Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 486325, 15 pages
doi:10.1155/2010/486325

*Research Article*

# Superstability and Stability of the Pexiderized Multiplicative Functional Equation

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Received 14 August 2009; Accepted 28 December 2009

Academic Editor: Yeol Je Cho



We obtain the superstability of the Pexiderized multiplicative functional equation $f(xy) = g(x)h(y)$ and investigate the stability of this equation in the following form: $1/(1+\varphi(x,y)) \le f(xy)/g(x)h(y) \le 1+\varphi(x,y)$.

## 1. Introduction

The superstability of the functional equation $f(x+y) = f(x)f(y)$ was studied by Baker et al. [1]. They proved that if $f$ is a functional on a real vector space $W$ satisfying $|f(x+y) - f(x)f(y)| \le \delta$ for some fixed $\delta > 0$ and all $x, y \in W$, then $f$ is either bounded or else $f(xy) = f(x)f(y)$ for all $x, y \in W$. This result was genealized with a simplified proof by Baker [2] as follows.

**Theorem 1.1** (Baker [2]). *Let $\delta > 0$, $S$ be a semigroup and $f : S \to C$ satisfying*

$$\left|f(xy) - f(x)f(y)\right| \le \delta \tag{1.1}$$

*for all $x, y \in S$. Put $\beta := (1+\sqrt{1+4\delta})/2$. Then $|f(x)| \le \beta$ for all $x \in S$ or else $f(xy) = f(x)f(y)$ for all $x, y \in S$.*

A different generalization of the result of Baker et al. was given by Székelyhidi [3]. It involves an interesting generalization of the class of bounded functions on a group or semigroup and may be stated as follows.

**Theorem 1.2** (Székelyhidi [3]). *Let $G$ be a commutative group with identity 1 and let $f, m : S \to C$ be functions such that there exist functions $M_1, M_2 : G \to [0,\infty)$ with*

$$|f(xy) - f(x)m(y)| \leq \min\{M_1(x), M_2(y)\} \tag{1.2}$$

*for all $x, y \in G$. Then $f$ is bounded or $m$ is an exponential and $f = f(1)m$.*

In this paper, we prove the superstability of the Pexiderized multiplicative functional equation (PMFE)

$$f(xy) = g(x)h(y). \tag{1.3}$$

That is, we prove that if $f$, $g$, $h$ are functional on a semigroup $S$ with identity 1 satisfying $g(1) = 1$ and

$$|f(xy) - g(x)h(y)| \leq \varphi(x,y) \tag{1.4}$$

for all $x, y \in S$ and for a function $\varphi : S \times S \to [0,\infty)$ with some coditions, then $g$ is bounded or else $g$ is an exponential and $h = h(1)g$. This is a generalization of the result of Székelyhidi. Also we investigate the stability of the Pexiderized multipicative functional equation (1.3) in the sense of Ger [4].

## 2. Superstability of the PMFE

In this section, let $(S,\cdot)$ be a semigroup with identity 1 and $\varphi : S \times S \to [0,\infty)$ a function with

$$\Phi_w(x) := \sum_{k=0}^{\infty} \frac{\varphi(w, x^{k+2}) + \varphi(wx, x^{k+1})}{2^k} < \infty \tag{2.1}$$

for all $x, w \in S$ and

$$\lim_{k\to\infty} \varphi\left(wx, zy^k\right) \quad \text{exists} \tag{2.2}$$

for all $x, y, z, w \in S$.

*Example 2.1.* The following functions satisfy conditions (2.1) and (2.2) above.

(a) $\varphi(x,y) = \delta$, for every $x, y \in R$ and $\delta \geq 0$.

(b) $\varphi(x,y) = |t(x)|$, for every $x, y \in S$ and $t$ is a functional on $S$.

(c) $\varphi(x,y) = |x| + 1/(1+|y|)$, for every $x, y \in R$.

(d) $\varphi(x,y) = 1/(1+|x|+|y|)$, for every $x, y \in R$.

*Example 2.2.* Let $(S,\cdot) = ([0,\infty),+)$ and also $g(x) = e^x, h(x) = e^{x+c}$,

$$f(x) = e^{x+c} + \frac{1}{1+x} \tag{2.3}$$

for all $x \in S$ and for some $c \in S$. Let $\varphi(x,y) = 1/(1+x+y)$. Then $f$, $g$, $h$, $\varphi$ satisfy the conditions (1.4), (2.1), (2.2) and

$$\left|f(x+y) - g(x)h(y)\right| = \frac{1}{1+x+y}. \tag{2.4}$$

In particular, we know that $g(0) = 1$, $g(x+y) = g(x)g(y)$, $h = h(0)g$, and $f(x+y) \neq f(x)f(y)$.

