claim that $\omega_w(\{x_n\}) \subset T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP$, where

$$\omega_w(\{x_n\}) := \big\{\hat{x} \in C : x_{n_k} \rightharpoonup \hat{x} \text{ for some subsequence } \{n_k\} \subset \{n\} \text{ with } n_k \uparrow \infty\big\}. \tag{3.43}$$

Indeed, since $\{x_n\}$ is bounded and X is reflexive, we know that $\omega_w(\{x_n\}) \neq \emptyset$. Take $\hat{x} \in \omega_w(\{x_n\})$ arbitrarily. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Hence it follows from $x_n - \tilde{x}_n \to 0$, $x_n - J_{r_n} x_n \to 0$, and $\tilde{x}_n - \tilde{J}_{r_n}\tilde{x}_n \to 0$ that $\{\tilde{x}_{n_k}\}$, $\{J_{r_{n_k}} x_{n_k}\}$, and $\{\tilde{J}_{r_{n_k}}\tilde{x}_{n_k}\}$ converge weakly to the same point $\hat{x}$. On the other hand, from (3.35), (3.36), and $\liminf_{n\to\infty} r_n > 0$ we obtain that

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - JJ_{r_n} x_n\| = 0, \tag{3.44}$$

$$\lim_{n\to\infty} \left\|\tilde{A}_{r_n}\tilde{x}_n\right\| = \lim_{n\to\infty} \frac{1}{r_n} \left\|J\tilde{x}_n - J\tilde{J}_{r_n}\tilde{x}_n\right\| = 0. \tag{3.45}$$

If $z^* \in Tz$ and $\tilde{z}^* \in \tilde{T}\tilde{z}$, then it follows from (2.17) and the monotonicity of the operators $T, \tilde{T}$ that for all $k \geq 1$

$$\left\langle z - J_{r_{n_k}} x_{n_k}, z^* - A_{r_{n_k}} x_{n_k} \right\rangle \geq 0, \qquad \left\langle \tilde{z} - \tilde{J}_{r_{n_k}}\tilde{x}_{n_k}, \tilde{z}^* - \tilde{A}_{r_{n_k}}\tilde{x}_{n_k} \right\rangle \geq 0. \tag{3.46}$$

Letting $k \to \infty$, we have that $\langle z - \hat{x}, z^* \rangle \geq 0$ and $\langle \tilde{z} - \hat{x}, \tilde{z}^* \rangle \geq 0$. Then the maximality of the operators $T, \tilde{T}$ implies that $\hat{x} \in T^{-1}0$ and $\hat{x} \in \tilde{T}^{-1}0$. Next, let us show that $\hat{x} \in EP$. Since we have by (3.32)

$$\phi(u, y_n) \leq \left(\alpha_n + \tilde{\beta}_n\right)\phi(u, x_0) + \phi(u, x_n), \tag{3.47}$$

from $u_n = K_{r_n} y_n$ and Proposition 2.10 it follows that

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, K_{r_n} y_n) \\ &\leq \left(\alpha_n + \tilde{\beta}_n\right)\phi(u, x_0) + \phi(u, x_n) - \phi(u, u_n). \end{aligned} \tag{3.48}$$

Also, since

$$\begin{aligned} |\phi(u, x_n) - \phi(u, u_n)| &= \left|\|x_n\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_n\rangle\right| \\ &\leq |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\| \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\|, \end{aligned} \tag{3.49}$$

so we get

$$\lim_{n\to\infty} (\phi(u, x_n) - \phi(u, u_n)) = 0. \tag{3.50}$$

Thus from (3.47), $\alpha_n \to 0$, $\widetilde{\beta}_n \to 0$, and $\phi(u, x_n) - \phi(u, u_n) \to 0$, we have $\lim_{n\to\infty}\phi(u_n, y_n) = 0$. Since X is uniformly convex and smooth, we conclude from Lemma 2.1 that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.51}$$

From $x_{n_k} \rightharpoonup \widehat{x}$, $x_n - u_n \to 0$, and (3.51), we have $y_{n_k} \rightharpoonup \widehat{x}$ and $u_{n_k} \rightharpoonup \widehat{x}$. Since J is uniformly norm-to-norm continuous on bounded subsets of X, from (3.51) we derive

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{3.52}$$

From $\liminf_{n\to\infty} r_n > 0$, it follows that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.53}$$

By the definition of $u_n := K_{r_n} y_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \tag{3.54}$$

where

$$F(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle. \tag{3.55}$$

Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k}\rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C. \tag{3.56}$$

Since $y \mapsto f(x, y) + \langle Ax, y - x\rangle$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (3.53) and (A4) we have

$$F(y, \widehat{x}) \leq 0, \quad \forall y \in C. \tag{3.57}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\widehat{x}$. Since $y \in C$ and $\widehat{x} \in C$, $y_t \in C$ and hence $F(y_t, \widehat{x}) \leq 0$. So, from (A1) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \widehat{x}) \leq tF(y_t, y). \tag{3.58}$$

Dividing by t, we have

$$F(y_t, y) \geq 0, \quad \forall y \in C. \tag{3.59}$$

Letting $t \downarrow 0$, from (A3) it follows that

$$F(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{3.60}$$

Thus $\hat{x} \in EP$. Therefore, we obtain that $\omega_w(\{x_n\}) \subset T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$ by the arbitrariness of $\hat{x}$.

Step 6. We claim that $\{x_n\}$ converges strongly to $w = \Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$.

Indeed, from $x_{n+1} = \Pi_{H_n \cap W_n} x_0$ and $w \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP \subset H_n \cap W_n$, It follows that

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0). \tag{3.61}$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned}\phi(\hat{x}, x_0) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx_0\rangle + \|x_0\|^2 \leq \liminf_{k\to\infty}\left(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2\right) \\ &= \liminf_{k\to\infty}\phi(x_{n_k}, x_0) \leq \limsup_{k\to\infty}\phi(x_{n_k}, x_0) \leq \phi(w, x_0).\end{aligned} \tag{3.62}$$

From the definition of $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}$, we have $\hat{x} = w$. Hence $\lim_{k\to\infty}\phi(x_{n_k}, x_0) = \phi(w, x_0)$ and

$$\begin{aligned}0 &= \lim_{k\to\infty}\left(\phi(x_{n_k}, x_0) - \phi(w, x_0)\right) \\ &= \lim_{k\to\infty}\left(\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0\rangle\right) \\ &= \lim_{k\to\infty}\left(\|x_{n_k}\|^2 - \|w\|^2\right),\end{aligned} \tag{3.63}$$

which implies that $\lim_{k\to\infty}\|x_{n_k}\| = \|w\|$. Since X has the Kadec-Klee property, then $x_{n_k} \to w = \Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$. Therefore, $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_0$.

