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Research Article

Weighted Decomposition Estimates for Differential Forms

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After introducing the definition of $A^{\beta}_{r,\lambda}$-weights, we establish the $A_r(\Omega)$-weighted decomposition estimates and $A^{\beta}_{r,\lambda}(\Omega)$-weighted Caccioppoli-type estimates for A-harmonic tensors. Furthermore, by Whitney covering lemma, we obtain the global results in domain $\Omega \subset \mathbb{R}^n$. These results can be used to study the integrability of differential forms and to estimate the integrals for differential forms.

1. Introduction

Let $e_1, e_2, \dots, e_n$ denote the standard orthogonal basis of $\mathbb{R}^n$. Suppose that $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ is the linear space of all l-vectors, spanned by the exterior product $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$ corresponding to all ordered l-tuples $I = (i_1, i_2, \dots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$. Throughout this paper, we always assume that Ω is an open subset of $\mathbb{R}^n$. We use $D'(M, \Lambda^l)$ to denote the space of all differential l-forms and $L^p(M, \Lambda^l)$ to denote the l-forms $\omega(x) = \sum_I \omega_I dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_l$ on M with the coefficient $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered l-tuples I, where M is a manifold. Thus $L^p(M, \Lambda^l)$ is a Banach space with the norm

$$\|\omega\|_{p,M} = \left(\int_M |\omega(x)|^p dx \right)^{1/p} = \left(\int_M \left(\sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}. \tag{1.1}$$

For a differential l-form $\omega \in D'(\Omega, \wedge^l)$, its vector-valued differential form $\nabla\omega = (\partial\omega/\partial x_1, \partial\omega/\partial x_2, \dots, \partial\omega/\partial x_n)$ is composed of the differential l-form $\partial\omega/\partial x_i \in D'(\Omega, \wedge^l)$, $i = 1, 2, \dots, n$, here the partial derivatives are with respect to the coefficients of ω. Usually, suppose that $L_1^p(\Omega, \wedge^l)$ is the space consisting of all $\partial\omega/\partial x_i \in L^p(\Omega, \wedge^l)$, the ith one of $\nabla\omega$. We denote the exterior differential operator of l-forms by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ and define the Hodge differential operator $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ with $d^\star = (-1)^{nl+1} \star d\star$, where $l = 0, 1, \ldots, n-1$, and $\star$ is the Hodge star operator. We denote a ball or cube by B and the ball or cube with the same center as B and $\operatorname{diam}(\rho B) = \rho \operatorname{diam}(B)$ by ρB. The following nonlinear elliptic equation:

$$d^\star A(x, d\omega) = B(x, d\omega) \tag{1.2}$$

is called the nonhomogeneous A-harmonic equation of differential forms, where

$$A : \Omega \times \wedge^l(\mathbb{R}^n) \longrightarrow \wedge^l(\mathbb{R}^n), \qquad B : \Omega \times \wedge^l(\mathbb{R}^n) \longrightarrow \wedge^{l-1}(\mathbb{R}^n) \tag{1.3}$$

are operators satisfying the following conditions for almost all $x \in \Omega$ and $\xi \in \wedge^l(\mathbb{R}^n)$:

$$\begin{aligned} |A(x,\xi)| &\leqslant a|\xi|^{p-1}, \\ |B(x,\xi)| &\leqslant b|\xi|^{p-1}, \\ \langle A(x,\xi), \xi\rangle &\geq |\xi|^p, \end{aligned} \tag{1.4}$$

where $a > 0$ and $b > 0$ are two constants, and $1 < p < \infty$ is a fixed exponent dependent on (1.2).

If the operator $B = 0$ in (1.2), it degenerates into the homogeneous A-harmonic equation

$$d^\star A(x, d\omega) = 0. \tag{1.5}$$

The solutions to (1.5) are called A-harmonic tensors. See [1] for the recent research on A-harmonic equation.

2. $A_{r,\lambda}^{\beta}$-Weighted Caccioppoli-Type Inequalities

Caccioppoli-type estimates have been widely studied and frequently used in analysis and related fields, including partial differential equations and the theory of elasticity. These inequalities provide upper bounds for the L^p-norm of ∇u if u is a function or du if u is a form with the L^p-norm of the differential form u. Different versions of the Caccioppoli-type inequality have been established in [2, 3].