**Theorem 2.3.** *Let $(S,\cdot)$ be a semigroup with identity 1. If $f,g,h : S \to C$ are functions with $|g(m)| \geq \max\{2, 2\Phi_1(m)/|h(m)|\}$ for some $m \in S$ satisfying $g(1) = 1$ and condition (1.4), that is,*

$$\left|f(xy) - g(x)h(y)\right| \leq \varphi(x,y), \tag{2.5}$$

*then*

$$g(xy) = g(x)g(y) \tag{2.6}$$

*for all $x, y \in S$ and $h = h(1)g$.*

*Proof.* If we replace $x$ by $m$ and also $y$ by $m$ in (1.4), we get

$$\left|f\left(m^2\right) - g(m)h(m)\right| \leq \varphi(m,m). \tag{2.7}$$

Also we replace $x$ by 1 in (1.4), then we have

$$\left|f(y) - h(y)\right| \leq \varphi(1,y) \tag{2.8}$$

for all $y \in S$. An induction argument implies that for all $n \geq 2$,

$$\left|f(m^n) - g(m)^{n-1}h(m)\right| \leq \varphi\left(m, m^{n-1}\right) + \sum_{k=1}^{n-2}|g(m)|^k\left(\varphi\left(1, m^{n-k}\right) + \varphi\left(m, m^{n-k-1}\right)\right). \tag{2.9}$$

Indeed, if inequality (2.9) holds, using inequality (1.4) and (2.8) we have

$$\begin{aligned}
&\left|f\left(m^{n+1}\right)-g(m)^{n}h(m)\right| \\
&\quad\leq \left|f(mm^{n})-g(m)h(m^{n})\right|+\left|g(m)\right|\left|h(m^{n})-f(m^{n})\right| \\
&\qquad+\left|g(m)\right|\left|f(m^{n})-g(m)^{n-1}h(m)\right| \\
&\quad\leq \varphi(m,m^{n})+\left|g(m)\right|\varphi(1,m^{n}) \\
&\qquad+\left|g(m)\right|\left(\varphi\left(m,m^{n-1}\right)+\sum_{k=1}^{n-2}\left|g(m)\right|^{k}\left(\varphi\left(1,m^{n-k}\right)+\varphi\left(m,m^{n-k-1}\right)\right)\right) \\
&\quad= \varphi(m,m^{n})+\sum_{k=1}^{n-1}\left|g(m)\right|^{k}\left(\varphi\left(1,m^{n+1-k}\right)+\varphi\left(m,m^{n-k}\right)\right)
\end{aligned} \tag{2.10}$$

for all $n \geq 2$. By (2.9), we have

$$\begin{aligned}
&\left|\frac{f(m^{n})}{g(m)^{n-1}h(m)}-1\right| \\
&\quad\leq \frac{1}{|h(m)|\left|g(m)\right|}\left(\frac{\varphi(m,m^{n-1})}{\left|g(m)\right|^{n-2}}+\sum_{k=1}^{n-2}\frac{1}{g(m)^{n-k-2}}\left(\varphi\left(1,m^{n-k}\right)+\varphi\left(m,m^{n-k-1}\right)\right)\right) \\
&\quad\leq \frac{1}{|h(m)|\left|g(m)\right|}\left(\left(\varphi\left(1,m^{2}\right)+\varphi(m,m)\right)+\frac{1}{2}\left(\varphi\left(1,m^{3}\right)+\varphi\left(m,m^{2}\right)\right)+\cdots\right. \\
&\qquad\qquad\left.+\frac{1}{2^{n-3}}\left(\varphi\left(1,m^{n-1}\right)+\varphi\left(m,m^{n-2}\right)\right)+\left(\frac{1}{2^{n-2}}\varphi\left(m,m^{n-1}\right)\right)\right) \\
&\quad\leq \frac{1}{|h(m)|\left|g(m)\right|}\left(\sum_{k=0}^{n-3}\frac{1}{2^{k}}\left(\varphi\left(1,m^{k+2}\right)+\varphi\left(m,m^{k+1}\right)\right)+\frac{1}{2^{n-2}}\varphi\left(m,m^{n-1}\right)+\varphi(1,m^{n})\right) \\
&\quad\leq \frac{1}{|h(m)|\left|g(m)\right|}\sum_{k=0}^{\infty}\frac{1}{2^{k}}\left(\varphi\left(1,m^{k+2}\right)+\varphi\left(m,m^{k+1}\right)\right) \\
&\quad= \frac{\Phi_{1}(m)}{|h(m)|\left|g(m)\right|}\leq\frac{1}{2}.
\end{aligned} \tag{2.11}$$

Thus we can easily show that $|f(m^n)| \to \infty$ from $|g(m)^{n-1}h(m)| \to \infty$ as $n \to \infty$ and thus $|h(m^n)| \to \infty$ as $n \to \infty$. By (1.4),

$$\left|\frac{f(xm^{n})}{h(m^{n})}-g(x)\right|\leq\frac{\varphi(x,m^{n})}{|h(m^{n})|}, \tag{2.12}$$

