Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 209309, 13 pages
doi:10.1155/2010/209309

Research Article

Oscillatory Criteria for the Two-Dimensional Difference Systems

Jin-Fa Cheng[1] and Yu-Ming Chu[2]

[1] *Department of Mathematics, Xiamen University, Xiamen 361005, China*
[2] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

Received 5 April 2010; Revised 3 June 2010; Accepted 17 June 2010

Academic Editor: Alberto Cabada

We establish some necessary and sufficient conditions for oscillation of the solutions of the following two-dimensional difference system: $\Delta x_n = f(n, y_n)$, $\Delta y_n = -g(n, x_n)$, where $f(n,u)$ and $g(n,u)$ are strongly superlinear or sublinear functions.

1. Introduction

We consider the following two-dimensional nonlinear difference system as follows:

$$\begin{aligned} \Delta x_n &= f(n, y_n), \\ \Delta y_n &= -g(n, x_n), \end{aligned} \tag{1.1}$$

where $\Delta x_n = x_{n+1} - x_n$, $\Delta y_n = y_{n+1} - y_n$, $f(n,u)$ and $g(n,u)$ are strongly superlinear or sublinear functions.

Now we pose some conditions on functions f and g:

$(H1)$: $uf(n,u) > 0$ and $ug(n,u) > 0$ for $u \neq 0$;

$(H2)$: $f(n,u)$ and $g(n,u)$ are continuous real-valued functions, and nondecreasing with respect to u;

$(H3)$: it is

$$\sum_{n=n_0}^{\infty} f(n, \pm c) = \pm\infty \tag{1.2}$$

for each $c > 0$.

Definition 1.1. Suppose that $f, g : N \times R \to R$ are real-valued functions. α and β are the quotients of positive odd numbers.

(1) f and g are said to be strongly superlinear if there exist constants $\alpha > 0$ and $\beta > 0$ with $\alpha\beta > 1$, such that $f(n,u)/|u|^{\alpha} \operatorname{sgn} u$ and $g(n,u)/|u|^{\beta} \operatorname{sgn} u$ are nondecreasing with respect to $|u|$ for each fixed $n \in N$.

(2) f and g are said to be strongly sublinear if there exist constants $\alpha > 0$ and $\beta > 0$ with $\alpha\beta < 1$, such that $f(n,u)/|u|^{\alpha} \operatorname{sgn} u$ and $g(n,u)/|u|^{\beta} \operatorname{sgn} u$ are nonincreasing with respect to $|u|$ for each fixed $n \in N$.

The solutions of (1.1) are said to be nonoscillatory if $\{x_n\}$ or $\{y_n\}$ is eventually positive or negative. Otherwise the solutions are called oscillatory.

Some oscillation results for the difference system (1.1) in the case of $g(n, x_n) = a_n x_n^{\beta}$ with $a_n > 0$ have been established by many authors. In particular, if $f(n, y_n) = b_n y_n$ and $b_n > 0$, then the difference system (1.1) is reduced to the well-known second-order nonlinear difference equation:

$$\Delta\left(\frac{1}{b_n}\Delta x_n\right) + a_n x_n^{\beta} = 0. \tag{1.3}$$

Also, if $b_n = 1$, then (1.3) becomes

$$\Delta^2 x_n + a_n x_n^{\beta} = 0. \tag{1.4}$$

Furthermore, if $f(n, y_n) = b_n y_n^{\alpha}$ and α is a ratio of odd positive integers, then (1.1) reduces to the well-known quasilinear difference equation:

$$\Delta\left(\frac{1}{b_n^{1/\alpha}}(\Delta x_n)^{1/\alpha}\right) + a_n x_n^{\beta} = 0. \tag{1.5}$$

For (1.4), the following well-known Theorem A was established by Hooker and Patula [1, 2].