We first introduce the following definition of $A_{r,\lambda}^{\beta}$-weights (or the two-weight), and then establish the local $A_{r,\lambda}^{\beta}$-weighted Caccioppoli-type inequality for solutions to the homogeneous A-harmonic equation.

Definition 2.1. Assume that $1 \leq r \leq \lambda < \infty$ and $0 \leq \beta < n$. One says a pair of weights $(\omega_1(x), \omega_2(x))$ satisfies the $A^{\beta}_{r,\lambda}$-condition, writes $(\omega_1(x), \omega_2(x)) \in A^{\beta}_{r,\lambda}(\Omega)$, if for all balls $B \subset \Omega$, one has that

$$\begin{gathered}\frac{1}{|B|^{1-\beta/n}}\left(\int_B \omega_1(x)dx\right)^{1/\lambda}\left(\int_B \omega_2(x)^{1-r'}dx\right)^{1/r'} \leqslant C, \quad \text{for } 1 < r < \infty,\ \frac{1}{r}+\frac{1}{r'} = 1, \\ \frac{1}{|B|^{1-\beta/n}}\left(\int_B \omega_1(x)dx\right)^{1/\lambda} \leqslant \omega_2(x), \quad \text{for } r = 1 \text{ almost all } x \in B,\end{gathered} \tag{2.1}$$

where $C > 0$ is a constant.

In [4], Nolder obtains the following local Caccioppoli-type estimate.

Lemma 2.2. *Let u be an A-harmonic tensor in Ω and let $\sigma > 1$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B} \leq C \operatorname{diam}(B)^{-1}\|u - c\|_{s,\sigma B} \tag{2.2}$$

for all balls or cubes B with $\sigma B \subset \Omega$ and all closed forms c. Here $1 < s < \infty$.

The next lemma is the generalized Hölder inequality which will be widely used in this paper.

Lemma 2.3. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If u and v are measurable functions on $\mathbb{R}^n$, then*

$$\|uv\|_{s,E} \leq \|u\|_{\alpha,E}\|v\|_{\beta,E} \tag{2.3}$$

for any measurable set $E \subset \mathbb{R}^n$.

The following weak reverse Hölder inequality plays an important role in founding the integral estimate of the nonhomogeneous and homogeneous A-harmonic tensor; see [4].

Lemma 2.4. *Let u be a solution to (1.2) in Ω, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, depending only on s, t, a, p, σ, and n, such that*

$$\|u\|_{s,B} \leq C|B|^{(s-t)/st}\|u\|_{t,\sigma B} \tag{2.4}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

The following Whitney covering lemma appears in [4].

Lemma 2.5. *Each Ω has a modified Whitney cover of cubes $\nu = \{Q_i\}$ such that*

$$\bigcup_i Q_i = \Omega,$$
$$\sum_{Q\in\nu} \chi_{\sqrt{(5/4)}Q} \le N\chi_\Omega \tag{2.5}$$

for all $x \in \mathbb{R}^n$ and some $N > 1$, where χ_E is the characteristic function for a set E. Moreover, if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube does not need to be a member of ν) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$.

Now we are ready to prove the local weighted Caccioppoli-type inequality for homogeneous A-harmonic tensors.

Theorem 2.6. *Let $u \in L^s(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n-1$, be a solution to the homogeneous A-harmonic equation (1.5) and $(\omega_1(x), \omega_2(x)) \in A^{\beta}_{r,\lambda}(\Omega)$ for some $0 \le \beta < n$ and $1 \le r \le \lambda < \infty$. Then there exists a constant C, independent of u and du, such that*

$$\left(\int_B |du|^s \omega_1^{\alpha} dx\right)^{1/s} \le C|B|^{(\lambda\alpha/s)(1/r-\beta/n)-1/n-\alpha/s}\left(\int_{\sigma B} |u-c|^s \omega_2^{\lambda\alpha/r} dx\right)^{1/s} \tag{2.6}$$

for all balls or cubes B with $\sigma B \subset \Omega$ and all closed forms c.

Specially, if $\lambda = r$, one has

$$\left(\int_B |du|^s \omega_1^{\alpha} dx\right)^{1/s} \le C|B|^{-\alpha\beta r/ns-1/n}\left(\int_{\sigma B} |u-c|^s \omega_2^{\alpha} dx\right)^{1/s}. \tag{2.7}$$

Here α is any constant with $0 < \alpha < 1$.