and thus we have

$$g(x) = \lim_{n\to\infty} \frac{f(xm^n)}{h(m^n)} \tag{2.13}$$

for all $x \in S$. Then, by (2.2),

$$\begin{aligned} \left|g(xy) - g(x)g(y)\right| &= \lim_{n\to\infty} \frac{1}{|h(m^n)|} \left|f(xym^n) - g(x)f(ym^n)\right| \\ &= \lim_{n\to\infty} \frac{1}{|h(m^n)|} \left(\left|f(xym^n) - g(x)h(ym^n)\right| + \left|g(x)\right|\left|h(ym^n) - f(ym^n)\right|\right) \\ &\le \lim_{n\to\infty} \frac{1}{|h(m^n)|} \left(\varphi(x, ym^n) + \left|g(x)\right|\varphi(1, ym^n)\right) = 0, \end{aligned} \tag{2.14}$$

and so

$$g(xy) = g(x)g(y) \tag{2.15}$$

for all $x, y \in S$. Thus we have $|g(m^n)| = |g(m)^n| \to \infty$ as $n \to \infty$. Since

$$\left|\frac{f(xm^n)}{g(m^n)} - h(x)\right| \le \frac{\varphi(x, m^n)}{\left|g(m^n)\right|} \longrightarrow 0 \tag{2.16}$$

as $n \to \infty$, we can define $h$ by

$$h(x) = \lim_{n\to\infty} \frac{f(xm^n)}{g(m^n)} \tag{2.17}$$

for all $x \in S$. Then

$$h(1)g(x) = \lim_{n\to\infty} \frac{f(m^n)}{g(m^n)} \cdot \frac{f(xm^n)}{h(m^n)} = g(1)h(x) = h(x) \tag{2.18}$$

for all $x \in S$. □

**Corollary 2.4.** *Let $(S, +)$ be a semigroup with identity 0 and $f, g, h : S \to C$ functions satisfying the inequality*

$$\left|f(x+y) - g(x)h(y)\right| \le \varphi(x, y) \tag{2.19}$$

*for all $x, y \in S$. If $g(0) = 1$, then $g$ is bounded or else $g$ is exponential and $h = h(0)g$.*

**Theorem 2.5.** *Let $(S,\cdot)$ be a semigroup with identity 1 and $f, g, h : S \to C$ functions satisfying condition* (1.4)*, that is,*

$$|f(xy) - g(x)h(y)| \leq \varphi(x,y). \tag{2.20}$$

*If $g$ satisfies that $g(s) \neq 0$ for some $s \in S$ and $|g(sm)| \geq \max\{2|g(s)|, 2\Phi_s(m)/|h(m)|\}$ for some $m \in S$, then*

$$g(sxy) = \frac{1}{g(s)} g(sx)g(sy) \tag{2.21}$$

*for all $x, y \in S$ and $h(x) = (h(1)/g(s))g(sx)$.*

*Proof.* Let $\overline{f}(x) = f(sx)$, $\overline{g}(x) = g(sx)/g(s)$ and $\overline{h}(x) = g(s)h(x)$ for every $x \in S$ and $\overline{\varphi}(x,y) = \varphi(sx,y)$. Then

$$\left|\overline{f}(xy) - \overline{g}(x)\overline{h}(y)\right| \leq \overline{\varphi}(x,y) \tag{2.22}$$

for all $x, y \in S$, $|\overline{g}(m)| \geq \max\{2, 2\overline{\Phi}_1(m)/|\overline{h}(m)|\}$ and $\overline{g}(1) = 1$ where $\overline{\Phi}_1(m) = \Phi_s(m)$. By Theorem 2.3, we complete the proof. □

**Corollary 2.6.** *Let $(S,\cdot)$ be a semigroup with identity 1. If $f, g, h : S \to C$ are nonzero functions satisfying condition* (1.4)*, that is,*

$$|f(xy) - g(x)h(y)| \leq \varphi(x,y), \tag{2.23}$$

*then either $g$ is bounded, or else*

$$g(sxy) = \frac{1}{g(s)} g(sx)g(sy) \tag{2.24}$$

*for all $x, y \in S$ and $h(x) = (h(1)/g(s))g(sx)$.*

*Proof.* Let $g(s) \neq 0$ for some $s \in S$. If $g$ is unbounded, then there exists $m$ such that $|g(sm)| \geq \max\{2|g(s)|, 2\Phi_s(m)/|h(m)|\}$. By Theorem 2.5, we complete the proof. □

## 3. Stability of the PMFE

In 1940, Ulam gave a wide-ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems [5]. One of those was the question concerning the stability of homomorphisms.

> *Let $G_1$ be a group and let $G_2$ be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

In the next year, Hyers [6] answered the Ulam's question for the case of the additive mapping on the Banach spaces $G_1, G_2$. Thereafter, the result of Hyers has been generalized by Rassias [7]. Since then, the stability problems of various functional equations have been investigated by many authors (see [6, 8–18]).

Ger [4] suggested another type of stability for the exponential equation in the following type:

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \le \delta. \tag{3.1}$$

In this section, the stability problem for the Pexiderized multiplicative functional equation in the following form:

$$\frac{1}{1+\varphi(x,y)} \le \frac{f(xy)}{g(x)h(y)} \le 1+\varphi(x,y) \tag{3.2}$$

will be investigated.

