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Research Article

Some Maximal Elements' Theorems in FC-Spaces

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Let I be a finite or infinite index set, let X be a topological space, and let $(Y_i, \varphi_{N_i})_{i\in I}$ be a family of FC-spaces. For each $i \in I$, let $A_i : X \to 2^{Y_i}$ be a set-valued mapping. Some new existence theorems of maximal elements for a set-valued mapping and a family of set-valued mappings involving a better admissible set-valued mapping are established under noncompact setting of FC-spaces. Our results improve and generalize some recent results.

1. Introduction

It is well known that many existence theorems of maximal elements for various classes of set-valued mappings have been established in different spaces. For their applications to mathematical economies, generalized games, and other branches of mathematics, the reader may consult [1–12] and the references therein.

In most of the known existence results of maximal elements, the convexity assumptions play a crucial role which strictly restrict the applicable area of these results. In this paper, we will continue to study existence theorems of maximal elements in general topological spaces without convexity structure. We introduce a new class of generalized $G_{\mathcal{B}}$-majorized mappings $A_i : X \to 2^{Y_i}$ for each $i \in I$ which involve a set-valued mapping $F \in \mathcal{B}(Y, X)$. The notion of generalized $G_{\mathcal{B}}$-majorized mappings unifies and generalizes the corresponding notions of $G_{\mathcal{B}}$-majorized mappings in [4]; $L_{\mathcal{S}}$-majorized mappings in [2, 13]; H-majorized mappings in [14]. Some new existence theorems of maximal elements for generalized $G_{\mathcal{B}}$-majorized mappings are proved under noncompact setting of FC-spaces. Our results improve and generalize the corresponding results in [2, 4, 14–16].

2. Preliminaries

Let X and Y be two nonempty sets. We denote by 2^Y and $\langle X\rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of X, respectively. For each $A \in \langle X\rangle$, we denote by $|A|$ the cardinality of A. Let Δ_n denote the standard n-dimensional simplex with the vertices $\{e_0, \dots, e_n\}$. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we will denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$.

Let X and Y be two sets, and let $T : X \to 2^Y$ be a set-valued mapping. We will use the following notations in the sequel:

(i) $T(x) = \{y \in Y : y \in T(x)\}$,

(ii) $T(A) = \bigcup_{x\in A} T(x)$,

(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$.

For topological spaces X and Y, a subset A of X is said to be compactly open (resp., compactly closed) if for each nonempty compact subset K of X, $A\cap K$ is open (resp., closed) in K. The compact closure of A and the compact interior of A (see [17]) are defined, respectively, by

$$\begin{aligned} \operatorname{ccl} A &= \bigcap\{B \subset X : A \subset B,\ B \text{ is compactly closed in } X\}, \\ \operatorname{cint} A &= \bigcup\{B \subset X : B \subset A,\ B \text{ is compactly open in } X\}. \end{aligned} \tag{2.1}$$

It is easy to see that $\operatorname{ccl}(X \setminus A) = X \setminus \operatorname{cint} A$, $\operatorname{int} A \subset \operatorname{cint} A \subset A$, $A \subset \operatorname{ccl} A \subset \operatorname{cl} A$, A is compactly open (resp., compactly closed) in X if and only if $A = \operatorname{cint} A$ (resp., $A = \operatorname{ccl} A$). For each nonempty compact subset K of X, $\operatorname{ccl} A\bigcap K = \operatorname{cl}_K(A\bigcap K)$ and $\operatorname{cint} A\bigcap K = \operatorname{int}_K(A\bigcap K)$, where $cl_K(A\bigcap K)$ (resp., $\operatorname{int}_K(A\bigcap K)$) denotes the closure (resp., interior) of $A\bigcap K$ in K. A set-valued mapping $T : X \to 2^Y$ is transfer compactly open valued on X (see [17]) if for each $x \in X$ and $y \in T(x)$, there exists $x' \in X$ such that $y \in \operatorname{cint} T(x')$. Let A_i $(i = 1, \dots, m)$ be transfer compactly open valued, then $\bigcap_{i=1}^m \operatorname{cint} A_i = \operatorname{cint} \bigcap_{i=1}^m A_i$. It is clear that each transfer open valued correspondence is transfer compactly open valued. The inverse is not true in general.

The definition of FC-space and the class $\mathcal{B}(Y, X)$ of better admissible mapping were introduced by Ding in [8]. Note that the class $\mathcal{B}(Y, X)$ of better admissible mapping includes many important classes of mappings, for example, the class $\mathcal{B}(Y, X)$ in [18], $\mathcal{U}_c^k(Y, X)$ in [19] and so on as proper subclasses. Now we introduce the following definition.

Definition 2.1. An FC-space (Y, φ_N) is said to be an CFC-space if for each $N \in \langle Y\rangle$, there exists a compact FC-subspace L_N of Y containing N.

(Y, φ_N) be a G-convex space, let the notion of CG-convex space was introduced by Ding in [4].

Lemma 2.2 ([8]). *Let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an FC-space, $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I}\varphi_{N_i}$. Then (Y, φ_N) is also an FC-space.*

Let X be a topological space, and let I be any index set. For each $i \in I$, let $(Y_i, \varphi_{N_i})_{i\in I}$ be an FC-space, and let $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 2.2.