Remark 3.4. In Theorem 3.2, put $A \equiv 0$, $\widetilde{T} \equiv 0$, and $\widetilde{\beta}_n = 0$, $\forall n \geq 0$. Then, for all $\alpha, r \in (0, \infty)$ and $x, y \in C$, we have that

$$\begin{aligned}&\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \\ &K_r(x) = \left\{u \in C : f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\right\} \\ &\quad = \left\{u \in C : f(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\right\} = T_r(x).\end{aligned} \tag{3.64}$$

Moreover, the following hold:

$$\begin{aligned} H_n &= \Big\{z \in C : \phi(z, K_{r_n} y_n) \le \Big[\alpha_n + \widetilde{\beta}_n - \alpha_n\widetilde{\beta}_n - \widetilde{\alpha}_n\widetilde{\beta}_n + \alpha_n\widetilde{\alpha}_n\widetilde{\beta}_n\Big]\phi(z, x_0) \\ &\qquad + \Big[(1-\alpha_n)(1-\widetilde{\alpha}_n)\Big(1-\widetilde{\beta}_n\Big) + \widetilde{\alpha}_n(1-\alpha_n)\Big]\phi(z, x_n)\Big\}, \\ &= \{z \in C : \phi(z, T_{r_n} y_n) \le \alpha_n\phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\ y_n &= J^{-1}\Big(\widetilde{\alpha}_n J\widetilde{x}_n + (1-\widetilde{\alpha}_n) J\widetilde{J}_{r_n} J^{-1}\Big(\widetilde{\beta}_n J x_0 + \Big(1-\widetilde{\beta}_n\Big) J\widetilde{x}_n\Big)\Big) \\ &= J^{-1}(\widetilde{\alpha}_n J\widetilde{x}_n + (1-\widetilde{\alpha}_n)(0 J x_0 + (1-0) J\widetilde{x}_n)) \\ &= J^{-1}(\widetilde{\alpha}_n J\widetilde{x}_n + (1-\widetilde{\alpha}_n) J\widetilde{x}_n) \\ &= J^{-1} J\widetilde{x}_n = \widetilde{x}_n, \end{aligned} \tag{3.65}$$

and hence

$$y_n = \widetilde{x}_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J J_{r_n} x_n)). \tag{3.66}$$

In this case, the previous Theorem 3.2 reduces to [20, Theorem 3.1]. □

4. Weak Convergence Theorem

In this section, we present the following algorithm for finding a common element of the set of solutions for a generalized equilibrium problem and the set $T^{-1}0 \cap \widetilde{T}^{-1}0$ for two maximal monotone operators T and $\widetilde{T}$.

Let $x_0 \in X$ be chosen arbitrarily and consider the sequence $\{x_n\}$ generated by

$$\begin{aligned} \widetilde{x}_n &= J^{-1}(\alpha_n J x_0 + (1-\alpha_n)(\beta_n J K_{r_n} x_n + (1-\beta_n) J J_{r_n} K_{r_n} x_n)), \\ x_{n+1} &= J^{-1}\Big(\widetilde{\alpha}_n J K_{r_n}\widetilde{x}_n + (1-\widetilde{\alpha}_n) J\widetilde{J}_{r_n} J^{-1}\Big(\widetilde{\beta}_n J x_0 + \Big(1-\widetilde{\beta}_n\Big) J K_{r_n}\widetilde{x}_n\Big)\Big), \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\widetilde{\alpha}_n\}, \{\widetilde{\beta}_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$, and $K_r, r > 0$, is defined by (2.15).

Before proving a weak convergence theorem, we need the following proposition.

Proposition 4.1. *Suppose that Assumption A is fulfilled and let* $\{x_n\}$ *be a sequence defined by* (4.1), *where* $\{\alpha_n\}, \{\beta_n\}, \{\widetilde{\alpha}_n\}, \{\widetilde{\beta}_n\} \subset [0,1]$ *satisfy the following conditions:*

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \sum_{n=0}^{\infty}\widetilde{\beta}_n < \infty, \qquad \liminf_{n\to\infty}\beta_n(1-\beta_n) > 0, \qquad \liminf_{n\to\infty}\widetilde{\alpha}_n(1-\widetilde{\alpha}_n) > 0. \tag{4.2}$$

Then, $\{\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_n\}$ *converges strongly to* $z \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$*, where* $\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}$ *is the generalized projection of* X *onto* $T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$*.*

Proof. We set $\Omega := T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$ and

$$\begin{aligned} u_n &:= K_{r_n}x_n, \qquad y_n := J^{-1}(\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n), \\ \widetilde{u}_n &:= K_{r_n}\widetilde{x}_n, \qquad \widetilde{y}_n := \widetilde{J}_{r_n} J^{-1}\Big(\widetilde{\beta}_n J x_0 + \Big(1-\widetilde{\beta}_n\Big) J \widetilde{u}_n\Big), \end{aligned} \tag{4.3}$$

so that

$$\begin{aligned} \widetilde{x}_n &= J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J y_n), \\ x_{n+1} &= J^{-1}(\widetilde{\alpha}_n J \widetilde{u}_n + (1-\widetilde{\alpha}_n) J \widetilde{y}_n), \quad n = 0,1,2,\dots. \end{aligned} \tag{4.4}$$

Then, in terms of Lemma 2.4 and Proposition 2.10, Ω is a nonempty closed convex subset of X such that $\Omega \subset C$. We first prove that $\{x_n\}$ is bounded. Fix $u \in \Omega$. Note that by the first and third of (4.3), $u_n, \widetilde{u}_n \in C$, and

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J x_n\rangle &\geq 0, \quad \forall y \in C, \\ F(\widetilde{u}_n, y) + \frac{1}{r_n}\langle y - \widetilde{u}_n, J\widetilde{u}_n - J\widetilde{x}_n\rangle &\geq 0, \quad \forall y \in C. \end{aligned} \tag{4.5}$$