Throughout this section, we denote by $(S,\cdot)$ a commutative semigroup and by $\varphi : S \times S \to [0,\infty)$ a function such that

$$\Psi(x,y,z,w) = \sum_{n=0}^{\infty} \frac{1}{2^n} \ln\left(1+\varphi\left(xz^{2^n}, yw^{2^n}\right)\right) < \infty \tag{3.3}$$

for all $x, y, z, w \in S$. Also we let

$$u(x,y) := \ln(1+\varphi(x,y))(1+\varphi(y,x))(1+\varphi(x,x))(1+\varphi(y,y)) \tag{3.4}$$

for all $x, y \in S$. Inequality (3.3) implies that

(a) for all $x, z \in S$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} u\left(xz^{2^n}, xz^{2^n}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} \ln\left(1+\varphi\left(xz^{2^n}, xz^{2^n}\right)\right)^4 = 4\Psi(x,x,z,z) < \infty, \tag{3.5}$$

(b) for all $x, z \in S$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2^n} u\left(x^2, z^{2^n}\right) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \ln\left(1+\varphi\left(x^2, z^{2^n}\right)\right)\left(1+\varphi\left(z^{2^n}, x^2\right)\right) \\ &\qquad \cdot \left(1+\varphi\left(x^2, x^2\right)\right)\left(1+\varphi\left(z^{2^n}, z^{2^n}\right)\right) \\ &= \Psi\left(x^2,1,1,z\right) + \Psi\left(1,x^2,z,1\right) + \Psi\left(x^2,x^2,1,1\right) + \Psi(1,1,z,z) < \infty, \end{aligned} \tag{3.6}$$

(c) for all $x, z \in S$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} u\left(x^4, z^{2^n}\right) = \Psi\left(x^4, 1, 1, z\right) + \Psi\left(1, x^4, z, 1\right) + \Psi\left(x^4, x^4, 1, 1\right) + \Psi(1, 1, z, z) < \infty, \tag{3.7}$$

(d) for all $x, y \in S$ for

$$\begin{aligned} &\lim_{n \to \infty} \frac{1}{2^n} u\left(x^{2^n}, y^{2^n}\right) \\ &= \lim_{n \to \infty} \frac{1}{2^n} \ln\left(1 + \varphi\left(x^{2^n}, y^{2^n}\right)\right)\left(1 + \varphi\left(y^{2^n}, x^{2^n}\right)\right) \cdot \left(1 + \varphi\left(x^{2^n}, x^{2^n}\right)\right)\left(1 + \varphi\left(y^{2^n}, y^{2^n}\right)\right) = 0, \end{aligned} \tag{3.8}$$

because

$$\Psi(1, 1, x, y) + \Psi(1, 1, y, x) + \Psi(1, 1, x, x) + \Psi(1, 1, y, y) < \infty. \tag{3.9}$$

*Example 3.1.* The following functions satisfy condition (3.3) above.

(a) $\varphi(x, y) = \delta$, for every $x, y \in R$ and $\delta \geq 0$.

(b) $\varphi(x, y) = 1/(1 + |x| + |y|)$, for every $x, y \in R$.

*Example 3.2.* Let $(S, \cdot) = ([0, \infty), +)$ and also $g(x) = e^x, h(x) = e^{x+c}$,

$$f(x) = e^{x+c}\left(1 + \frac{1}{1 + x}\right) \tag{3.10}$$

for all $x \in S$ and for some $c \in S$. Let $\varphi(x, y) = 1/(1 + x + y)$. Then $f,\ g,\ h,\ \varphi$ satisfy condition (3.3) and

$$\frac{1}{1 + \varphi(x, y)} \leq \frac{f(x + y)}{g(x)h(y)} \leq 1 + \varphi(x, y). \tag{3.11}$$

In particular, we know that if we let $T(x) = e^x$ then

$$\frac{e^{-c}}{2} \leq \frac{T(x)}{f(x)} \leq e^{-c}, \qquad \frac{T(x)}{g(x)} = 1, \qquad \frac{T(x)}{h(x)} = e^{-c}. \tag{3.12}$$

**Theorem 3.3.** *If $f, g, h : S \to (0, \infty)$ are functions such that*

$$\frac{1}{1 + \varphi(x, y)} \leq \frac{f(xy)}{g(x)h(y)} \leq 1 + \varphi(x, y) \tag{3.13}$$

*for all $x, y \in S$, then there exists a function $T : S \to (0, \infty)$ and there exists a constant $M$ such that $T(xy) = T(x)T(y)$ for all $x, y \in S$ and*

$$e^{-M} \leq \frac{T(x)}{f(x)} \leq e^{M} \tag{3.14}$$

*for all $x \in S$. Moreover, if $\varphi$ is bounded, then*

$$\begin{aligned} e^{-M_1} &\leq \frac{T(x)}{g(x)} \leq e^{M_1}, \\ e^{-M_1} &\leq \frac{T(x)}{h(x)} \leq e^{M_1} \end{aligned} \tag{3.15}$$