Theorem A. *For* (1.4), *the following statements are true.*

(1) If $0 < \beta < 1$, then every solution of (1.4) *oscillates if and only if*

$$\sum_{n=1}^{\infty} n^{\beta} a_n = \infty. \tag{1.6}$$

(2) If $\beta > 1$, then every solution of (1.4) *oscillates if and only if*

$$\sum_{n=1}^{\infty} n a_n = \infty. \tag{1.7}$$

For (1.3), *if one denotes*

$$B_n = \sum_{s=0}^{n-1} b_s \tag{1.8}$$

and assumes that

$$\lim_{n \to \infty} B_n = \sum_{s=0}^{\infty} b_s = \infty, \tag{1.9}$$

then the following theorem is proved in [3].

Theorem B. *If* (1.9) *holds, then the following statements are true.*

(1) If $0 < \beta < 1$, then every solution of (1.3) *oscillates if and only if*

$$\sum_{n=1}^{\infty} B_n^{\beta} a_n = \infty. \tag{1.10}$$

(2) If $\beta > 1$, then every solution of (1.3) *oscillates if and only if*

$$\sum_{n=1}^{\infty} B_n a_n = \infty. \tag{1.11}$$

The problem of oscillation of second-order nonlinear difference equations has attracted the attention of many mathematicians because of its physical applications [2, 4]. For some results regarding the growth of solutions of some equations related to the above mentioned see book [5], as well as the following papers [6–8]. It is an interesting problem to extend oscillation criteria for second-order nonlinear difference equations to the case of nonlinear two-dimensional difference systems since such systems include, in particular, the second-order nonlinear, half-linear, and quasilinear difference equations that are the special cases of the nonlinear two-dimensional difference systems [5, 9, 10].

The main purpose of this paper is to establish some necessary and sufficient conditions for oscillation of the nonlinear two-dimensional difference systems.

2. Main Results

In order to establish our main results, we need the following lemma.

Lemma 2.1. *Suppose that conditions* $(H1)$–$(H3)$ *are satisfied. If* $\{x_n\}$ *and* $\{y_n\}$ *are nonoscillatory solutions of* (1.1) *for* $n > n_0$, *then*

$$\operatorname{sgn} x_n = \operatorname{sgn} y_n. \tag{2.1}$$

Proof. Without loss of generality, we assume that x_n is eventually positive; that is, $x_n > 0$ for $n > n_0 > 0$. From (1.1), we clearly see that $\Delta y_n < 0$, then we know that either $y_n > 0$ or $y_n < 0$ eventually holds.

If $y_n < y_{N_1} < 0$ for $n > N_1 > n_0$, then we have

$$\Delta x_n = f(n, y_n) \leq f(n, y_N) < 0, \tag{2.2}$$

summing up from N_1 to n, and by (1.2) of $(H3)$, we get

$$x_n - x_{N_1} \leq \sum_{s=N_1}^{n-1} f(s, y_N) \longrightarrow -\infty \quad (n \longrightarrow \infty). \tag{2.3}$$

This contradiction completes the proof of the lemma. □

Theorem 2.2. *If f and g are strongly sublinear (i.e., $0 < \alpha\beta < 1$), then a necessary and sufficient condition for* (1.1) *to oscillate is that*

$$\sum_{n=n_1}^{\infty} g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right) = +\infty \tag{2.4}$$

for every $c > 0$, where $n_1 > n_0$.

Proof. Sufficiency. If (1.1) has a nonoscillatory solution x_n, then without loss of generality, we assume that x_n is eventually positive. Then, by Lemma 2.1, for n_0 sufficiently large,

$$y_n > 0, \quad \Delta x_n > 0, \quad \Delta y_n < 0, \quad \text{for } n \geq n_0. \tag{2.5}$$