Here, each K_{r_n} is relatively nonexpansive. Then from Proposition 2.10 we obtain

$$\begin{aligned} \phi(u, y_n) &= \phi\Big(u, J^{-1}(\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n)\Big) \\ &= \|u\|^2 - 2\langle u, \beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\rangle + \left\|\beta_n J u_n + (1-\beta_n) J J_{r_n} u_n\right\|^2 \\ &\leq \|u\|^2 - 2\beta_n\langle u, J u_n\rangle - 2(1-\beta_n)\langle u, J J_{r_n} u_n\rangle + \beta_n\|u_n\|^2 + (1-\beta_n)\|J_{r_n} u_n\|^2 \\ &= \beta_n\phi(u, u_n) + (1-\beta_n)\phi(u, J_{r_n} u_n) \\ &\leq \beta_n\phi(u, u_n) + (1-\beta_n)\phi(u, u_n) \\ &= \phi(u, u_n) = \phi(u, K_{r_n} x_n) \leq \phi(u, x_n), \end{aligned} \tag{4.6}$$

$$\begin{aligned}
\phi(u,\widetilde{y}_n) &= \phi\Big(u,\widetilde{J}_{r_n}J^{-1}\Big(\widetilde{\beta}_nJx_0+\Big(1-\widetilde{\beta}_n\Big)J\widetilde{u}_n\Big)\Big)\\
&\le \phi\Big(u,J^{-1}\Big(\widetilde{\beta}_nJx_0+\Big(1-\widetilde{\beta}_n\Big)J\widetilde{u}_n\Big)\Big)\\
&= \|u\|^2-2\Big\langle u,\widetilde{\beta}_nJx_0+\Big(1-\widetilde{\beta}_n\Big)J\widetilde{u}_n\Big\rangle+\Big\|\widetilde{\beta}_nJx_0+\Big(1-\widetilde{\beta}_n\Big)J\widetilde{u}_n\Big\|^2\\
&\le \|u\|^2-2\widetilde{\beta}_n\langle u,Jx_0\rangle-2\Big(1-\widetilde{\beta}_n\Big)\langle u,J\widetilde{u}_n\rangle+\widetilde{\beta}_n\|x_0\|^2+\Big(1-\widetilde{\beta}_n\Big)\|\widetilde{u}_n\|^2\\
&= \widetilde{\beta}_n\phi(u,x_0)+\Big(1-\widetilde{\beta}_n\Big)\phi(u,\widetilde{u}_n)\\
&\le \widetilde{\beta}_n\phi(u,x_0)+\phi(u,\widetilde{u}_n)\\
&= \widetilde{\beta}_n\phi(u,x_0)+\phi(u,K_{r_n}\widetilde{x}_n)\\
&\le \widetilde{\beta}_n\phi(u,x_0)+\phi(u,\widetilde{x}_n),
\end{aligned}\tag{4.7}$$

and hence by Proposition 2.10, we have

$$\begin{aligned}
\phi(u,\widetilde{x}_n) &= \phi\Big(u,J^{-1}(\alpha_nJx_0+(1-\alpha_n)Jy_n)\Big)\\
&= \|u\|^2-2\langle u,\alpha_nJx_0+(1-\alpha_n)Jy_n\rangle+\|\alpha_nJx_0+(1-\alpha_n)Jy_n\|^2\\
&\le \|u\|^2-2\alpha_n\langle u,Jx_0\rangle-2(1-\alpha_n)\langle u,Jy_n\rangle+\alpha_n\|x_0\|^2+(1-\alpha_n)\|y_n\|^2\\
&= \alpha_n\phi(u,x_0)+(1-\alpha_n)\phi(u,y_n)\\
&\le \alpha_n\phi(u,x_0)+\phi(u,y_n)\\
&\le \phi(u,x_n)+\alpha_n\phi(u,x_0),
\end{aligned}\tag{4.8}$$

$$\begin{aligned}
\phi(u,x_{n+1}) &= \phi\Big(u,J^{-1}(\widetilde{\alpha}_nJ\widetilde{u}_n+(1-\widetilde{\alpha}_n)J\widetilde{y}_n)\Big)\\
&= \|u\|^2-2\langle u,\widetilde{\alpha}_nJ\widetilde{u}_n+(1-\widetilde{\alpha}_n)J\widetilde{y}_n\rangle+\|\widetilde{\alpha}_nJ\widetilde{u}_n+(1-\widetilde{\alpha}_n)J\widetilde{y}_n\|^2\\
&\le \|u\|^2-2\widetilde{\alpha}_n\langle u,J\widetilde{u}_n\rangle-2(1-\widetilde{\alpha}_n)\langle u,J\widetilde{y}_n\rangle+\widetilde{\alpha}_n\|\widetilde{u}_n\|^2+(1-\widetilde{\alpha}_n)\|\widetilde{y}_n\|^2\\
&= \widetilde{\alpha}_n\phi(u,\widetilde{u}_n)+(1-\widetilde{\alpha}_n)\phi(u,\widetilde{y}_n)\\
&\le \widetilde{\alpha}_n\phi(u,\widetilde{x}_n)+(1-\widetilde{\alpha}_n)\Big[\widetilde{\beta}_n\phi(u,x_0)+\phi(u,\widetilde{x}_n)\Big]\\
&\le \phi(u,\widetilde{x}_n)+\widetilde{\beta}_n\phi(u,x_0).
\end{aligned}\tag{4.9}$$

Consequently, the last two inequalities yield that

$$\begin{aligned}\phi(u, x_{n+1}) &\leq \phi(u, \tilde{x}_n) + \tilde{\beta}_n \phi(u, x_0)\\ &\leq \phi(u, x_n) + \alpha_n \phi(u, x_0) + \tilde{\beta}_n \phi(u, x_0)\\ &= \phi(u, x_n) + \left(\alpha_n + \tilde{\beta}_n\right) \phi(u, x_0)\end{aligned} \tag{4.10}$$

for all $n \geq 0$. So, from $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\sum_{n=0}^{\infty} \tilde{\beta}_n < \infty$, and Lemma 2.12, we deduce that $\lim_{n \to \infty} \phi(u, x_n)$ exists. This implies that $\{\phi(u, x_n)\}$ is bounded. Thus, $\{x_n\}$ is bounded and so are $\{u_n\}, \{\tilde{u}_n\}, \{J_{r_n} u_n\}$, and $\{\tilde{J}_{r_n} \tilde{u}_n\}$.

Define $z_n = \Pi_\Omega x_n$ for all $n \geq 0$. Let us show that $\{z_n\}$ is bounded. Indeed, observe that

$$\begin{aligned}(\|z_n\| - \|x_n\|)^2 \leq \phi(z_n, x_n) = \phi(\Pi_\Omega x_n, x_n) &\leq \phi(p, x_n) - \phi(p, \Pi_\Omega x_n)\\ &= \phi(p, x_n) - \phi(p, z_n) \leq \phi(p, x_n),\end{aligned} \tag{4.11}$$

for each $p \in \Omega$. This, together with the boundedness of $\{x_n\}$, implies that $\{z_n\}$ is bounded and so is $\phi(z_n, x_0)$. Furthermore, from $z_n \in \Omega$ and (4.10) we have

$$\phi(z_n, x_{n+1}) \leq \phi(z_n, x_n) + \left(\alpha_n + \tilde{\beta}_n\right) \phi(z_n, x_0). \tag{4.12}$$

Since Π_Ω is the generalized projection, then, from Lemma 2.3 we obtain

$$\begin{aligned}\phi(z_{n+1}, x_{n+1}) = \phi(\Pi_\Omega x_{n+1}, x_{n+1}) &\leq \phi(z_n, x_{n+1}) - \phi(z_n, \Pi_\Omega x_{n+1})\\ &= \phi(z_n, x_{n+1}) - \phi(z_n, z_{n+1}) \leq \phi(z_n, x_{n+1}).\end{aligned} \tag{4.13}$$

Hence, from (4.12), it follows that $\phi(z_{n+1}, x_{n+1}) \leq \phi(z_n, x_n) + (\alpha_n + \tilde{\beta}_n) \phi(z_n, x_0)$.