*for all $x \in S$ and for some constant $M_1$.*

*Proof.* If we define functions $F, G, H : S \to R$ by

$$F(x) = \ln f(x), \qquad G(x) = \ln g(x), \qquad H(x) = \ln h(x) \tag{3.16}$$

for all $x \in S$, then equality (3.13) may be transformed into

$$\ln \frac{1}{1 + \varphi(x,y)} \leq F(xy) - G(x) - H(y) \leq \ln(1 + \varphi(x,y)), \tag{3.17}$$

and thus

$$|F(xy) - G(x) - H(y)| \leq \ln(1 + \varphi(x,y)), \tag{3.18}$$

for all $x, y \in S$. For the case of $x = y$, the above inequality implies

$$\left|F\left(x^2\right) - G(x) - H(x)\right| \leq \ln(1 + \varphi(x,x)) \tag{3.19}$$

and so

$$\begin{aligned} &\left|2F(xy) - F\left(x^2\right) - F\left(y^2\right)\right| \\ &\quad \leq |F(xy) - G(x) - H(y)| + |F(xy) - G(y) - H(x)| \\ &\qquad + \left|G(x) + H(x) - F\left(x^2\right)\right| + \left|G(y) + H(y) - F\left(y^2\right)\right| \\ &\quad \leq \ln(1 + \varphi(x,y))(1 + \varphi(y,x)) \cdot (1 + \varphi(x,x))(1 + \varphi(y,y)) := u(x,y) \end{aligned} \tag{3.20}$$

for all $x, y \in S$. Putting $xz$ instead of $x$ and $yz$ instead of $y$ in (3.20), respectively, we get

$$\left|2F\left(xz^2y\right) - F\left(x^2z^2\right) - F\left(y^2z^2\right)\right| \leq u(xz, yz). \tag{3.21}$$

Letting $x$ by $x^2$ and $y$ by $z^2$ in (3.20), we have

$$\left|F\left(x^2z^2\right)-\frac{1}{2}F\left(x^4\right)-\frac{1}{2}F\left(z^4\right)\right|\leq\frac{1}{2}u\left(x^2,z^2\right), \tag{3.22}$$

and also

$$\left|F\left(y^2z^2\right)-\frac{1}{2}F\left(y^4\right)-\frac{1}{2}F\left(z^4\right)\right|\leq\frac{1}{2}u\left(y^2,z^2\right). \tag{3.23}$$

From (3.21), (3.22) and (3.23),

$$\left|2F\left(xz^2y\right)-\frac{1}{2}F\left(x^4\right)-\frac{1}{2}F\left(y^4\right)-F\left(z^4\right)\right|\leq u(xz,yz)+\frac{1}{2}u\left(x^2,z^2\right)+\frac{1}{2}u\left(y^2,z^2\right) \tag{3.24}$$

for all $x,y,z\in S$. Now replacing $x$ by $xz$ and $y$ by $yz$, respectively, we have

$$\begin{aligned}&\left|2F\left(xz^4y\right)-\frac{1}{2}F\left(x^4y^4\right)-\frac{1}{2}F\left(y^4z^4\right)-F\left(z^4\right)\right|\\&\quad\leq u\left(xz^2,yz^4\right)+\frac{1}{2}u\left(x^2z^2,z^2\right)+\frac{1}{2}u\left(y^2z^2,z^2\right)\end{aligned} \tag{3.25}$$

for all $x,y,z\in S$. Replacing $x$ by $xz$ and $y$ by $yz$ in (3.21), (3.22), and (3.23), respectively, one obtains

$$\begin{aligned}&\left|2F\left(xz^4y\right)-F\left(x^2z^4\right)-F\left(y^2z^4\right)\right|\leq u\left(xz^2,yz^2\right),\\&\left|F\left(x^2z^4\right)-\frac{1}{2}F\left(x^4z^4\right)-\frac{1}{2}F\left(z^4\right)\right|\leq\frac{1}{2}u\left(x^2z^2,z^2\right),\\&\left|F\left(y^2z^4\right)-\frac{1}{2}F\left(y^4z^4\right)-\frac{1}{2}F\left(z^4\right)\right|\leq\frac{1}{2}u\left(y^2z^2,z^2\right)\end{aligned} \tag{3.26}$$

for all $x,y,z\in S$. Also from (3.22) and (3.23), we have

$$\begin{aligned}&\left|\frac{1}{2}F\left(x^4z^4\right)-\frac{1}{4}F\left(x^8\right)-\frac{1}{4}F\left(z^8\right)\right|\leq\frac{1}{4}u\left(x^4,z^4\right),\\&\left|\frac{1}{2}F\left(y^4z^4\right)-\frac{1}{4}F\left(y^8\right)-\frac{1}{4}F\left(z^8\right)\right|\leq\frac{1}{4}u\left(y^4,z^4\right)\end{aligned} \tag{3.27}$$