Let $F \in \mathcal{B}(Y, X)$ and for each $i \in I$, let $A_i : X \to 2^{Y_i}$ be a set-valued mapping. For each $i \in I$,

(1) $A_i : X \to 2^{Y_i}$ is said to be a generalized $G_{\mathcal{B}}$-mapping if

  (a) for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$, $F(\varphi_N(\Delta_k)) \bigcap (\bigcap_{j=0}^{k} \operatorname{cint} A_i^{-1}(\pi_i(y_{i_j}))) = \emptyset$, where π_i is the projection of Y onto Y_i and $\Delta_k = \operatorname{co}(\{e_{i_j} : j = 0, \ldots, k\})$;

  (b) $A_i^{-1}(y_i) = \{x \in X : y_i \in A_i(x)\}$ is transfer compactly open in Y_i for each $y_i \in Y_i$;

(2) $A_{x,i} : X \to 2^{Y_i}$ is said to be a generalized $G_{\mathcal{B}}$-majorant of A_i at $x \in X$ if $A_{x,i}$ is a generalized $G_{\mathcal{B}}$-mapping and there exists an open neighborhood $N(x)$ of x in X such that $A_i(z) \subset A_{x,i}(z)$ for all $z \in N(x)$;

(3) A_i is said to be a generalized $G_{\mathcal{B}}$-majorized if for each $x \in X$ with $A_i(x) \neq \emptyset$, there exists a generalized $G_{\mathcal{B}}$-majorant $A_{x,i}$ of A_i at x, and for any $N \in \langle \{x \in X : A_i(x) \neq \emptyset\} \rangle$, the mapping $\bigcap_{x \in N} A_{x,i}^{-1}$ is transfer compactly open in Y_i;

(4) A_i is said to be a generalized $G_{\mathcal{B}}$-majorized if for each $x \in X$, there exists a generalized $G_{\mathcal{B}}$-majorant $A_{x,i}$ of A_i at x.

Then $\{A_i\}_{i \in I}$ is said to be a family of generalized $G_{\mathcal{B}}$-mappings (resp., $G_{\mathcal{B}}$-majorant mappings) if for each $i \in I$, $A_i : X \to 2^{Y_i}$ is a generalized $G_{\mathcal{B}}$-mapping (resp., $G_{\mathcal{B}}$-majorant mapping).

If for each $i \in I$, let (Y_i, φ_{N_i}) be a G-convex space, a family of $G_{\mathcal{B}}$-mappings (resp., $G_{\mathcal{B}}$-majorant mappings) were introduced by Ding in [4]. Clearly, each family of generalized $G_{\mathcal{B}}$-mappings must be a family of generalized $G_{\mathcal{B}}$-majorant mappings. If $F = S$ is a single-valued mapping and $A_i(x)$ is an FC-subspace of Y_i for each $x \in X$, then condition $y_i \notin A_i(S(y))$ for each $y \in Y$ implies that condition (a) in (1) holds. Indeed, if (a) is false, then there exist $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, $\{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$, and $\overline{y} \in \varphi_N(\Delta_k)$ such that $F(\overline{y}) = S\overline{y} \in \bigcap_{j=0}^{k} A_i^{-1}(\pi_i(y_{i_j}))$ and hence $\pi_i(y_{i_j}) \in A_i(S\overline{y})$ for each $j = 0, \ldots, k$. It follows from $\overline{y} \in \varphi_N(\Delta_k)$ that $\pi_i(\overline{y}) \in (\varphi_{N_i}(\Delta_k))$ where $N_i = \pi_i(N)$. It follows from $A_i(S\overline{y})$ being an FC-subspace of Y_i that $\pi_i(\overline{y}) \in (\varphi_{N_i}(\Delta_k)) \subset A_i(S\overline{y})$ which contradicts condition $y_i \notin A_i(S(y))$ for each $y \in Y$. Hence each $L_{\mathcal{S}}$-mapping (resp., $L_{\mathcal{S}}$-majorant mapping) introduced by Deguire et al. (see [2, page 934]) must be a generalized $G_{\mathcal{B}}$-mapping (resp., $G_{\mathcal{B}}$-majorant mapping). The inverse is not true in general.

3. Maximal Elements

In order to obtain our main results, we need the following lemmas.

Lemma 3.1 ([3]). *Let X and Y be topological spaces, let K be a nonempty compact subset of X, and let $G : X \to 2^Y$ be a set-valued mapping such that $G(x) \neq \emptyset$ for each $x \in K$. Then the following conditions are equivalent:*

(1) *G have the compactly local intersection property;*

(2) *for each $y \in Y$, there exists an open subset O_y of X (which may be empty) such that $O_y \bigcap K \subset G^{-1}(y)$ and $K = \bigcup_{y \in Y}(O_y \bigcap K)$;*

(3) *there exists a set-valued mapping $F : X \to 2^Y$ such that for each $y \in Y$, $F^{-1}(y)$ is open or empty in X, $F^{-1}(y) \bigcap K \subset G^{-1}(y), \forall y \in Y$, and $K = \bigcup_{y \in Y}(F^{-1}(y) \bigcap K)$;*

(4) *for each* $x \in K$, *there exists* $y \in Y$ *such that* $x \in \operatorname{cint} G^{-1}(y) \bigcap K$ *and* $K = \bigcup_{y \in Y}(\operatorname{cint} G^{-1}(y) \bigcap K) = \bigcup_{y \in Y}(G^{-1}(y) \bigcap K)$;

(5) $G^{-1} : Y \to 2^X$ *is transfer compactly open valued on* Y.