Note that $\sum_{n=0}^{\infty} \alpha_n < \infty, \sum_{n=0}^{\infty} \tilde{\beta}_n < \infty$, and $\{\phi(z_n, x_0)\}$ is bounded, so that $\sum_{n=0}^{\infty} (\alpha_n + \tilde{\beta}_n) \phi(z_n, x_0) < \infty$. Therefore, $\{\phi(z_n, x_n)\}$ is a convergent sequence. On the other hand, from (4.10) we derive, for all $m \geq 0$,

$$\phi(u, x_{n+m}) \leq \phi(u, x_n) + \sum_{j=0}^{m-1} \left(\alpha_{n+j} + \tilde{\beta}_{n+j}\right) \phi(u, x_0). \tag{4.14}$$

In particular, we have

$$\phi(z_n, x_{n+m}) \leq \phi(z_n, x_n) + \sum_{j=0}^{m-1} \left(\alpha_{n+j} + \tilde{\beta}_{n+j}\right) \phi(z_n, x_0), \tag{4.15}$$

Consequently, from $z_{n+m} = \Pi_\Omega x_{n+m}$ and Lemma 2.3, we have

$$\phi(z_n, z_{n+m}) + \phi(z_{n+m}, x_{n+m}) \le \phi(z_n, x_{n+m}) \le \phi(z_n, x_n) + \sum_{j=0}^{m-1}\left(\alpha_{n+j} + \widetilde{\beta}_{n+j}\right)\phi(z_n, x_0) \tag{4.16}$$

and hence

$$\phi(z_n, z_{n+m}) \le \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + \sum_{j=0}^{m-1}\left(\alpha_{n+j} + \widetilde{\beta}_{n+j}\right)\phi(z_n, x_0). \tag{4.17}$$

Let $r = \sup\{\|z_n\| : n \ge 0\}$. From Lemma 2.6, there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that

$$g(\|x - y\|) \le \phi(x, y), \quad \forall x, y \in B_r. \tag{4.18}$$

So, we have

$$\begin{aligned} g(\|z_n - z_{n+m}\|) &\le \phi(z_n, z_{n+m}) \\ &\le \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + \sum_{j=0}^{m-1}\left(\alpha_{n+j} + \widetilde{\beta}_{n+j}\right)\phi(z_n, x_0). \end{aligned} \tag{4.19}$$

Since $\{\phi(z_n, x_n)\}$ is a convergent sequence, $\{\phi(z_n, x_0)\}$ is bounded and $\sum_{n=0}^{\infty}(\alpha_n + \widetilde{\beta}_n)$ is convergent, from the property of g we have that $\{z_n\}$ is a Cauchy sequence. Since Ω is closed, $\{z_n\}$ converges strongly to $z \in \Omega$. This completes the proof. □

Now, we are in a position to prove the following theorem.

Theorem 4.2. *Suppose that Assumption A is fulfilled and let $\{x_n\}$ be a sequence defined by (4.1), where $\{\alpha_n\}, \{\beta_n\}, \{\widetilde{\alpha}_n\}, \{\widetilde{\beta}_n\} \subset [0,1]$ satisfy the following conditions:*

$$\sum_{n=0}^{\infty}\alpha_n < \infty, \qquad \sum_{n=0}^{\infty}\widetilde{\beta}_n < \infty, \qquad \liminf_{n\to\infty}\beta_n(1 - \beta_n) > 0, \qquad \liminf_{n\to\infty}\widetilde{\alpha}_n(1 - \widetilde{\alpha}_n) > 0, \tag{4.20}$$

and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP$, where $z = \lim_{n\to\infty}\Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP}x_n$.

Proof. We consider the notations (4.3). As in the proof of Proposition 4.1, we have that $\{x_n\}, \{u_n\}, \{J_{r_n}u_n\}, \{\tilde{x}_n\}, \{\tilde{u}_n\}$, and $\{\tilde{J}_{r_n}\tilde{u}_n\}$ are bounded sequences. Let

$$r = \sup\{\|u_n\|, \|J_{r_n}u_n\|, \|\tilde{u}_n\|, \|\tilde{y}_n\| : n \geq 0\}. \tag{4.21}$$

From Lemma 2.5 and as in the proof of Theorem 3.2, there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that

$$\|\alpha x^* + (1-\alpha)y^*\|^2 \leq \alpha\|x^*\|^2 + (1-\alpha)\|y^*\|^2 - \alpha(1-\alpha)g(\|x^* - y^*\|) \tag{4.22}$$

for $x^*, y^* \in B_r^*$ and $\alpha \in [0,1]$. Observe that for $u \in \Omega := T^{-1}0 \cap \tilde{T}^{-1}0 \cap EP$,