Since $\{y_n\}$ is decreasing, hence there exists $c > 0$ such that

$$y_n \leq y_N \leq c, \quad \text{for } N \geq n_0. \tag{2.6}$$

Summing up

$$\Delta x_n = f(n, y_n) \tag{2.7}$$

from $s = n_0$ to $n-1$, we obtain

$$x_n - x_{n_0} = \sum_{s=n_0}^{n-1} f(s, y_s) \geq y_n^\alpha \sum_{s=n_0}^{n-1} \frac{f(s, y_s)}{y_s^\alpha} \geq y_n^\alpha \sum_{s=n_0}^{n-1} \frac{f(s, c)}{c^\alpha},$$
$$x_n \geq c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c), \tag{2.8}$$

and so

$$g(n, x_n) \geq g\left(n, c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c)\right). \tag{2.9}$$

Therefore

$$-\Delta y_n = g(n, x_n) \geq g\left(n, c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c)\right). \tag{2.10}$$

Since

$$c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c) \leq \sum_{s=n_0}^{n-1} f(s, c), \tag{2.11}$$

we have

$$\begin{aligned}
-\Delta y_n \geq g\left(n, c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c)\right) &= \frac{g\left(n, c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c)\right)}{\left(c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c)\right)^\beta} \left(c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c)\right)^\beta \\
&\geq \frac{g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right)}{\left(\sum_{s=n_0}^{n-1} f(s, c)\right)^\beta} \left(c^{-\alpha} y_n^\alpha \sum_{s=n_0}^{n-1} f(s, c)\right)^\beta \\
&= (y_n)^{\alpha\beta} c^{-\alpha\beta} g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right), \\
-\frac{\Delta y_n}{(y_n)^{\alpha\beta}} \geq c^{-\alpha\beta} g\left(n, \sum_{s=n_0}^{n-1} f(s, c)\right)&,
\end{aligned} \tag{2.12}$$

and let $n_1 > n_0$, we have

$$\sum_{i=n_1}^{n-1} \left(-\frac{\Delta y_i}{(y_i)^{\alpha\beta}}\right) \geq c^{-\alpha\beta} \sum_{i=n_1}^{n-1} g\left(i, \sum_{s=n_0}^{i-1} f(s, c)\right). \tag{2.13}$$

From

$$\int_{y_{n+1}}^{y_n} \frac{1}{u^{\alpha\beta}} du = \frac{1}{\xi^{\alpha\beta}} (y_n - y_{n+1}) \geq -\frac{\Delta y_n}{(y_n)^{\alpha\beta}}, \quad y_{n+1} < \xi < y_n, \tag{2.14}$$

we get

$$\begin{gathered} \sum_{i=n_1}^{\infty} \int_{y_{i+1}}^{y_i} \frac{1}{u^{\alpha\beta}} du \geq \sum_{i=n_1}^{\infty} - \frac{\Delta y_i}{(y_i)^{\alpha\beta}} \geq c^{-\alpha\beta} \sum_{i=n_1}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s,c) \right), \\ \sum_{i=n_1}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s,c) \right) \leq c^{\alpha\beta} \int_{c}^{y_{n_1}} \frac{1}{u^{\alpha\beta}} du < +\infty, \end{gathered} \tag{2.15}$$

which leads to a contradiction.

Necessity. If

$$\sum_{n=n_1}^{\infty} g\left(n, \sum_{s=n_0}^{n-1} f(s,c) \right) < +\infty \tag{2.16}$$

for some $c > 0$, then there exist $M > n_0 > 0$ and $c/2 > d > 0$, such that

$$\sum_{n=M}^{\infty} g\left(n, \sum_{s=n_0}^{n-1} f(s,c) \right) < d. \tag{2.17}$$

Let X be the Banach space of all the real-valued sequences $\{x_n\}$ with the norm

$$\|x\| = \sup_{n \geq M} \frac{|x_n|}{\sum_{s=n_0}^{n-1} f(s,c)}, \tag{2.18}$$

let Ψ be the subset of X defined by

$$\Psi = \left\{ \{x_n\} \in X : \sum_{s=n_0}^{n-1} f(s,d) \leq x_n \leq \sum_{s=n_0}^{n-1} f(s,2d) \right\} \tag{2.19}$$

and let $F : \Psi \to X$ be the operator defined by

$$(Fx)_n = \sum_{s=n_0}^{n-1} f\left(s, d + \sum_{i=s}^{\infty} g(i, x_i) \right). \tag{2.20}$$