Proof. Choosing $t = s/(1-\alpha)$ and by Lemma 2.3, we have

$$\begin{aligned}\left(\int_B |du|^s \omega_1^{\alpha} dx\right)^{1/s} &= \left(\int_B \left(|du|\omega_1^{\alpha/s}\right)^s dx\right)^{1/s}\\ &\le \left(\int_B |du|^t dx\right)^{1/t}\left(\int_B \left(\omega_1^{\alpha/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st}\\ &= \|du\|_{t,B}\cdot\left(\int_B \omega_1 dx\right)^{(t-s)/st}.\end{aligned} \tag{2.8}$$

Choosing $1 < \sigma_1 < \sigma$ and by Lemma 2.2, we know that

$$\|du\|_{t,B} \le C_1 \operatorname{diam}(B)^{-1}\|u-c\|_{t,\sigma_1 B}, \tag{2.9}$$

where c is a closed form. Putting (2.9) into (2.8), we have

$$\begin{aligned}\left(\int_B |du|^s \omega_1^{\alpha} dx\right)^{1/s} &\le \|du\|_{t,B} \cdot \left(\int_B \omega_1 dx\right)^{(t-s)/st} \\ &\le C_1 \operatorname{diam}(B)^{-1} \|u-c\|_{t,\sigma_1 B} \left(\int_B \omega_1 dx\right)^{\alpha/s}.\end{aligned} \tag{2.10}$$

Since c is a closed form and u is a solution to (1.5), $u - c$ is still a solution to (1.5).

When $r > 1$, taking $m = s/(1 + (\lambda/r)\alpha(r-1)) = rs/(r + \lambda\alpha(r-1))$ and by Lemma 2.4, we obtain

$$\|u-c\|_{t,\sigma_1 B} \le C_2 |B|^{(m-t)/mt} \|u-c\|_{m,\sigma B}. \tag{2.11}$$

By Lemma 2.3, we have

$$\begin{aligned}\|u-c\|_{m,\sigma B} &= \left(\int_{\sigma B} \left(|u-c| \omega_2^{\lambda\alpha/rs} \omega_2^{-\lambda\alpha/rs}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\sigma B} |u-c|^s \omega_2^{(\lambda/r)\alpha} dx\right)^{1/s} \left(\int_{\sigma B} \left(\frac{1}{\omega_2}\right)^{\lambda\alpha m/r(s-m)} dx\right)^{(s-m)/sm} \\ &= \|u-c\|_{s,\sigma B,\omega_2^{(\lambda/r\alpha)}} \left(\int_{\sigma B} \left(\frac{1}{\omega_2}\right)^{1/(r-1)} dx\right)^{(\lambda\alpha/s)(1-1/r)}.\end{aligned} \tag{2.12}$$

By the condition $(\omega_1(x), \omega_2(x)) \in A_{r,\lambda}^{\beta}(\Omega)$, we obtain

$$\begin{aligned}&\left(\int_B \omega_1 dx\right)^{\alpha/s} \left(\int_{\sigma B} \left(\frac{1}{\omega_2}\right)^{1/(r-1)} dx\right)^{(\lambda\alpha/s)(1-1/r)} \\ &\quad = \left\{ |B|^{1-\beta/n} |B|^{\beta/n-1} \left(\int_B \omega_1 dx\right)^{1/\lambda} \left(\int_{\sigma B} \left(\frac{1}{\omega_2}\right)^{1/(r-1)} dx\right)^{1-1/r} \right\}^{\lambda\alpha/s} \\ &\quad \le \left\{ |B|^{1-\beta/n} |B|^{\beta/n-1} \left(\int_{\sigma B} \omega_1 dx\right)^{1/\lambda} \left(\int_{\sigma B} \left(\frac{1}{\omega_2}\right)^{1/(r-1)} dx\right)^{1-1/r} \right\}^{\lambda\alpha/s} \\ &\quad = \left\{ |B|^{1-\beta/n} |B|^{\beta/n-1} \left(\int_{\sigma B} \omega_1 dx\right)^{1/\lambda} \left(\int_{\sigma B} \left(\frac{1}{\omega_2}\right)^{r'-1} dx\right)^{1/r'} \right\}^{\lambda\alpha/s} \\ &\quad \le C_3 |B|^{(1-\beta/n)(\lambda/s)\alpha},\end{aligned} \tag{2.13}$$