$$\begin{aligned}
\phi(u, y_n) &= \phi\left(u, J^{-1}(\beta_n J u_n + (1-\beta_n)JJ_{r_n}u_n)\right) \\
&= \|u\|^2 - 2\langle u, \beta_n J u_n + (1-\beta_n)JJ_{r_n}u_n\rangle + \|\beta_n J u_n + (1-\beta_n)JJ_{r_n}u_n\|^2 \\
&\leq \|u\|^2 - 2\beta_n\langle u, Ju_n\rangle - 2(1-\beta_n)\langle u, JJ_{r_n}u_n\rangle \\
&\quad + \beta_n\|u_n\|^2 + (1-\beta_n)\|J_{r_n}u_n\|^2 - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&\leq \beta_n\phi(u, u_n) + (1-\beta_n)\phi(u, J_{r_n}u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&\leq \beta_n\phi(u, u_n) + (1-\beta_n)\phi(u, u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) \\
&= \phi(u, u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|), \\
\phi(u, \tilde{y}_n) &= \phi\left(u, \tilde{J}_{r_n}J^{-1}\left(\tilde{\beta}_n Jx_0 + \left(1-\tilde{\beta}_n\right)J\tilde{u}_n\right)\right) \\
&\leq \phi\left(u, J^{-1}\left(\tilde{\beta}_n Jx_0 + \left(1-\tilde{\beta}_n\right)J\tilde{u}_n\right)\right) \\
&= \|u\|^2 - 2\left\langle u, \tilde{\beta}_n Jx_0 + \left(1-\tilde{\beta}_n\right)J\tilde{u}_n\right\rangle + \left\|\tilde{\beta}_n Jx_0 + \left(1-\tilde{\beta}_n\right)J\tilde{u}_n\right\|^2 \\
&\leq \|u\|^2 - 2\tilde{\beta}_n\langle u, Jx_0\rangle - 2\left(1-\tilde{\beta}_n\right)\langle u, J\tilde{u}_n\rangle + \tilde{\beta}_n\|x_0\|^2 + \left(1-\tilde{\beta}_n\right)\|\tilde{u}_n\|^2 \\
&= \tilde{\beta}_n\phi(u, x_0) + \left(1-\tilde{\beta}_n\right)\phi(u, \tilde{u}_n) \\
&= \tilde{\beta}_n\phi(u, x_0) + \left(1-\tilde{\beta}_n\right)\phi(u, K_{r_n}\tilde{x}_n) \\
&\leq \tilde{\beta}_n\phi(u, x_0) + \phi(u, \tilde{x}_n).
\end{aligned} \tag{4.23}$$

Hence,

$$
\begin{aligned}
\phi(u,\widetilde{x}_n) &= \phi\Big(u, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jy_n)\Big)\\
&= \|u\|^2 - 2\langle u, \alpha_n Jx_0 + (1-\alpha_n)Jy_n\rangle + \big\|\alpha_n Jx_0 + (1-\alpha_n)Jy_n\big\|^2\\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_0\rangle - 2(1-\alpha_n)\langle u, Jy_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\big\|y_n\big\|^2\\
&= \alpha_n\phi(u,x_0) + (1-\alpha_n)\phi(u,y_n)\\
&\le \alpha_n\phi(u,x_0) + \phi(u,y_n)\\
&\le \alpha_n\phi(u,x_0) + \phi(u,u_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|)\\
&= \alpha_n\phi(u,x_0) + \phi(u,K_{r_n}x_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|)\\
&\le \alpha_n\phi(u,x_0) + \phi(u,x_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|),
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
\phi(u,x_{n+1}) &= \phi\Big(u, J^{-1}(\widetilde{\alpha}_n J\widetilde{u}_n + (1-\widetilde{\alpha}_n)J\widetilde{y}_n)\Big)\\
&= \|u\|^2 - 2\langle u, \widetilde{\alpha}_n J\widetilde{u}_n + (1-\widetilde{\alpha}_n)J\widetilde{y}_n\rangle + \big\|\widetilde{\alpha}_n J\widetilde{u}_n + (1-\widetilde{\alpha}_n)J\widetilde{y}_n\big\|^2\\
&\le \|u\|^2 - 2\widetilde{\alpha}_n\langle u, J\widetilde{u}_n\rangle - 2(1-\widetilde{\alpha}_n)\langle u, J\widetilde{y}_n\rangle + \widetilde{\alpha}_n\|\widetilde{u}_n\|^2 + (1-\widetilde{\alpha}_n)\big\|\widetilde{y}_n\big\|^2\\
&\quad - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big)\\
&= \widetilde{\alpha}_n\phi(u,\widetilde{u}_n) + (1-\widetilde{\alpha}_n)\phi(u,\widetilde{y}_n) - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big)\\
&\le \widetilde{\alpha}_n\phi(u,K_{r_n}\widetilde{x}_n) + (1-\widetilde{\alpha}_n)\Big[\widetilde{\beta}_n\phi(u,x_0) + \phi(u,\widetilde{x}_n)\Big] - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big)\\
&\le \phi(u,\widetilde{x}_n) + \widetilde{\beta}_n\phi(u,x_0) - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big).
\end{aligned}
\tag{4.25}
$$

Consequently, the last two inequalities yield that

$$
\begin{aligned}
\phi(u,x_{n+1}) &\le \phi(u,\widetilde{x}_n) + \widetilde{\beta}_n\phi(u,x_0) - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big)\\
&\le \alpha_n\phi(u,x_0) + \phi(u,x_n) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|)\\
&\quad + \widetilde{\beta}_n\phi(u,x_0) - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big)\\
&= \phi(u,x_n) + \Big(\alpha_n + \widetilde{\beta}_n\Big)\phi(u,x_0) - \beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|)\\
&\quad - \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big).
\end{aligned}
\tag{4.26}
$$

Thus, we have

$$
\begin{aligned}
&\beta_n(1-\beta_n)g(\|Ju_n - JJ_{r_n}u_n\|) + \widetilde{\alpha}_n(1-\widetilde{\alpha}_n)g\big(\big\|J\widetilde{u}_n - J\widetilde{y}_n\big\|\big)\\
&\quad \le \phi(u,x_n) - \phi(u,\mathrm{x}_{n+1}) + \Big(\alpha_n + \widetilde{\beta}_n\Big)\phi(u,x_0).
\end{aligned}
\tag{4.27}
$$

By the proof of Proposition 4.1, it is known that $\{\phi(u, x_n)\}$ is convergent; since $\lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\tilde{\beta}_n = 0, \liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, and $\liminf_{n\to\infty}\tilde{\alpha}_n(1-\tilde{\alpha}_n) > 0$, then we have

$$\lim_{n\to\infty} g(\|Ju_n - JJ_{r_n}u_n\|) = \lim_{n\to\infty} g\left(\left\|J\tilde{u}_n - J\tilde{y}_n\right\|\right) = 0. \tag{4.28}$$