Lemma 3.2 ([8]). *Let* X *be a topological space, and let* (Y, φ_N) *be an FC-space,* $F \in \mathcal{B}(Y, X)$ *and* $A : X \to 2^Y$ *such that*

(i) *for each* $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ *and for each* $\{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$,

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{cint} A^{-1}\left(y_{i_j}\right) \right) = \emptyset, \tag{3.1}$$

(ii) $A^{-1} : Y \to 2^X$ *is transfer compactly open valued;*

(iii) *there exists a nonempty set* $Y_0 \subset Y$ *and for each* $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, *there exists a compact FC-subspace* L_N *of* Y *containing* $Y_0 \cup N$ *such that* $K = \bigcap_{y \in Y_0} (\operatorname{cint} A^{-1}(y))^c$ *is empty or compact in* X, *where* $(\operatorname{cint} A^{-1}(y))^c$ *denotes the complement of* $\operatorname{cint} A^{-1}(y)$.

Then there exists a point $\hat{x} \in X$ *such that* $A(\hat{x}) = \emptyset$.

Theorem 3.3. *Let* X *be a topological space, let* K *be a nonempty compact subset of* X, *and let* (Y, φ_N) *be an FC-space,* $F \in \mathcal{B}(Y, X)$ *and* $A : X \to 2^Y$ *be a generalized* $G_{\mathcal{B}}$*-mapping such that*

(i) *for each* $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, *there exists a compact FC-subspace* L_N *of* Y *containing* N *such that for each* $x \in X \setminus K$, $L_N \bigcap \operatorname{cint} A(x) \neq \emptyset$.

Then there exists a point $\hat{x} \in K$ *such that* $A(\hat{x}) = \emptyset$.

Proof. Suppose that $A(x) \neq \emptyset$ for each $x \in X$. Since A is a generalized $G_{\mathcal{B}}$-mapping, A^{-1} is transfer compactly open valued. By Lemma 3.1, we have

$$K = \bigcup_{y \in Y} \left(\operatorname{cint} A^{-1}(y) \bigcap K \right). \tag{3.2}$$

Since K is compact, there exists a finite set $N = \{y_o, \ldots, y_n\} \in \langle Y \rangle$ such that

$$K = \bigcup_{i=0}^{n} \left(\operatorname{cint} A^{-1}(y_i) \bigcap K \right). \tag{3.3}$$

By condition (i) and $F \in \mathcal{B}(Y, X)$, there exists a compact FC-subspace L_N of Y containing N and $F(L_N)$ is compact in X, and hence we have

$$F(L_N) = \bigcup_{y \in L_N} \left(\operatorname{cint} A^{-1}(y) \bigcap F(L_N) \right). \tag{3.4}$$

By using similar argument as in the proof of Lemma 3.2, we can show that there exists $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$. Condition (i) implies that $\hat{x}$ must be in K. This completes the proof. □

Remark 3.4. Theorem 3.3 generalizes in [4, Theorem 2.2] in the following several aspects: (a) from G-convex space to FC-space without linear structure; (b) from $G_{\mathcal{B}}$-mappings to generalized $G_{\mathcal{B}}$-mappings.

Theorem 3.5. *Let X be a topological space, and let (Y, φ_N) be an FC-space. Let $F \in \mathcal{B}(Y, X)$ and $A : X \to 2^Y$ be a generalized $G_{\mathcal{B}}$-majorized mapping such that*

(i) *there exists a paracompact subset E of X such that $\{x \in X : A(x) \neq \emptyset\} \subset E$;*

(ii) *there exists a nonempty set $Y_0 \subset Y$ and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing $Y_0 \cup N$ such that the set $K = \bigcap_{y \in Y_0} (\operatorname{cint} A^{-1}(y))^c$ is empty or compact.*

Then there exists a point $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$.

Proof. Suppose that $A(x) \neq \emptyset$ for each $x \in X$. Since A is a generalized $G_{\mathcal{B}}$-majorized, for each $x \in X$, there exists an open neighborhood $N(x)$ of x in X and a generalized $G_{\mathcal{B}}$-mapping $A_x : X \to 2^Y$ such that

(a) $A(z) \subset A_x(z)$ for each $z \in N(x)$,

(b) for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $\{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$, $F(\varphi_N(\Delta_k)) \bigcap (\bigcap_{j=0}^{k} \operatorname{cint} A_x^{-1}(y_{i_j})) = \emptyset$,

(c) A_x^{-1} is transfer compactly open in Y,

(d) for any $N \in \langle \{x \in X : A(x) \neq \emptyset\} \rangle$, the mapping $\bigcap_{x \in N} A_x^{-1}$ is transfer compactly open in X.