where $1/r + 1/r' = 1$. By (2.10), (2.11), (2.12), and (2.13), we conclude that

$$\begin{aligned}\left(\int_B |du|^s \omega_1^\alpha dx\right)^{1/s} &\le C_1 \operatorname{diam}(B)^{-1} C_2 |B|^{(m-t)/mt} \|u-c\|_{m,\sigma B}\left(\int_B \omega_1 dx\right)^{\alpha/s}\\ &\le C_4 \operatorname{diam}(B)^{-1}|B|^{(m-t)/mt}\|u-c\|_{s,\sigma B,\omega_2^{(\lambda/r)\alpha}}\left(\int_B \omega_1 dx\right)^{\alpha/s}\\ &\quad\times\left(\int_{\sigma B}\left(\frac{1}{\omega_2}\right)^{1/(r-1)} dx\right)^{(\lambda\alpha/s)(1-1/r)}\\ &\le C_5 \operatorname{diam}(B)^{-1}|B|^{1/t-1/m}|B|^{(1-\beta/n)(\lambda/s)\alpha}\|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha/r}}.\end{aligned} \tag{2.14}$$

Since $\operatorname{diam}(B) = C_6|B|^{1/n}$, inequality (2.14) can be rewritten as

$$\left(\int_B |du|^s \omega_1^\alpha dx\right)^{1/s} \leqslant C_7 |B|^{-1/n}|B|^{1/t-1/m}|B|^{(1-\beta/n)(\lambda/s)\alpha}\|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha/r}}. \tag{2.15}$$

By simple computation, we know that

$$\frac{1}{t}-\frac{1}{m} = \frac{1-\alpha}{s} - \frac{1+(\lambda\alpha/r)(r-1)}{s} = -\frac{\alpha}{s} - \frac{\lambda\alpha}{s}\left(1-\frac{1}{r}\right). \tag{2.16}$$

Then we can change inequality (2.15) into

$$\left(\int_B |du|^s \omega_1^\alpha dx\right)^{1/s} \le C|B|^{(\lambda\alpha/s)(1/r-\beta/n)-1/n-\alpha/s}\|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha/r}}. \tag{2.17}$$

When $r = 1$, taking $m < s$ and by Lemma 2.4, we obtain

$$\|u-c\|_{t,\sigma_1 B} \le C_8 |B|^{(m-t)/mt}\|u-c\|_{m,\sigma B}. \tag{2.18}$$

By Lemma 2.3, it follows that

$$\begin{aligned}\|u-c\|_{m,\sigma B} &= \left(\int_{\sigma B}\left(|u-c|\omega_2^{\lambda\alpha/s}\omega_2^{-\lambda\alpha/s}\right)^m dx\right)^{1/m}\\ &\le \left(\int_{\sigma B}|u-c|^s\omega_2^{\lambda\alpha}dx\right)^{1/s}\left(\int_{\sigma B}\left(\frac{1}{\omega_2}\right)^{\lambda\alpha m/(s-m)}dx\right)^{(s-m)/sm}\\ &= \|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha}}\left\{|\sigma B|^{1-\beta/n}\left(\int_{\sigma B}\omega_1(x)dx\right)^{-1/\lambda}\right\}^{\lambda\alpha/s}|\sigma B|^{(s-m)/sm}\\ &= \|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha}}|\sigma B|^{(\lambda\alpha/s)(1-\beta/n)}\left(\int_{\sigma B}\omega_1(x)dx\right)^{-\alpha/s}|\sigma B|^{(s-m)/sm}.\end{aligned} \tag{2.19}$$

By (2.10), (2.18), and (2.19), we conclude that

$$\begin{aligned}\left(\int_B |du|^s \omega_1^\alpha dx\right)^{1/s} &\le C_1 \operatorname{diam}(B)^{-1} C_8 |B|^{(m-t)/mt} \|u-c\|_{m,\sigma B} \left(\int_B \omega_1 dx\right)^{\alpha/s} \\ &\le C_9 \operatorname{diam}(B)^{-1} |B|^{(m-t)/mt} \|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha}} \left(\int_B \omega_1 dx\right)^{\alpha/s} \\ &\quad \times |\sigma B|^{(\lambda\alpha/s)(1-\beta/n)} \left(\int_{\sigma B} \omega_1(x) dx\right)^{-\alpha/s} |\sigma B|^{(s-m)/sm} \\ &\le C_{10} \operatorname{diam}(B)^{-1} |B|^{1/t-1/m} |\sigma B|^{(\lambda\alpha/s)(1-\beta/n)} |\sigma B|^{1/m-1/s} \|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha}}.\end{aligned} \tag{2.20}$$