Taking into account the properties of g, as in the proof of Theorem 3.2, we have

$$\begin{aligned} \lim_{n\to\infty}\|Ju_n - JJ_{r_n}u_n\| &= \lim_{n\to\infty}\|u_n - J_{r_n}u_n\| = 0, \\ \lim_{n\to\infty}\left\|J\tilde{u}_n - J\tilde{y}_n\right\| &= \lim_{n\to\infty}\left\|\tilde{u}_n - \tilde{y}_n\right\| = 0, \end{aligned} \tag{4.29}$$

since J^{-1} is uniformly norm-to-norm continuous on bounded subsets of X^*. Note that $\|\tilde{\beta}_n Jx_0 + (1-\tilde{\beta}_n)J\tilde{u}_n - J\tilde{u}_n\| = \tilde{\beta}_n\|Jx_0 - J\tilde{u}_n\| \to 0$. Hence, from the uniform norm-to-norm continuity of J^{-1} on bounded subsets of X^* we obtain $\|J^{-1}(\tilde{\beta}_n Jx_0 + (1-\tilde{\beta}_n)J\tilde{u}_n) - \tilde{u}_n\| \to 0$. Also, observe that

$$\begin{aligned} \left\|\tilde{J}_{r_n}\tilde{u}_n - \tilde{u}_n\right\| &\le \left\|\tilde{J}_{r_n}\tilde{u}_n - \tilde{J}_{r_n}J^{-1}\left(\tilde{\beta}_n Jx_0 + \left(1-\tilde{\beta}_n\right)J\tilde{u}_n\right)\right\| \\ &\quad + \left\|\tilde{J}_{r_n}J^{-1}\left(\tilde{\beta}_n Jx_0 + \left(1-\tilde{\beta}_n\right)J\tilde{u}_n\right) - \tilde{u}_n\right\| \\ &\le \left\|\tilde{u}_n - J^{-1}\left(\tilde{\beta}_n Jx_0 + \left(1-\tilde{\beta}_n\right)J\tilde{u}_n\right)\right\| + \left\|\tilde{y}_n - \tilde{u}_n\right\|. \end{aligned} \tag{4.30}$$

From $\|\tilde{u}_n - \tilde{y}_n\| \to 0$ it follows that $\|\tilde{J}_{r_n}\tilde{u}_n - \tilde{u}_n\| \to 0$. Since J is uniformly norm-to-norm continuous on bounded subsets of X, we have

$$\begin{aligned} \lim_{n\to\infty}\|Ju_n - JJ_{r_n}u_n\| &= \lim_{n\to\infty}\|u_n - J_{r_n}u_n\| = 0, \\ \lim_{n\to\infty}\left\|J\tilde{u}_n - J\tilde{J}_{r_n}\tilde{u}_n\right\| &= \lim_{n\to\infty}\left\|\tilde{u}_n - \tilde{J}_{r_n}\tilde{u}_n\right\| = 0. \end{aligned} \tag{4.31}$$

Now let us show that

$$\lim_{n\to\infty}\phi(u, x_n) = \lim_{n\to\infty}\phi(u, \tilde{x}_n) = \lim_{n\to\infty}\phi(u, u_n) = \lim_{n\to\infty}\phi(u, \tilde{u}_n). \tag{4.32}$$

Indeed, from (4.10) we get

$$\phi(u, x_{n+1}) - \tilde{\beta}_n\phi(u, x_0) \le \phi(u, \tilde{x}_n) \le \phi(u, x_n) + \alpha_n\phi(u, x_0), \tag{4.33}$$

which, together with $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\tilde{\beta}_n = 0$, yields that

$$\lim_{n\to\infty}\phi(u, \tilde{x}_n) = \lim_{n\to\infty}\phi(u, x_n). \tag{4.34}$$

From (4.9) it follows that

$$\begin{aligned} \phi(u, x_{n+1}) &\leq \widetilde{\alpha}_n \phi(u, \widetilde{u}_n) + (1 - \widetilde{\alpha}_n)\phi(u, \widetilde{y}_n) \\ &= \phi(u, \widetilde{y}_n) + \widetilde{\alpha}_n(\phi(u, \widetilde{u}_n) - \phi(u, \widetilde{y}_n)) \leq \phi(u, \widetilde{x}_n) + \widetilde{\beta}_n \phi(u, x_0). \end{aligned} \tag{4.35}$$

Note that

$$\begin{aligned} \left|\phi(u, \widetilde{u}_n) - \phi(u, \widetilde{y}_n)\right| &= \left|\|\widetilde{u}_n\|^2 - \|\widetilde{y}_n\|^2 + 2\langle u, J\widetilde{y}_n - J\widetilde{u}_n\rangle\right| \\ &\leq \left|\|\widetilde{u}_n\| - \|\widetilde{y}_n\|\right|(\|\widetilde{u}_n\| + \|\widetilde{y}_n\|) + 2\|u\|\|J\widetilde{y}_n - J\widetilde{u}_n\| \\ &\leq \|\widetilde{u}_n - \widetilde{y}_n\|(\|\widetilde{u}_n\| + \|\widetilde{y}_n\|) + 2\|u\|\|J\widetilde{y}_n - J\widetilde{u}_n\|. \end{aligned} \tag{4.36}$$

Since $\|\widetilde{u}_n - \widetilde{y}_n\| \to 0$ and $\|J\widetilde{u}_n - J\widetilde{y}_n\| \to 0$, we obtain $\lim_{n\to\infty}(\phi(u, \widetilde{u}_n) - \phi(u, \widetilde{y}_n)) = 0$, which, together with $\lim_{n\to\infty}\phi(u, \widetilde{x}_n) = \lim_{n\to\infty}\phi(u, x_n)$, yields that

$$\lim_{n\to\infty} \phi(u, \widetilde{y}_n) = \lim_{n\to\infty} \phi(u, x_n). \tag{4.37}$$

We have from (4.8) that

$$\phi(u, \widetilde{x}_n) - \alpha_n \phi(u, x_0) \leq \phi(u, y_n) \leq \phi(u, x_n), \tag{4.38}$$

which, together with $\lim_{n\to\infty}\phi(u, \widetilde{x}_n) = \lim_{n\to\infty}\phi(u, x_n)$, yields that

$$\lim_{n\to\infty} \phi(u, y_n) = \lim_{n\to\infty} \phi(u, x_n). \tag{4.39}$$

Also from (4.7) it follows that

$$\phi(u, \widetilde{y}_n) - \widetilde{\beta}_n \phi(u, x_0) \leq \phi(u, \widetilde{u}_n) \leq \phi(u, \widetilde{x}_n), \tag{4.40}$$

which, together with $\lim_{n\to\infty}\phi(u, \widetilde{x}_n) = \lim_{n\to\infty}\phi(u, \widetilde{y}_n) = \lim_{n\to\infty}\phi(u, x_n)$, yields that

$$\lim_{n\to\infty} \phi(u, \widetilde{u}_n) = \lim_{n\to\infty} \phi(u, x_n). \tag{4.41}$$