Then the mapping F satisfies the assumptions of Knaster's fixed-point theorem (see [11, page 8]): F maps Ψ into itself and F is increasing. The latter statement is easy to see, and the former statement follows from

$$\begin{aligned}
(Fx)_n &\geq \sum_{n=n_0}^{n-1} f(s,d),\\
(Fx)_n &= \sum_{s=n_0}^{n-1} f\left(s, d + \sum_{i=s}^{\infty} g(i, x_i)\right) \leq \sum_{s=n_0}^{n-1} f\left(s, d + \sum_{i=s}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s, 2d)\right)\right)\\
&\leq \sum_{s=n_0}^{n-1} f\left(s, d + \sum_{i=s}^{\infty} g\left(i, \sum_{s=n_0}^{i-1} f(s, c)\right)\right) \leq \sum_{s=n_0}^{n-1} f(s, 2d)
\end{aligned} \tag{2.21}$$

for any $\{x_n\} \in \Psi$. From Knaster's fixed-point theorem, we know that there exists $\{x_n\} \in \Psi$ such that $x_n = (Fx)_n$. Let

$$y_n = d + \sum_{s=n}^{\infty} g(s, x_s), \tag{2.22}$$

then $\lim_{n\to\infty} y_n = d$ and $\Delta y_n = -g(n, x_n)$. On the other hand, we have

$$x_n = (Fx)_n = \sum_{i=n_0}^{n-1} f(i, y_i). \tag{2.23}$$

Then by (1.2) and the continuity of function f, we have that $\lim_{n\to\infty} x_n = \infty$ and $\Delta x_n = f(n, x_n)$, which leads to a contradiction and the proof of Theorem 2.2 is completed. □

Example 2.3. Considering the difference system,

$$\begin{aligned}
\Delta x_n &= 2(n+1)^{1/3} y_n^{1/3},\\
\Delta y_n &= -\frac{2n+3}{(n+1)(n+2)} x_n^{5/3}, \quad n \geq n_0.
\end{aligned} \tag{2.24}$$

Here $\alpha = 1/3$, $\beta = 5/3$, and f and g are strongly sublinear. It is easy to verify that the conditions of Theorem 2.2 are satisfied and hence all solutions are oscillatory. In fact, we clearly see that the sequence $\{(x_n, y_n)\} = \{((-1)^n, (-1)^{n+1}/(n+1))\}$ is such a solution for the difference system.

Example 2.4. Considering the difference system,

$$\begin{aligned}
\Delta x_n &= \left(\frac{n}{n+1}\right)^{1/3} y_n^{1/3},\\
\Delta y_n &= -\frac{1}{n^{5/3}} \frac{1}{n(n+1)} x_n^{5/3}, \quad n \geq n_0.
\end{aligned} \tag{2.25}$$

Here $\alpha = 1/3$, $\beta = 5/3$, and f and g are strong sublinear. We clearly see that the conditions of Theorem 2.2 are not satisfied and hence there exists a nonoscillatory solution. In fact, the sequence $\{(x_n, y_n)\} = \{(n, (n+1)/n)\}$ is such a solution.

Theorem 2.5. *If f and g are strongly superlinear (i.e., $\alpha\beta > 1$), then a necessary and sufficient condition for* (1.1) *to oscillate is that*

$$\sum_{n=1}^{\infty} f\left(n, \sum_{s=n}^{\infty} g(s, c)\right) = +\infty \tag{2.26}$$

for every $c > 0$.