for all $x, y, z \in S$. Thus we have

$$\begin{aligned}
&\left|2F\left(xz^4y\right) - \frac{1}{4}F\left(x^8\right) - \frac{1}{4}F\left(y^8\right) - \frac{1}{2}F\left(z^8\right) - F\left(z^4\right)\right| \\
&\quad\le \left|2F\left(xz^4y\right) - F\left(x^2z^4\right) - F\left(y^2z^4\right)\right| + \left|F\left(x^2z^4\right) - \frac{1}{2}F\left(x^4z^4\right) - \frac{1}{2}F\left(z^4\right)\right| \\
&\qquad + \left|F\left(y^2z^4\right) - \frac{1}{2}F\left(y^4z^4\right) - \frac{1}{2}F\left(z^4\right)\right| + \left|\frac{1}{2}F\left(x^4z^4\right) - \frac{1}{4}F\left(x^8\right) - \frac{1}{4}F\left(z^8\right)\right| \\
&\qquad + \left|\frac{1}{2}F\left(y^4z^4\right) - \frac{1}{4}F\left(y^8\right) - \frac{1}{4}F\left(z^8\right)\right| \\
&\quad\le u\left(xz^2, yz^2\right) + \frac{1}{2}u\left(x^2z^2, z^2\right) + \frac{1}{2}u\left(y^2z^2, z^2\right) + \frac{1}{4}u\left(x^4, z^4\right) + \frac{1}{4}u\left(y^4, z^4\right),
\end{aligned} \tag{3.28}$$

for all $x, y, z \in S$. For arbitrary positive integer $n$, putting $z^{2^n}$ instead of $z$ in (3.24) and $z^{2^{n-1}}$ instead of $z$ in (3.28), respectively, we see that

$$\begin{aligned}
&\left|2F\left(xz^{2^{n+1}}y\right) - F\left(z^{2^{n+2}}\right)\right| \\
&\quad\le u\left(xz^{2^n}, yz^{2^n}\right) + \frac{1}{2}u\left(x^2, z^{2^{n+1}}\right) + \frac{1}{2}u\left(y^2, z^{2^{n+1}}\right) + \frac{1}{2}\left|F\left(x^4\right)\right| + \frac{1}{2}\left|F\left(y^4\right)\right|, \\
&\left|2F\left(xz^{2^{n+1}}y\right) - \frac{1}{2}F\left(z^{2^{n+2}}\right) - F\left(z^{2^{n+1}}\right)\right| \\
&\quad\le u\left(xz^{2^n}, yz^{2^n}\right) + \frac{1}{2}u\left(x^2z^{2^n}, z^{2^n}\right) + \frac{1}{2}u\left(y^2z^{2^n}, z^{2^n}\right) \\
&\qquad + \frac{1}{4}u\left(x^4, z^{2^{n+1}}\right) + \frac{1}{4}u\left(y^4, z^{2^{n+1}}\right) + \frac{1}{4}\left|F\left(x^8\right)\right| + \frac{1}{4}\left|F\left(y^8\right)\right|
\end{aligned} \tag{3.29}$$

for all $x, y, z \in S$. By (3.29) with $x = y$,

$$\begin{aligned}
&\left|\frac{F\left(z^{2^{n+2}}\right)}{2^{n+2}} - \frac{F\left(z^{2^{n+1}}\right)}{2^{n+1}}\right| \\
&\quad\le \frac{1}{2^{n+1}}\left|\frac{1}{2}F\left(z^{2^{n+2}}\right) - F\left(z^{2^{n+1}}\right)\right| \\
&\quad\le \frac{1}{2^{n+1}}\left(\left|F\left(z^{2^{n+2}}\right) - 2F\left(xz^{2^{n+1}}x\right)\right| + \left|2F\left(xz^{2^{n+1}}x\right) - \frac{1}{2}F\left(z^{2^{n+2}}\right) - F\left(z^{2^{n+1}}\right)\right|\right) \\
&\quad\le \frac{1}{2^{n+1}}\left(2u\left(xz^{2^n}, xz^{2^n}\right) + u\left(x^2, z^{2^{n+1}}\right) + u\left(x^2z^{2^n}, z^{2^n}\right)\right. \\
&\qquad\qquad \left. + \frac{1}{2}u\left(x^4, z^{2^{n+1}}\right) + \left|F\left(x^4\right)\right| + \frac{1}{2}\left|F\left(x^8\right)\right|\right)
\end{aligned} \tag{3.30}$$

for all $x, z \in S$. By (3.30), for every positive integer $k, m$ with $k \geq m \geq 2$, we have