Since $A(x) \neq \emptyset$ for each $x \in X$, it follows from condition (i) that $X = \{x \in X : A(x) \neq \emptyset\} = E$ is paracompact. By Dugundji in [20, Theorem VIII.1.4], the open covering $\{N(x) : x \in X\}$ has an open precise locally finite refinement $\{O(x) : x \in X\}$, and for each $x \in X, \overline{O(x)} \subset N(x)$ since X is normal. For each $x \in X$, define a mapping $B_x : X \to 2^Y$ by

$$B_x(z) = \begin{cases} A_x(z), & \text{if } z \in \overline{O(x)}, \\ Y, & \text{if } z \in X \setminus \overline{O(x)}. \end{cases} \tag{3.5}$$

Then for each $y \in Y$, we have

$$\begin{aligned} B_x^{-1}(y) &= \left\{ z \in \overline{O(x)} : y \in A_x(z) \right\} \bigcup \left\{ z \in X \setminus \overline{O(x)} : y \in Y \right\} \\ &= \left(A_x^{-1}(y) \bigcap \overline{O(x)} \right) \bigcup \left(X \setminus \overline{O(x)} \right) \\ &= \left[A_x^{-1}(y) \bigcup \left(X \setminus \overline{O(x)} \right) \right] \bigcap \left[\overline{O(x)} \bigcup \left(X \setminus \overline{O(x)} \right) \right] = A_x^{-1}(y) \bigcup X \setminus \overline{O(x)}. \end{aligned} \tag{3.6}$$

Hence $B_x^{-1}(y)$ is transfer compactly open in Y by (c).

Now define a mapping $B : X \to 2^Y$ by

$$B(z) = \bigcap_{x \in X} B_x(z), \quad \forall z \in X. \tag{3.7}$$

We claim that B is a generalized $G_{\mathcal{B}}$-mapping and $A(z) \subset B(z)$ for each $z \in X$. Indeed, for any nonempty compact subset C of X and each $y \in Y$ with $B^{-1}(y) \cap C \neq \emptyset$, we may take any fixed $u \in B^{-1}(y) \cap C$. Since $\{O(x) : x \in X\}$ is locally finite, there exists an open neighborhood V_u of u in X such that $\{x \in X : V_u \cap O(x) \neq \emptyset\} = \{x_1, \ldots, x_n\}$ is a finite set. If $x \notin \{x_1, \ldots, x_n\}$, then $\emptyset = V_u \cap O(x) = V_u \cap \overline{O(x)}$, and hence $B_x(z) = Y$ for all $z \in V_u$ which implies that $B(z) = \bigcap_{x \in X} B_x(z) = \bigcap_{i=1}^{n} B_{x_i}(z)$ for all $z \in V_u$. It follows that for each $y \in Y$,

$$\begin{aligned} B^{-1}(y) &= \{z \in X : y \in B(z)\} \supset \{z \in V_u : y \in B(z)\} \\ &= \left\{ z \in V_u : y \in \bigcap_{i=1}^{n} B_{x_i}(z) \right\} = V_u \bigcap \left(\bigcap_{i=1}^{n} B_{x_i}^{-1}(y) \right). \end{aligned} \tag{3.8}$$

For any nonempty compact subset C of X and each $y \in Y$, if $v \in V_u \cap (\bigcap_{i=1}^{n} B_{x_i}^{-1}(y)) \bigcap C \subset B^{-1}(y) \bigcap C$. Since V_u is open in X, it follows from (d) that there exists $y' \in Y$ such that

$$\begin{aligned} v \in V_u \bigcap \operatorname{cint}\left(\bigcap_{i=1}^{n} B_{x_i}^{-1}(y') \right) \bigcap C &= \operatorname{cint}\left(V_u \bigcap \bigcap_{i=1}^{n} B_{x_i}^{-1}(y') \right) \bigcap C \\ &= \operatorname{cint} B^{-1}(y') \bigcap C. \end{aligned} \tag{3.9}$$

This proves that $B^{-1} : Y \to 2^X$ is transfer compactly open valued in Y.

On the other hand, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $N_1 = \{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$, if $t \in \bigcap_{j=0}^{k} \operatorname{cint} B^{-1}(y_{i_j})$, then $N_1 \subset \operatorname{cint} B(t)$. Since there exists $x_0 \in X$ such that $t \in \overline{O(x_0)}$ and $N_1 \subset \operatorname{cint} B(t) \subset \operatorname{cint} B_{x_0}(t) = \operatorname{cint} A_{x_0}(t)$, we have $t \in \bigcap_{j=0}^{k} \operatorname{cint} A_{x_0}^{-1}(y_{i_j})$, and hence $t \notin F(\varphi_N(\Delta_k))$ by (b). Hence we have

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{cint} B^{-1}\left(y_{i_j}\right) \right) = \emptyset \tag{3.10}$$

for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $N_1 = \{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$. This shows that B is a generalized $G_{\mathcal{B}}$-mapping.