Since $\operatorname{diam}(B) = C_{11}|B|^{1/n}$ and $|\sigma B| = \sigma^n |B|$, inequality (2.20) can be rewritten as

$$\begin{aligned}\left(\int_B |du|^s \omega_1^\alpha dx\right)^{1/s} &\le C_{12} |B|^{-1/n} |B|^{1/t-1/m} |B|^{(1-\beta/n)(\lambda\alpha/s)} |B|^{1/m-1/s} \|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha}} \\ &= C_{12} |B|^{-1/n+1/t-1/s+(1-\beta/n)(\lambda\alpha/s)} \|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha}}.\end{aligned} \tag{2.21}$$

Because of $t = s/(1-\alpha)$, we have that

$$\left(\int_B |du|^s \omega_1^\alpha dx\right)^{1/s} \le C |B|^{(\lambda\alpha/s)(1-\beta/n)-1/n-\alpha/s} \|u-c\|_{s,\sigma B,\omega_2^{\lambda\alpha}}. \tag{2.22}$$

The proof of Theorem 2.6 is completed. □

Since there are fore real parameters, $\alpha, \lambda, \beta,$ and r, in Theorem 2.6, we can obtain some desired versions of weighted Caccioppoli-type estimates by different choices of them. Let $\beta = n-1$ in Theorem 2.6, we obtain the following corollaries.

Corollary 2.7. *Let $u \in L^s(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n-1$, be a solution to the homogeneous A-harmonic equation* (1.5) *and $(\omega_1(x), \omega_2(x)) \in A_{r,r}^{n-1}(\Omega)$ for some $0 < \alpha < 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u and du, such that*

$$\left(\int_B |du|^s \omega_1^\alpha dx\right)^{1/s} \le C |B|^{\alpha r/ns-1/n-\alpha r/s} \left(\int_{\sigma B} |u-c|^s \omega_2^\alpha dx\right)^{1/s} \tag{2.23}$$

for all closed forms c.

Furthermore, choosing $\alpha = 1/r$ in Corollary 2.7, we get the following result.

Corollary 2.8. *Let $u \in L^s(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n-1$, be a solution to the homogeneous A-harmonic equation* (1.5) *and $(\omega_1(x), \omega_2(x)) \in A^{n-1}_{r,r}(\Omega)$ for some $1 < r < \infty$. Then there exists a constant C, independent of u and du, such that*

$$\left(\int_B |du|^s \omega_1^{1/r} dx\right)^{1/s} \leq C|B|^{1/ns-1/n-1/s}\left(\int_{\sigma B} |u-c|^s \omega_2^{1/r} dx\right)^{1/s} \tag{2.24}$$

for all closed forms c.

3. The Local Weighted Estimates for the Decomposition

The Hodge decomposition theorem has been playing an important part in partial differential equation, the operator theory, and so on. In the recent years, there are some interesting conclusions on the Hodge decomposition of differential forms; see [5, 6]. In [7], there is the following decomposition theorem for differential form $u \in L^p(\mathbb{R}^n, \wedge^l)$.

Lemma 3.1. *Let $u \in L^p(\mathbb{R}^n, \wedge^l)$, $1 < p < \infty$, $l = 1, 2, \ldots, n-1$. Then, there exist differential forms $Q \in L^p_1(\mathbb{R}^n, \wedge^{l-1})$ and $R \in L^p_1(\mathbb{R}^n, \wedge^{l+1})$, such that*

$$u = dQ + d^\star R, \qquad d^\star Q = dR = 0, \tag{3.1}$$

$$\|\nabla Q\|_p + \|\nabla R\|_p \leq C\|u\|_p, \tag{3.2}$$

here C is a positive constant.

Definition 3.2. The weight $\omega(x)$ satisfies the A_r-condition on the set $E \subset \mathbb{R}^n$, write $\omega \in A_r(E)$, if $\omega(x) > 0$, a.e., and for all balls $B \subset E$, one has that

$$\sup_B \left(\frac{1}{|B|}\int_B \omega dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{r-1} < \infty, \tag{3.3}$$

where $r > 1$.

The next lemma states the reverse Hölder inequality for $A_r(E)$-weight; see [8].