Similarly from (4.6) it follows that

$$\phi(u, y_n) \leq \phi(u, u_n) \leq \phi(u, x_n) \tag{4.42}$$

which, together with $\lim_{n\to\infty}\phi(u, y_n) = \lim_{n\to\infty}\phi(u, x_n)$, yields that

$$\lim_{n\to\infty}\phi(u, u_n) = \lim_{n\to\infty}\phi(u, x_n). \tag{4.43}$$

On the other hand, let us show that

$$\lim_{n\to\infty}\|x_n - \tilde{x}_n\| = 0. \tag{4.44}$$

Indeed, let $s = \sup\{\|x_n\|, \|u_n\|, \|\tilde{x}_n\|, \|\tilde{u}_n\| : n \geq 0\}$. From Lemma 2.6, there exists a continuous, strictly increasing, and convex function g_1 with $g_1(0) = 0$ such that

$$g_1(\|x - y\|) \leq \phi(x, y), \quad \forall x, y \in B_s. \tag{4.45}$$

Since $u_n = K_{r_n}x_n$ and $\tilde{u}_n = K_{r_n}\tilde{x}_n$, we deduce from Proposition 2.10 that for $u \in \Omega$,

$$\begin{gathered} g_1(\|u_n - x_n\|) \leq \phi(u_n, x_n) \leq \phi(u, x_n) - \phi(u, u_n), \\ g_1(\|\tilde{u}_n - \tilde{x}_n\|) \leq \phi(\tilde{u}_n, \tilde{x}_n) \leq \phi(u, \tilde{x}_n) - \phi(u, \tilde{u}_n). \end{gathered} \tag{4.46}$$

This implies that

$$\lim_{n\to\infty} g_1(\|u_n - x_n\|) = \lim_{n\to\infty} g_1(\|\tilde{u}_n - \tilde{x}_n\|) = 0. \tag{4.47}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of X, from the properties of g_1 we obtain

$$\begin{gathered} \lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|Ju_n - Jx_n\| = 0, \\ \lim_{n\to\infty}\|\tilde{u}_n - \tilde{x}_n\| = \lim_{n\to\infty}\|J\tilde{u}_n - J\tilde{x}_n\| = 0. \end{gathered} \tag{4.48}$$

Note that

$$\begin{aligned}
\phi(x_n,u_n)-\phi(u_n,x_n) &= \|x_n\|^2-2\langle x_n,Ju_n\rangle+\|u_n\|^2-\left[\|x_n\|^2-2\langle u_n,Jx_n\rangle+\|u_n\|^2\right]\\
&= -2\langle x_n,Ju_n\rangle+2\langle u_n,Jx_n\rangle\\
&= 2\langle x_n,Jx_n-Ju_n\rangle+2\langle u_n-x_n,Jx_n\rangle\\
&\le 2\|x_n\|\|Jx_n-Ju_n\|+2\|u_n-x_n\|\|x_n\|,
\end{aligned}\tag{4.49}$$

$$\begin{aligned}
\phi(x_n,J_{r_n}u_n) &= \|x_n\|^2-2\langle x_n,JJ_{r_n}u_n\rangle \ +\|J_{r_n}u_n\|^2\\
&= \|x_n\|^2-\|x_n\|^2+\|J_{r_n}u_n\|^2-\|x_n\|^2+2\langle x_n,Jx_n-JJ_{r_n}u_n\rangle\\
&= (\|J_{r_n}u_n\|-\|x_n\|)(\|J_{r_n}u_n\|+\|x_n\|)+2\langle x_n,Jx_n-JJ_{r_n}u_n\rangle\\
&\le \|J_{r_n}u_n-x_n\|(\|J_{r_n}u_n\|+\|x_n\|)+2\|x_n\|\|Jx_\mathrm{n}-JJ_{r_n}u_n\|\\
&= \|J_{r_n}u_n-u_n+u_n-x_n\|(\|J_{r_n}u_n\|+\|x_n\|)\\
&\quad +2\|x_n\|\|Jx_n-Ju_n+Ju_n-JJ_{r_n}u_n\|\\
&\le (\|J_{r_n}u_n-u_n\|+\|u_n-x_n\|)(\|J_{r_n}u_n\|+\|x_n\|)\\
&\quad +2\|x_n\|(\|Jx_n-Ju_n\|+\|Ju_n-JJ_{r_n}u_n\|).
\end{aligned}\tag{4.50}$$

Since $\phi(u_n,x_n)\to 0$, it follows from (4.31) and (4.35) that $\phi(x_n,u_n)\to 0$ and $\phi(x_n,J_{r_n}u_n)\to 0$. Also, observe that

$$\begin{aligned}
\phi(x_n,y_n) &= \phi\Big(x_n,J^{-1}(\beta_nJu_n+(1-\beta_n)JJ_{r_n}u_n)\Big)\\
&= \|x_n\|^2-2\langle x_n,\beta_nJu_n+(1-\beta_n)JJ_{r_n}u_n\rangle+\big\|\beta_nJu_n+(1-\beta_n)JJ_{r_n}u_n\big\|^2\\
&\le \|x_n\|^2-2\beta_n\langle x_n,Ju_n\rangle-2(1-\beta_n)\langle x_n,JJ_{r_n}u_n\rangle+\beta_n\|u_n\|^2+(1-\beta_n)\|J_{r_n}u_n\|^2\\
&= \beta_n\phi(x_n,u_n)+(1-\beta_n)\phi(x_n,J_{r_n}u_n)\\
&\le \phi(x_n,u_n)+\phi(x_n,J_{r_n}u_n),
\end{aligned}\tag{4.51}$$

and hence

$$\begin{aligned}
\phi(x_n,\tilde{x}_n) &= \phi\Big(x_n,J^{-1}(\alpha_nJx_0+(1-\alpha_n)Jy_n)\Big)\\
&= \|x_n\|^2-2\langle x_n,\alpha_nJx_0+(1-\alpha_n)Jy_n\rangle+\big\|\alpha_nJx_0+(1-\alpha_n)Jy_n\big\|^2\\
&\le \|x_n\|^2-2\alpha_n\langle x_n,Jx_0\rangle-2(1-\alpha_n)\langle x_n,Jy_n\rangle+\alpha_n\|x_0\|^2+(1-\alpha_n)\big\|y_n\big\|^2\\
&= \alpha_n\phi(x_n,x_0)+(1-\alpha_n)\phi(x_n,y_n)\\
&\le \alpha_n\phi(x_n,x_0)+\phi(x_n,y_n)\\
&\le \alpha_n\phi(x_n,x_0)+\phi(x_n,u_n)+\phi(x_n,J_{r_n}u_n).
\end{aligned}\tag{4.52}$$

Thus, from $\alpha_n \to 0, \phi(x_n, u_n) \to 0$, and $\phi(x_n, J_{r_n} u_n) \to 0$, it follows that $\phi(x_n, \widetilde{x}_n) \to 0$. In terms of Lemma 2.1, we derive $\|x_n - \widetilde{x}_n\| \to 0$.