Proof. Sufficiency. If (1.1) has a nonoscillatory solution x_n, then without loss of generality, we may assume that x_n is eventually positive. Then by Lemma 2.1, we have for N_1 sufficiently large,

$$x_n > 0, \quad \Delta x_n > 0, \quad \Delta y_n < 0, \quad \text{for } n > N_1. \tag{2.27}$$

Since $\{x_n\}$ is increasing, hence there exists $c > 0$ such that

$$x_n \geq y_N \geq c, \quad \text{for } N > n_0. \tag{2.28}$$

Summing up

$$\Delta y_n = -g(n, x_n) \tag{2.29}$$

from $s = n$ to ∞, we have

$$\begin{gathered} -y_n \leq y_\infty - y_n = -\sum_{s=n}^{\infty} g(s, x_s), \\ y_n \geq \sum_{s=n}^{\infty} g(s, x_s) \geq \sum_{s=n+1}^{\infty} \frac{g(s, x_s)}{x_s^\beta} x_s^\beta \geq x_{n+1}^\beta \sum_{s=n+1}^{\infty} \frac{g(s, c)}{c^\beta} = c^{-\beta} x_{n+1}^\beta \sum_{s=n+1}^{\infty} g(s, c). \end{gathered} \tag{2.30}$$

Therefore

$$\Delta x_n = f(n, y_n) \geq f\left(n, c^{-\beta} x_{n+1}^\beta \sum_{s=n+1}^{\infty} g(s, c)\right). \tag{2.31}$$

From $x_n \geq c$ and $c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c) \geq \sum_{s=n+1}^{\infty} g(s,c)$, we get

$$\begin{aligned}
\Delta x_n &\geq \frac{f\left(n, c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c)\right)}{\left(c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c)\right)^{\alpha}} \left(c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c) \right)^{\alpha} \\
&\geq \frac{f\left(n, \sum_{s=n+1}^{\infty} g(s,c)\right)}{\left(\sum_{s=n+1}^{\infty} g(s,c)\right)^{\alpha}} \left(c^{-\beta} x_{n+1}^{\beta} \sum_{s=n+1}^{\infty} g(s,c) \right)^{\alpha} \\
&= x_{n+1}^{\alpha\beta} c^{-\alpha\beta} f\left(n, \sum_{s=n+1}^{\infty} g(s,c) \right), \\
\frac{\Delta x_n}{x_{n+1}^{\alpha\beta}} &\geq c^{-\alpha\beta} f\left(n, \sum_{s=n+1}^{\infty} g(s,c) \right), \\
\sum_{i=n}^{\infty} \frac{\Delta x_i}{x_{i+1}^{\alpha\beta}} &\geq c^{-\alpha\beta} \sum_{i=n}^{\infty} f\left(i, \sum_{s=i+1}^{\infty} g(s,c) \right).
\end{aligned} \tag{2.32}$$

But

$$\begin{gathered}
\int_{x_n}^{x_{n+1}} \frac{1}{u^{\alpha\beta}} du = \frac{1}{\xi^{\alpha\beta}} (x_{n+1} - x_n) \geq \frac{\Delta x_n}{(x_{n+1})^{\alpha\beta}}, \quad x_n < \xi < x_{n+1}, \\
\sum_{s=n}^{\infty} \int_{x_s}^{x_{s+1}} \frac{1}{u^{\alpha\beta}} du \geq \sum_{s=n}^{\infty} \frac{\Delta x_s}{(x_{s+1})^{\alpha\beta}} \geq c^{-\alpha\beta} \sum_{s=n}^{\infty} f\left(s, \sum_{t=s+1}^{\infty} g(t,c) \right).
\end{gathered} \tag{2.33}$$

Therefore

$$\sum_{s=N_1}^{\infty} f\left(s, \sum_{t=s+1}^{\infty} g(t,c) \right) \leq c^{\alpha\beta} \int_{x_{N_1}}^{\infty} \frac{1}{u^{\alpha\beta}} du < +\infty, \tag{2.34}$$

which leads to a contradiction.