For each $z \in X$, if $y \notin B(z)$, then there exists an $x_0 \in X$ such that $y \notin B_{x_0}(z) = A_{x_0}(z)$ and $z \in \overline{O(x_0)} \subset N(x_0)$. It follows from (a) that $y \notin A(z)$. Hence we have $A(z) \subset B(z)$ for each $z \in X$. By condition (ii), there exists a nonempty set $Y_0 \subset Y$ and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing $Y_0 \cup N$ such that the set $K = \bigcap_{y \in Y_0} (\operatorname{cint} A^{-1}(y))^c$ is empty or compact. Note that $A(z) \subset B(z)$ for each $z \in X$ implies $(\operatorname{cint} B^{-1}(y))^c \subset (\operatorname{cint} A^{-1}(y))^c$ for each $y \in Y$. Hence $K' = \bigcap_{y \in Y_0} (\operatorname{cint} B^{-1}(y))^c \subset K$ and K' is empty or compact. By Lemma 3.2, there exists a point $\overline{x} \in X$ such that $B(\overline{x}) = \emptyset$, and hence $A(\overline{x}) = \emptyset$ which contradicts the assumption that $A(x) \neq \emptyset$ for each $x \in X$. Therefore, there exists $\widehat{x} \in X$ such that $A(\widehat{x}) = \emptyset$. □

Theorem 3.6. *Let X be a topological space, let K be a nonempty compact subset of X and (Y, φ_N) be an FC-space. Let $F \in \mathcal{B}(Y, X)$ and $A : X \to 2^Y$ be a generalized $G_{\mathcal{B}}$-majorized mapping such that*

(i) *there exists a paracompact subset E of X such that $\{x \in X : A(x) \neq \emptyset\} \subset E$;*

(ii) *for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing N such that for each $x \in X \setminus K$, $L_N \bigcap \operatorname{cint} A(x) \neq \emptyset$.*

Then there exists $\hat{x} \in K$ such that $A(\hat{x}) = \emptyset$.

Proof. Suppose that $A(x) \neq \emptyset$ for each $x \in X$. By using similar argument as in the proof of Theorem 3.5, we can show that there exists a generalized $G_{\mathcal{B}}$-mapping $B : X \to 2^Y$ such that $A(x) \subset B(x)$ for each $x \in X$. It follows from condition (ii) that for each $x \in X \setminus K, L_N \cap \operatorname{cint} B(x) \neq \emptyset$. By Theorem 3.3, there exists $\overline{x} \in K$ such that $B(\overline{x}) = \emptyset$, and hence $A(\overline{x}) = \emptyset$ which contradicts the assumption that $A(x) \neq \emptyset$ for each $x \in X$. Therefore, there exists $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$. Condition (ii) implies $\hat{x} \in K$. This completes the proof. □

Remark 3.7. Theorem 3.5 generalizes [4, Theorem 2.3] in several aspects: Section 1(1) from G-convex space to FC-space without linear structure; Section 1(2) from a $G_{\mathcal{B}}$-majorized mapping to a generalized $G_{\mathcal{B}}$-majorized mapping; Section 1(3) condition (ii) of Theorem 3.5 is weaker than condition (ii) of [4, Theorem 2.3]. If X is compact, condition (i) is satisfied trivially. If $X = (Y, \varphi_N)$ is a compact FC-space, then by letting $K = X = Y = L_N$ for all $N \in \langle X \rangle$, conditions (i) and (ii) are satisfied automatically. Theorem 3.6 unifies and generalizes Shen's [14, Theorem 2.1, Corollary 2.2 and Theorem 2.3] in the following ways: Section 2(1) from CH-convex space to FC-space without linear structure; Section 2(2) from H-majorized correspondences to generalized $G_{\mathcal{B}}$-majorized mapping; Section 2(3) condition (ii) of Theorem 3.6 is weaker than that in the corresponding results of Shen in [14]. Theorem 3.6 also generalizes in [4, Theorem 2.4], Ding in [15, Theorem 5.3], and Ding and Yuan in [16, Theorem 2.3] in several aspects.

Corollary 3.8. *Let X be a compact topological space, and let (Y, φ_N) be an CFC-space. Let $F \in \mathcal{B}(Y, X)$ and $A : X \to 2^Y$ be a generalized $G_{\mathcal{B}}$-majorized mapping. Then there exists a point $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$.*

Proof. The conclusion of Corollary 3.8 follows from Theorem 3.6 with $E = K = X$. □

Corollary 3.9. *Let X be a topological space, and let (Y, φ_N) be an CFC-space. Let $F \in \mathcal{B}(Y, X)$ be a compact mapping and $A : X \to 2^Y$ be a generalized $G_{\mathcal{B}}$-majorized mapping. Then there exists a point $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$.*

Proof. Since F is a compact mapping, there exists a compact subset X_0 of X such that $F(Y) \subset X_0$. The mapping $A|_{X_0} : X_0 \to 2^Y$ be the restriction of A to X_0. It is easy to see that $A|_{X_0}$ is also generalized $G_{\mathcal{B}}$-majorized. By Corollary 3.8, there exists $\hat{x} \in X_0$ such that $A|_{X_0}(\hat{x}) = A(\hat{x}) = \emptyset$. □

Remark 3.10. Corollary 3.8 generalizes Deguire et al. [2, Theorem 1] in the following ways: (1.1) from a convex subset of Hausdorff topological vector space to an FC-space without linear structure; (1.2) from a $L_{\mathcal{S}}$-majorized mapping to a generalized $G_{\mathcal{B}}$-majorized mapping. Corollary 3.8 also generalizes [4, Corollary 2.3] from CG-convex space to CFC-space and from a $G_{\mathcal{B}}$-majorized mapping to a generalized $G_{\mathcal{B}}$-majorized mapping. Corollary 3.9 generalizes [2, Theorem 2] and [4, Corollary 2.4] in several aspects.