Next, let us show that $x_n \rightharpoonup z$, where $z = \lim_{n\to\infty} \Pi_{T^{-1}0\cap\widetilde{T}^{-1}0\cap EP} x_n$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \widehat{x} \in C$. Hence it follows from (4.31), (4.35), and $\|x_n - \widetilde{x}_n\| \to 0$ that both $\{u_{n_k}\}, \{\widetilde{u}_{n_k}\}, \{J_{r_{n_k}} u_{n_k}\}$ and $\widetilde{J}_{r_{n_k}} \widetilde{u}_{n_k}$ converge weakly to the same point $\widehat{x}$. Furthermore, from $\liminf_{n\to\infty} r_n > 0$ and (4.31) we have that

$$\lim_{n\to\infty} \|A_{r_n} u_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|J u_n - J J_{r_n} u_n\| = 0,$$
$$\lim_{n\to\infty} \left\|\widetilde{A}_{r_n} \widetilde{u}_n\right\| = \lim_{n\to\infty} \frac{1}{r_n} \left\|J\widetilde{u}_n - J\widetilde{J}_{r_n} \widetilde{u}_n\right\| = 0. \tag{4.53}$$

If $z^* \in Tz$ and $\widetilde{z}^* \in \widetilde{T}\widetilde{z}$, then it follows from (2.17) and the monotonicity of the operators $T, \widetilde{T}$ that for all $k \geq 1$

$$\left\langle z - J_{r_{n_k}} u_{n_k}, z^* - A_{r_{n_k}} u_{n_k} \right\rangle \geq 0, \qquad \left\langle \widetilde{z} - \widetilde{J}_{r_{n_k}} \widetilde{u}_{n_k}, \widetilde{z}^* - \widetilde{A}_{r_{n_k}} \widetilde{u}_{n_k} \right\rangle \geq 0. \tag{4.54}$$

Letting $k \to \infty$, we obtain that

$$\langle z - \widehat{x}, z^* \rangle \geq 0, \qquad \langle \widetilde{z} - \widehat{x}, \widetilde{z}^* \rangle \geq 0. \tag{4.55}$$

Then the maximality of the operators $T, \widetilde{T}$ implies that $\widehat{x} \in T^{-1}0 \cap \widetilde{T}^{-1}0$.

Now, by the definition of $u_n := K_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J x_n \rangle \geq 0, \quad \forall y \in C, \tag{4.56}$$

where $F(x, y) = f(x, y) + \langle Ax, y - x\rangle$. Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}} \langle y - u_{n_k}, J u_{n_k} - J x_{n_k} \rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C. \tag{4.57}$$

Since $y \mapsto F(x, y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (4.35) and (A4) we have

$$F(y, \widehat{x}) \leq 0, \quad \forall y \in C. \tag{4.58}$$

For t, with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\widehat{x}$. Since $y \in C$ and $\widehat{x} \in C$, then $y_t \in C$ and hence $F(y_t, \widehat{x}) \leq 0$. So, from (A1) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \widehat{x}) \leq tF(y_t, y). \tag{4.59}$$

Dividing by t, we get $F(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, from (A3) it follows that $F(\widehat{x}, y) \geq 0$, $\forall y \in C$. So, $\widehat{x} \in EP$. Therefore, $\widehat{x} \in \Omega$. Let $z_n = \Pi_{\Omega} x_n$. From Lemma 2.2 and $\widehat{x} \in \Omega$, we get

$$\langle z_{n_k} - \widehat{x}, Jx_{n_k} - Jz_{n_k} \rangle \geq 0. \tag{4.60}$$

From Proposition 4.1, we also know that $z_n \to z \in \Omega$. Note that $x_{n_k} \rightharpoonup \widehat{x}$. Since J is weakly sequentially continuous, then $\langle z - \widehat{x}, J\widehat{x} - Jz \rangle \geq 0$ as $k \to \infty$. In addition, taking into account the monotonicity of J, we conclude that $\langle z - \widehat{x}, J\widehat{x} - Jz \rangle \leq 0$. Hence

$$\langle z - \widehat{x}, J\widehat{x} - Jz \rangle = 0. \tag{4.61}$$

From the strict convexity of X, it follows that $z = \widehat{x}$. Therefore, $x_n \rightharpoonup \widehat{x}$, where $\widehat{x} = \lim_{n\to\infty} \Pi_{T^{-1}0 \cap \widetilde{T}^{-1}0 \cap EP} x_n$. This completes the proof. □

Remark 4.3. In Theorem 4.2, put $A \equiv 0$, $\widetilde{T} \equiv 0$, and $\widetilde{\beta}_n = 0$, $\forall n \geq 0$. Then, for all $\alpha, r \in (0, \infty)$ and $x, y \in C$, we have that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \tag{4.62}$$

$$\begin{aligned} K_r(x) &= \left\{ u \in C : f(u, y) + \langle Au, y - u \rangle + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0,\ \forall y \in C \right\} \\ &= \left\{ u \in C : f(u, y) + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0,\ \forall y \in C \right\} = T_r(x). \end{aligned} \tag{4.63}$$

Moreover, the following hold:

$$\begin{aligned} x_{n+1} &= J^{-1}\left(\widetilde{\alpha}_n JK_{r_n}\widetilde{x}_n + (1 - \widetilde{\alpha}_n) J\widetilde{J}_{r_n} J^{-1}\left(\widetilde{\beta}_n Jx_0 + \left(1 - \widetilde{\beta}_n\right) JK_{r_n}\widetilde{x}_n\right)\right) \\ &= J^{-1}(\widetilde{\alpha}_n JT_{r_n}\widetilde{x}_n + (1 - \widetilde{\alpha}_n)(0Jx_0 + (1 - 0)JT_{r_n}\widetilde{x}_n)) \\ &= J^{-1}(\widetilde{\alpha}_n JT_{r_n}\widetilde{x}_n + (1 - \widetilde{\alpha}_n)JT_{r_n}\widetilde{x}_n) \\ &= J^{-1}JT_{r_n}\widetilde{x}_n = T_{r_n}\widetilde{x}_n. \end{aligned} \tag{4.64}$$

In this case, Algorithm (4.1) reduces to the following one:

$$x_{n+1} = T_{r_n} J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)(\beta_n JT_{r_n} x_n + (1 - \beta_n) JJ_{r_n} T_{r_n} x_n)). \tag{4.65}$$

Corollary 4.4. *Suppose that conditions (A1)–(A5) are fulfilled and let $\{x_n\}$ be a sequence defined by* (4.65)*, where T_r, $r > 0$ is defined in Lemma 2.8, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy the conditions $\sum_{n=0}^{\infty} \alpha_n < \infty$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$, and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in T^{-1}0 \cap EP(f)$, where $z = \lim_{n\to\infty} \Pi_{T^{-1}0 \cap EP(f)} x_n$.*

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