Necessity. If

$$\sum_{n=1}^{\infty} f\left(n, \sum_{s=n}^{\infty} g(s,c) \right) < +\infty \tag{2.35}$$

for some $c > 0$, then there exists $M > 0$ large enough, such that

$$\sum_{n=M}^{\infty} f\left(n, \sum_{s=n}^{\infty} g(s,c) \right) < \frac{c}{2}. \tag{2.36}$$

Let X be the set of all bounded and real-valued sequences $\{x_n\}$ with the norm

$$\|x\| = \sup_{n\geq M}|x_n| \tag{2.37}$$

and Ψ be the subset of X defined by

$$\Psi = \left\{\{x_n\} \in X : \frac{c}{2} \leq x_n \leq c\right\}, \tag{2.38}$$

then Ψ is a bounded, convex, and closed subset of X. Let $F : \Psi \to X$ be the operator defined by

$$(Fx)_n = c - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right). \tag{2.39}$$

Then F maps Ψ into Ψ. In fact, if $\{x_n\} \in \Psi$, then

$$\begin{aligned} c \geq (Fx)_n &\geq c - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, c)\right) \\ &\geq c - \sum_{i=M}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, c)\right) \geq \frac{c}{2}. \end{aligned} \tag{2.40}$$

Next, we show that F is continuous. Let $\{x_n^{(j)}\}$ be a convergent sequence in Ψ such that $\lim_{i\to\infty}\|x_n^{(j)} - x_n\| = 0$, then from that Ψ is closed ($\{x_n\} \in \Psi$) and the definition of F, we have

$$\begin{aligned} \left|\left(Fx^j\right)_n - (Fx)_n\right| &= \left|\sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g\left(s, x_s^j\right)\right) - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right)\right| \\ &\leq \sum_{i=n}^{\infty}\left| f\left(i, \sum_{s=i}^{\infty} g\left(s, x_s^j\right)\right) - f\left(i, \sum_{s=i}^{\infty} g(s, x_s)\right)\right|. \end{aligned} \tag{2.41}$$

Since $\sum_{i=n}^{\infty} f(i, \sum_{s=i}^{\infty} g(s, x_s)) < \sum_{i=n}^{\infty} f(i, \sum_{s=i}^{\infty} g(s, c)) < \infty$, now from the continuity of f and g together with the well-known Lebesgue's dominated convergence theorem (see [11, page 263]), we know that $\lim_{j\to\infty}\|(Fx^j)_n - (Fx)_n\| = 0$ for $\|x^{(j)} - x\| \to 0$.

Finally, we show that $F\Psi$ is precompact. Let $\{x_m\} \in \Psi$, $\{x_n\} \in \Psi$, then for large enough $m > n$ we have

$$\begin{aligned}
|(Fx)_m - (Fx)_n| &= \left| \sum_{i=m}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s) \right) - \sum_{i=n}^{\infty} f\left(i, \sum_{s=i}^{\infty} g(s, x_s) \right) \right| \\
&= \left| \sum_{i=n}^{m} f\left(i, \sum_{s=i}^{\infty} g(s, x_s) \right) \right| \\
&\le \sum_{i=n}^{m} f\left(i, \sum_{s=i}^{\infty} g(s, c) \right) < \varepsilon
\end{aligned} \tag{2.42}$$

for any $\varepsilon > 0$. From Schauder's fixed-point theorem (see [11]), we know that there exists $\{x_n\} \in \Psi$ such that $x_n = (Fx)_n$.

Let

$$y_n = \sum_{s=n}^{\infty} g(s, x_s), \tag{2.43}$$

then $\lim_{n\to\infty} y_n = 0$ and $\Delta y_n = -g(n, x_n)$. On the other hand, we have

$$x_n = (Fx)_n = c - \sum_{s=n}^{\infty} f(s, y_s). \tag{2.44}$$

Therefore, $\lim_{n\to\infty} x_n = c$ and $\Delta x_n = f(n, y_n)$, which leads to a contradiction. The proof of Theorem 2.5 is completed. □

Example 2.6. Considering the difference system,

$$\begin{aligned}
\Delta x_n &= 2^{3n+1} y_n^3, \\
\Delta y_n &= -\frac{3}{2^n} x_n^3, \quad n \ge n_0.
\end{aligned} \tag{2.45}$$

Here $\alpha = 3$, $\beta = 3$, and f and g are strongly suplinear. We clearly see that the conditions of Theorem 2.5 are satisfied and hence all solutions are oscillatory. In fact, the sequence $\{(x_n, y_n)\} = \{((-1)^n, (-1)^{n+1}/2^n)\}$ is such a solution.