Theorem 3.11. *Let X be a topological space, and let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an FC-space, and let $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 2.2. Let $F \in \mathcal{B}(Y, X)$ such that for each $i \in I$,*

(i) *let $A_i : X \to 2^{Y_i}$ be a generalized $G_{\mathcal{B}}$-majorized mapping;*

(ii) $\bigcup_{i\in I}\{x \in X : A_i(x) \neq \emptyset\} = \bigcup_{i\in I} \operatorname{cint}\{x \in X : A_i(x) \neq \emptyset\}$;

(iii) *there exists a paracompact subset E_i of X such that $\{x \in X : A_i(x) \neq \emptyset\} \subset E_i$;*

(iv) *there exists a nonempty set $Y_0 \subset Y$ and for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing $Y_0 \bigcup N$ such that the set $\bigcap_{y\in Y_0} \operatorname{ccl}\{x \in X : \exists i \in I(x), \pi_i(y) \notin A_i(x)\}$ is empty or compact, where $I(x) = \{i \in I : A_i(x) \neq \emptyset\}$.*

Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

Proof. For each $x \in X$, $I(x) = \{i \in I : A_i(x) \neq \emptyset\}$. Define $A : X \to 2^Y$ by

$$A(x) = \begin{cases} \displaystyle\bigcap_{i\in I(x)} \pi_i^{-1}(A_i(x)), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset. \end{cases} \tag{3.11}$$

Then for each $x \in X$, $A(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. Let $x \in X$ with $A(x) \neq \emptyset$, then there exists $j_0 \in I(x)$ such that $A_{j_0}(x) \neq \emptyset$. By condition (ii), there exists $i_0 \in I(x)$ such that $x \in \operatorname{cint}\{x \in X : A_{i_0}(x) \neq \emptyset\}$. Since A_{i_0} is generalized $G_{\mathcal{B}}$-majorized, there exist an open neighborhood $N(x)$ of x in X and a generalized $G_{\mathcal{B}}$-majorant A_{x,i_0} of A_{i_0} at x such that

(a) $A_{i_0}(z) \subset A_{x,i_0}(z)$ for all $z \in N(x)$,

(b) for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ and $\{y_{r_0}, \dots, y_{r_k}\} \subset N$,

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{cint} A_{x,i_0}^{-1}\left(\pi_{i_0}\left(y_{r_j}\right)\right) \right) = \emptyset, \tag{3.12}$$

(c) $A_{x,i_0}^{-1} : Y_i \to 2^X$ is transfer compactly open in Y_i,

(d) for each $N \in \langle \{x \in X : A_{i_0}(x) \neq \emptyset\} \rangle$, the mapping $\bigcap_{x\in N} A_{x,i_0}^{-1}$ is transfer compactly open in Y_i.

Without loss of generality, we can assume that $N(x) \subset \operatorname{cint}\{x \in X : A_{i_0}(x) \neq \emptyset\}$. Hence, $A_{i_0}(z) \neq \emptyset$ for each $z \in N(x)$. Define $B_{x,i_0} : X \to 2^Y$ by

$$B_{x,i_0}(z) = \pi_{i_0}^{-1}(A_{x,i_0}(z)), \quad \forall z \in X. \tag{3.13}$$

We claim that B_{x,i_0} is a generalized $G_{\mathcal{B}}$-majorant of A at x. Indeed, we have

(a′) for each $z \in N(x)$, $A(z) = \bigcap_{i\in I(z)} \pi_i^{-1}(A_i(z)) \subset \pi_{i_0}^{-1}(A_{i_0}(z)) \subset \pi_{i_0}^{-1}(A_{x,i_0}(z)) = B_{x,i_0}(z)$, (b′) for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and $M = \{y_{r_0}, \ldots, y_{r_k}\} \subset N$, if $u \in \bigcap_{j=0}^{k} \operatorname{cint} B_{x,i_0}^{-1}(\pi_{i_0}(y_{r_j}))$, then $M \subset \operatorname{cint} B_{x,i_0}(u)$. It is easy to see that $\pi_{i_0}(M) \subset \operatorname{cint} \pi_{i_0}(B_{x,i_0}(u))$, so that $\pi_{i_0}(M) \subset \operatorname{cint} A_{x,i_0}(u)$, i.e., $u \in \bigcap_{j=0}^{k} \operatorname{cint} A_{x,i_0}^{-1}(\pi_{i_0}(y_{r_j}))$ and hence $u \notin F(\varphi_N(\Delta_k))$ by (b). It follows that

$$F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{cint} B_{x,i_0}^{-1}\left(\pi_{i_0}\left(y_{r_j}\right)\right) \right) = \emptyset, \tag{3.14}$$

(c′) for each $y \in Y$, we have that

$$B_{x,i_0}^{-1}(y) = A_{x,i_0}^{-1}(\pi_{i_0}(y)) \tag{3.15}$$

is transfer compactly open in Y by (c).