$$\begin{aligned}
&\left|\frac{F\left(z^{2^{m+k}}\right)}{2^{m+k}} - \frac{F\left(z^{2^m}\right)}{2^m}\right| \\
&\quad \leq \sum_{i=1}^{k}\left|\frac{F\left(z^{2^{m+i}}\right)}{2^{m+i}} - \frac{F\left(z^{2^{m+i-1}}\right)}{2^{m+i-1}}\right| \\
&\quad \leq \sum_{i=1}^{\infty}\frac{1}{2^{m+i-1}}\left(2u\left(xz^{2^{m+i-2}}, xz^{2^{m+i-2}}\right) + u\left(x^2, z^{2^{m+i-1}}\right) + u\left(x^2 z^{2^{m+i-2}}, z^{2^{m+i-2}}\right)\right. \\
&\qquad\qquad \left. + \frac{1}{2}u\left(x^4, z^{2^{m+i-1}}\right) + \left|F\left(x^4\right)\right| + \frac{1}{2}\left|F\left(x^8\right)\right|\right) \\
&\quad \leq \sum_{i=m-1}^{\infty}\frac{1}{2^i}u\left(xz^{2^i}, xz^{2^i}\right) + \sum_{i=m}^{\infty}\frac{1}{2^i}u\left(x^2, xz^{2^i}\right) + \sum_{i=m}^{\infty}\frac{1}{2^i}u\left(x^2 z^{2^{i-1}}, z^{2^i}\right) \\
&\qquad + \frac{1}{2}\sum_{i=m}^{\infty}\frac{1}{2^i}u\left(x^4, z^{2^{i-1}}\right) + \left(\left|F\left(x^4\right)\right| + \frac{1}{2}\left|F\left(x^8\right)\right|\right)\sum_{i=m}^{\infty}\frac{1}{2^i} \longrightarrow 0,
\end{aligned} \tag{3.31}$$

as $m \to \infty$. This proves that $\{F(z^{2^n})/2^n\}$ is a Cauchy sequence in $R$. Thus we can define a function $L : S \to R$ by

$$L(z) = \lim_{n\to\infty}\frac{F(z^{2^n})}{2^n} \tag{3.32}$$

for all $z \in S$. Then, by (3.20) and (3.31), we have

$$\begin{aligned}
\left|L(xy) - L(x) - L(y)\right| &\leq \lim_{n\to\infty}\left|\frac{F(x^{2^n}y^{2^n})}{2^n} - \frac{F(x^{2^n})}{2^n} - \frac{F(y^{2^n})}{2^n}\right| \\
&\leq \lim_{n\to\infty}\frac{1}{2^{n+1}}\left|2F\left(x^{2^n}y^{2^n}\right) - F\left(x^{2^{n+1}}\right) - F\left(y^{2^{n+1}}\right)\right| \\
&\quad + \lim_{n\to\infty}\left|\frac{F\left(x^{2^{n+1}}\right)}{2^{n+1}} - \frac{F(x^{2^n})}{2^n}\right| + \lim_{n\to\infty}\left|\frac{F\left(y^{2^{n+1}}\right)}{2^{n+1}} - \frac{F(y^{2^n})}{2^n}\right| \\
&\leq \lim_{n\to\infty}\frac{1}{2^{n+1}}u\left(x^{2^n}, y^{2^n}\right) + 0 + 0 = 0
\end{aligned} \tag{3.33}$$

for all $x, y \in S$. Thus

$$L(xy) = L(x) + L(y) \tag{3.34}$$

for all $x, y \in S$. Now replacing $x$ by $xz$ and then $y$ by $yz$ in (3.20), respectively, we obtain

$$\begin{aligned} &\left|2F(xyz) - F\left(x^2z^2\right) - F\left(y^2\right)\right| \le u(xz, y),\\ &\left|2F(xyz) - F\left(x^2\right) - F\left(y^2z^2\right)\right| \le u(x, yz) \end{aligned} \tag{3.35}$$

and so

$$\left|F\left(x^2\right) - F\left(y^2\right) + F\left(y^2z^2\right) - F\left(x^2z^2\right)\right| \le u(xz, y) + u(x, yz) \tag{3.36}$$

for all $x, y, z \in S$. By (3.22), (3.23), and (3.36), we have

$$\left|F\left(x^2\right) - F\left(y^2\right) - \frac{1}{2}F\left(x^4\right) + \frac{1}{2}F\left(y^4\right)\right| \le u(xz, y) + u(x, yz) + \frac{1}{2}u\left(x^2, z^2\right) + \frac{1}{2}u\left(y^2, z^2\right), \tag{3.37}$$

and thus

$$\left|F\left(x^4\right) - 2F\left(x^2\right)\right| \le 2u(xz, y) + 2u(x, yz) + u\left(x^2, z^2\right) + u\left(y^2, z^2\right) + \left|F\left(y^4\right) - 2F\left(y^2\right)\right| < \infty \tag{3.38}$$

for all $x \in S$ and for fixed $y, z \in S$. By (3.3) and (3.30),

$$\begin{aligned} \left|L\left(z^4\right) - F\left(z^4\right)\right| &= 4\left|L(z) - \frac{F(z^4)}{4}\right| = 4\lim_{n\to\infty}\left|\frac{F(z^{2^n})}{2^n} - \frac{F(z^4)}{4}\right| \\ &= 4\lim_{n\to\infty}\sum_{i=1}^{n-2}\left|\frac{F\left(z^{2^{i+2}}\right)}{2^{i+2}} - \frac{F\left(z^{2^{i+1}}\right)}{2^{i+1}}\right| < \infty \end{aligned} \tag{3.39}$$

for all $s \in S$. By (3.20), (3.38) and (3.39), for all $z \in S$ with $x = zw$, there exits a constant $M$ such that

$$\begin{aligned} |L(x) - F(x)| &= |L(zw) - F(zw)| \\ &= \left|F(zw) - \frac{1}{2}F\left(z^2\right) - \frac{1}{2}F\left(w^2\right)\right| + \frac{1}{4}\left|L\left(z^4\right) - F\left(z^4\right)\right| \\ &\quad + \frac{1}{4}\left|L\left(w^4\right) - F\left(w^4\right)\right| + \frac{1}{4}\left|F\left(w^4\right) - 2F\left(w^2\right)\right| \\ &\quad + \frac{1}{4}\left|F\left(z^4\right) - 2F\left(z^2\right)\right| \\ &\le M. \end{aligned} \tag{3.40}$$