Example 2.7. Considering the difference system,

$$\begin{aligned}
\Delta x_n &= \left(\frac{n}{n+1} \right)^{2/3} y_n^{2/3}, \\
\Delta y_n &= -\frac{1}{n^{5/3}} \frac{1}{n(n+1)} x_n^{5/3}, \quad n \ge n_0.
\end{aligned} \tag{2.46}$$

Here $\alpha = 2/3$, $\beta = 5/3$, and f and g are strong sublinear. However, it is easy to see that the conditions of Theorem 2.5 are not satisfied and hence there exists a nonoscillatory solution. In fact, the sequence $\{(x_n, y_n)\} = \{(n, (n+1)/n)\}$ is such a solution.

If we set $f(n, y_n) = b_n y_n^{\alpha}$, $g(n, x_n) = a_n x_n^{\beta}$, then the difference system (1.1) is reduced to (1.5). From Theorems 2.2 and 2.5, we get the following results for (1.5).

Corollary 2.8. *If* $0 < \alpha\beta < 1$, *then every solution of* (1.5) *oscillation if and only if*

$$\sum_{n=n_1}^{\infty} a_n \left(\sum_{s=n_0}^{n-1} b_s \right)^{\beta} = \infty, \tag{2.47}$$

where $n_1 > n_0$.

Corollary 2.9. *If* $\alpha\beta > 1$, *then every solution of* (1.5) *oscillation if and only if*

$$\sum_{n=1}^{\infty} b_n \left(\sum_{s=n}^{\infty} a_s \right)^{\alpha} = \infty. \tag{2.48}$$

Remark 2.10. It is easy to see that Theorems A and B are the special cases of our Corollaries 2.8 and 2.9, respectively.

Acknowledgment

The authors wish to thank the anonymous referees for their very careful reading of the paper and fruitful comments and suggestions.

References

[1] J. W. Hooker and W. T. Patula, "A second-order nonlinear difference equation: oscillation and asymptotic behavior," *Journal of Mathematical Analysis and Applications*, vol. 91, no. 1, pp. 9–29, 1983.
[2] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
[3] J. Cheng and Y. Chu, "Oscillation theorem for second-order difference equations," *Taiwanese Journal of Mathematics*, vol. 12, no. 3, pp. 623–633, 2008.
[4] R. B. Potts, "Exact solution of a difference approximation to Duffing's equation," *Journal of the Australian Mathematical Society. Series B*, vol. 23, no. 1, pp. 64–77, 1981.
[5] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, vol. 155 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1992.
[6] S. Stević, "Growth theorems for homogeneous second-order difference equations," *The ANZIAM Journal*, vol. 43, no. 4, pp. 559–566, 2002.
[7] S. Stević, "Asymptotic behaviour of second-order difference equations," *The ANZIAM Journal*, vol. 46, no. 1, pp. 157–170, 2004.
[8] S. Stević, "Growth estimates for solutions of nonlinear second-order difference equations," *The ANZIAM Journal*, vol. 46, no. 3, pp. 439–448, 2005.
[9] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, vol. 404 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[10] J. R. Graef and E. Thandapani, "Oscillation of two-dimensional difference systems," *Computers & Mathematics with Applications*, vol. 38, no. 7-8, pp. 157–165, 1999.

[11] R. P. Agarwal, M. Bohner, and W.-T. Li, *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, vol. 267 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2004.