Hence B_{x,i_0} is a generalized $G_{\mathcal{B}}$-majorant of A at x.

For each $N \in \langle \{x \in X : A_{i_0}(x) \neq \emptyset\} \rangle$ and $y \in Y$, by (3.15), we have

$$\bigcap_{x \in N} B_{x,i_0}^{-1}(y) = \bigcap_{x \in N} A_{x,i_0}^{-1}(\pi_{i_0}(y)). \tag{3.16}$$

It follows from (d) that $\bigcap_{x \in N} B_{x,i_0}^{-1}$ is transfer compactly open in Y.

Hence $A : X \to 2^Y$ is generalized $G_{\mathcal{B}}$-majorized. By condition (iii), we have

$$\{x \in X : A(x) \neq \emptyset\} \subset \{x \in X : A_{i_0}(x) \neq \emptyset\} \subset E_{i_0}. \tag{3.17}$$

By condition (iv), there exists a nonempty set $Y_0 \subset Y$ and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing $Y_0 \bigcup N$. By the definition of A, for each $y \in Y_0$, we have

$$\begin{aligned} A^{-1}(y) = \{x \in X : y \in A(x)\} &= \left\{ x \in X : y \in \bigcap_{i \in I(x)} \pi_i^{-1}(A_i(x)) \right\} \\ &= \left\{ x \in X : \pi_i(y) \in \bigcap_{i \in I(x)} (A_i(x)) \right\}. \end{aligned} \tag{3.18}$$

It follows from condition (iv) that $K = \bigcap_{y \in Y_0} (\operatorname{cint} A^{-1}(y))^c = \bigcap_{y \in Y_0} \operatorname{ccl}\{x \in X : \exists i \in I(x), \pi_i(y) \notin A_i(x)\}$ is empty or compact and hence all conditions of Theorem 3.5 are satisfied. By Theorem 3.5, there exists $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$ which implies $I(\hat{x}) = \emptyset$, that is, $A_i(\hat{x}) = \emptyset$ for each $i \in I$. □

Theorem 3.12. *Let X be a topological space, and let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an CFC-space, and let $Y = \prod_{i\in I} Y_i$. Let $F \in \mathcal{B}(y, x)$ be a compact mapping such that for each $i \in I$,*

(i) *let $A_i : X \to 2^{Y_i}$ be a generalized $G_{\mathcal{B}}$-majorized mapping;*

(ii) $\bigcup_{i\in I}\{x \in X : A_i(x) \neq \emptyset\} = \bigcup_{i\in I} \operatorname{cint}\{x \in X : A_i(x) \neq \emptyset\}$.

Then there exists $\widehat{x} \in X$ such that $A_i(\widehat{x}) = \emptyset$ for each $i \in I$.

Proof. Since for each $i \in I$, let (Y_i, φ_{N_i}) be an CFC-space, then for each $N_i \in \langle Y_i \rangle$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i. Let $L_N = \prod_{i\in I} L_{N_i}$ and $N = \prod_{i\in I} N_i \in \langle Y \rangle$, then L_N is a compact FC-subspace of Y for each $N \in \langle Y \rangle$, L_N is a compact FC-subspace of Y containing N. Hence (Y, φ_N) is also an CFC-space.

For each $x \in X$, $I(x) = \{i \in I : A_i(x) \neq \emptyset\}$. Define $A : X \to 2^Y$

$$A(x) = \begin{cases} \bigcap_{i\in I(x)} \pi_i^{-1}(A_i(x)), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset. \end{cases} \tag{3.19}$$

Then for each $x \in X$, $A(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. By using similar argument as in the proof of Theorem 3.11, we can show that $A : X \to 2^Y$ is a generalized $G_{\mathcal{B}}$-majorized mapping. By Corollary 3.9, there exists $\widehat{x} \in X$ such that $A(\widehat{x}) = \emptyset$, and so $I(\widehat{x}) = \emptyset$. Hence, we have $A_i(\widehat{x}) = \emptyset$ for each $i \in I$. □

Theorem 3.13. *Let X be a topological space, let K be a nonempty compact subset of X, and let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an FC-space, and let $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 2.2. Let $F \in \mathcal{B}(Y, X)$ such that for each $i \in I$, $A_i : X \to 2^{Y_i}$ be a generalized $G_{\mathcal{B}}$-mapping such that*

(i) *for each $i \in I$ and $N_i \in \langle Y_i \rangle$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap \operatorname{cint} A_i(x) \neq \emptyset$.*

Then there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$ for each $i \in I$.