Now we define a function $T : S \to (0,\infty)$ by

$$T(x) := e^{L(x)} \tag{3.41}$$

for all $x \in S$. Then

$$T(xy) = e^{L(xy)} = e^{L(x)+L(y)} = T(x)T(y) \tag{3.42}$$

for all $x, y \in S$. By (3.40), we have

$$-M \le L(x) - \ln f(x) \le M, \tag{3.43}$$

and thus for all $x \in S$

$$e^{-M} \le \frac{T(x)}{f(x)} \le e^{M}. \tag{3.44}$$

If $\psi$ is bounded, there exist constants $M_0, M_1$ such that

$$\begin{aligned} |G(x) - H(x)| &\le \left|G(x) + H(y_0) - F(xy_0)\right| + \left|F(xy_0) - G(y_0) - H(x)\right| + \left|G(y_0) - H(y_0)\right| \\ &\le \ln(1 + \psi(x, y_0)) + \ln(1 + \psi(y_0, x)) + \left|G(y_0) - H(y_0)\right| \le M_0, \end{aligned} \tag{3.45}$$

and so

$$\begin{aligned} |L(x) - G(x)| &\le \frac{1}{2}\left|L\left(x^2\right) - F\left(x^2\right)\right| + \frac{1}{2}\left|F\left(x^2\right) - H(x) - G(x)\right| + \frac{1}{2}|H(x) - G(x)| \\ &\le \frac{1}{2}(M + \ln(1 + \psi(x, x)) + M_0) \le M_1, \end{aligned} \tag{3.46}$$

and by the same method above, we have

$$|L(x) - H(x)| \le M_1 \tag{3.47}$$

for all $x \in S$. Therefore, we have

$$e^{-M_1} \le \frac{T(x)}{g(x)} \le e^{M_1}, \qquad e^{-M_1} \le \frac{T(x)}{h(x)} \le e^{M_1} \tag{3.48}$$

for all $x \in S$. □

## References


[1] J. A. Baker, J. Lawrence, and F. Zorzitto, "The stability of the equation $f(x+y) = f(x) + f(y)$," *Proceedings of the American Mathematical Society*, vol. 74, no. 2, pp. 242–246, 1979.

[2] J. A. Baker, "The stability of the cosine equation," *Proceedings of the American Mathematical Society*, vol. 80, no. 3, pp. 411–416, 1980.

[3] L. Székelyhidi, "On a theorem of Baker, Lawrence and Zorzitto," *Proceedings of the American Mathematical Society*, vol. 84, no. 1, pp. 95–96, 1982.

[4] R. Ger, "Superstability is not natural," *Rocznik Naukowo-Dydaktyczny WSP Krakkowie*, vol. 159, no. 13, pp. 109–123, 1993.

[5] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1964.

[6] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[7] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[8] G. L. Forti, "Hyers-Ulam stability of functional equations in several variables," *Aequationes Mathematicae*, vol. 50, no. 1-2, pp. 143–190, 1995.

[9] D. H. Hyers and Th. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[10] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, 34, Birkhäuser, Boston, Mass, USA, 1998.

[11] K. W. Jun, G. H. Kim, and Y. W. Lee, "Stability of generalized gamma and beta functional equations," *Aequationes Mathematicae*, vol. 60, no. 1-2, pp. 15–24, 2000.

[12] S.-M. Jung, "On a general Hyers-Ulam stability of gamma functional equation," *Bulletin of the Korean Mathematical Society*, vol. 34, no. 3, pp. 437–446, 1997.

[13] S.-M. Jung, "On the stability of gamma functional equation," *Results in Mathematics*, vol. 33, no. 3-4, pp. 306–309, 1998.

[14] G. H. Kim and Y. W. Lee, "The stability of the beta functional equation," *Babeş-Bolyai. Mathematica*, vol. 45, no. 1, pp. 89–96, 2000.

[15] Y. W. Lee, "On the stability of a quadratic Jensen type functional equation," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 2, pp. 590–601, 2002.

[16] Y. W. Lee, "The stability of derivations on Banach algebras," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 28, no. 2, pp. 113–116, 2000.

[17] Y. W. Lee and B. M. Choi, "The stability of Cauchy's gamma-beta functional equation," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 305–313, 2004.

[18] Th. M. Rassias, "The problem of S. M. Ulam for approximately multiplicative mappings," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 352–378, 2000.