Proof. Suppose that the conclusion is not true, then for each $x \in K$, there exists $i \in I$ such that $A_i(x) \neq \emptyset$. Since A_i is a generalized $G_{\mathcal{B}}$-mapping, A_i^{-1} is transfer compactly open valued. By Lemma 3.1, we have

$$K \subset \bigcup_{i\in I} \bigcup_{y_i\in Y_i} \left(\operatorname{cint} A_i^{-1}(y_i)\right). \tag{3.20}$$

Since K is compact, there exists a finite set $J \subset I$ such that for each $j \in J$, there exists $N_j = \{y_j^1, y_j^2, \ldots, y_j^{m_j}\} \subset Y_j$ with $K \subset \bigcup_{j\in J} \bigcup_{k=1}^{m_j} (\operatorname{cint} A_j^{-1}(y_j^k))$. It follows that for each $x \in K$, there exists a $j \in J \subset I$ such that $N_j \bigcap \operatorname{cint} A_j(x) \neq \emptyset$. We may take any fixed $y^0 = (y_i^0)_{i\in I} \in Y$. For each $i \in I \setminus J$, let $N_i = \{y_i^0\}$. By condition (i), for each $i \in I$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap \operatorname{cint} A_i(x) \neq \emptyset$. Hence for each $x \in X$, there exists $i \in I$ such that $L_{N_i} \bigcap \operatorname{cint} A_i(x) \neq \emptyset$. Let $L_N = \prod_{i\in I} L_{N_i}$, then L_N is a compact FC-subspace of Y and hence it is also a compact CFC-space. Let $X_0 = F(L_N)$, then X_0 is compact in X. Define $A_i' : X_0 \to 2^{L_{N_i}}$ by $A_i'(x) = L_{N_i} \bigcap A_i(x)$. For each $y_i \in L_{N_i}$, we have

$$(A_i')^{-1}(y_i) = \left\{x \in X_0 : y_i \in L_{N_i} \bigcap A_i(x)\right\} = X_0 \bigcap A_i^{-1}(y_i). \tag{3.21}$$

Since $A_i^{-1}(y_i)$ is transfer compactly open valued in Y_i for each $i \in I$ and $y_i \in Y_i$, so that we claim that $(A_i')^{-1}(y_i)$ is transfer open valued in L_{N_i}. Noting that each A_i is a generalized $G_{\mathcal{B}}$-mapping, for each $M = \{y^0, \ldots, y^m\} \in \langle L_N \rangle \subset \langle Y \rangle$ and $M_1 = \{y^{r_0}, \ldots, y^{r_k}\} \subset M$, we have

$$\begin{aligned} F(\varphi_M(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{cint} (A_i')^{-1}(\pi_i(y^{r_j})) \right) &= F(\varphi_M(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{cint}\left(X_0 \bigcap A_i^{-1}(\pi_i(y^{r_j})) \right) \right) \\ &\subset F(\varphi_M(\Delta_k)) \bigcap \left(\bigcap_{j=0}^{k} \operatorname{cint} A_i^{-1}(\pi_i(y^{r_j})) \right) = \emptyset, \end{aligned} \tag{3.22}$$

where $\Delta_k = \operatorname{co}(\{e_{i_j} : j = 0, \ldots, k\})$.

Hence for each $i \in I$, A_i' is a generalized $G_{\mathcal{B}}$-mapping and hence it is also a generalized $G_{\mathcal{B}}$-majorized mapping. All conditions of Corollary 3.8 are satisfied. By Corollary 3.8, there exists $\overline{x} \in X_0 \subset X$ such that $A_i'(\overline{x}) = L_{N_i} \bigcap A_i(\overline{x}) = \emptyset$ for each $i \in I$, so we have $L_{N_i} \bigcap \operatorname{cint} A_i(\overline{x}) \subset L_{N_i} \bigcap A_i(\overline{x}) = A_i'(\overline{x}) = \emptyset$ which contradicts the fact that for each $x \in X \setminus K$ there exists $i \in I$ such that $L_{N_i} \bigcap \operatorname{cint} A_i(x) \neq \emptyset$. Therefore, there exists $\widehat{x} \in K$ such that $A_i(\widehat{x}) = \emptyset$ for each $i \in I$. □

Remark 3.14. Theorem 3.11 generalizes [4, Theorem 2.5] in several aspects. Theorem 3.12 improves [2, Theorem 3] from convex subsets of topological vector spaces to CFC-spaces without linear structure and from a family of $L_{\mathcal{S}}$-majorized mappings to the family of generalized $G_{\mathcal{B}}$-majorized mappings. Theorem 3.13 generalizes [4, Theorem 2.6] in several aspects: (1.1) from G-convex spaces to FC-spaces without linear structure; (1.2) from a $G_{\mathcal{B}}$-mapping to a generalized $G_{\mathcal{B}}$-mapping; (1.3) condition (i) of Theorem 3.13 is weaker than condition (i) of [4, Theorem 2.6]. Theorem 3.13 improves and generalizes [2, Theorem 7] in the following ways: (2.1) from nonempty convex subsets of Hausdorff topological vector spaces to FC-space without linear structure; (2.2) from the family of $L_{\mathcal{S}}$-majorized mappings to the family of generalized $G_{\mathcal{B}}$-majorized mappings.

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