Lemma 3.3. *Assume that $\omega \in A_r(E)$. Then, there exists a constant C, independent of ω, for all balls $B \subset E$, such that*

$$\|\omega\|_{\beta,B} \leq C|B|^{(1-\beta)/\beta}\|\omega\|_{1,B}, \tag{3.4}$$

where $\beta > 1$ is a real number.

In this section, we will extend (3.2) to the weighted form.

Theorem 3.4. *Let $u \in L^p(\Omega, \wedge^l)$ be the solution to the nonhomogeneous A-harmonic equation* (1.2), $1 < p < \infty$, $l = 1, 2, \ldots, n-1$. *Then, there exist differential forms $Q \in L_1^p(\Omega, \wedge^{l-1})$ and $R \in L_1^p(\Omega, \wedge^{l+1})$, such that*

$$u = dQ + d^\star R, \qquad d^\star Q = dR = 0, \tag{3.5}$$

$$\|\nabla Q\|_{p,B,\omega^\alpha} + \|\nabla R\|_{p,B,\omega^\alpha} \le C\|u\|_{p,\sigma B,\omega^\alpha}, \tag{3.6}$$

here $\omega \in A_r(\Omega)$, $r > 1$, $\sigma > 1$, $0 < \alpha \le 1$, and $C > 0$ is a constant.

Proof. From (3.2), it follows that

$$\|\nabla Q\|_s \le C_1\|u\|_s, \tag{3.7}$$

$$\|\nabla R\|_s \le C_2\|u\|_s \tag{3.8}$$

for any $s > 1$.

When $0 < \alpha < 1$, take $s = p/(1-\alpha) > p > 1$. By Lemma 2.3 with $1/p = 1/s + (s-p)/sp$ and (3.7), we have

$$\begin{aligned}\|\nabla Q\|_{p,B,\omega^\alpha} &= \left(\int_B |\nabla Q|^p \omega^\alpha dx\right)^{1/p} \\ &= \left(\int_B \left(|\nabla Q|\omega^{\alpha/p}\right)^p dx\right)^{1/p} \\ &\le \left(\int_B (|\nabla Q|)^s dx\right)^{1/s} \left(\int_B (\omega^{\alpha/p})^{sp/(s-p)} dx\right)^{(s-p)/sp} \\ &= \|\nabla Q\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p} \\ &\le C_1\|u\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p}.\end{aligned} \tag{3.9}$$

Similarly, using Lemma 2.3 and (3.8), we have

$$\begin{aligned}\|\nabla R\|_{p,B,\omega^\alpha} &= \left(\int_B |\nabla R|^p \omega^\alpha dx\right)^{1/p} \\ &= \left(\int_B (|\nabla R|\omega^{\alpha/p})^p dx\right)^{1/p} \\ &\le \left(\int_B (|\nabla R|)^s dx\right)^{1/s} \left(\int_B \left(\omega^{\alpha/p}\right)^{sp/(s-p)} dx\right)^{(s-p)/sp} \\ &= \|\nabla R\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p} \\ &\le C_2\|u\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p}.\end{aligned} \tag{3.10}$$

Combining (3.9) with (3.10), we have

$$\begin{aligned}\|\nabla Q\|_{p,B,\omega^\alpha}+\|\nabla R\|_{p,B,\omega^\alpha}&\le C_1\|u\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p}+C_2\|u\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p}\\&\le C_3\|u\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p}.\end{aligned}\tag{3.11}$$

Take $t=p/(1+\alpha(r-1))$, then $t<p<s$ and $(p-t)/pt=\alpha(r-1)/p$. By Lemma 2.4, it follows that

$$\|u\|_{s,B}\le C_4|B|^{(t-s)/st}\|u\|_{t,\sigma B},\tag{3.12}$$

here $\sigma>1$ is a constant. Putting (3.11) into (3.12), we get

$$\begin{aligned}\|\nabla Q\|_{p,B,\omega^\alpha}+\|\nabla R\|_{p,B,\omega^\alpha}&\le C_3\|u\|_{s,B}\left(\int_B \omega dx\right)^{\alpha/p}\\&\le C_5|B|^{(t-s)/st}\|u\|_{t,\sigma B}\left(\int_B \omega dx\right)^{\alpha/p}.\end{aligned}\tag{3.13}$$

By Lemma 2.3, the following inequality holds:

$$\begin{aligned}\|u\|_{t,\sigma B}&=\left(\int_{\sigma B}|u|^t dx\right)^{1/t}\\&=\left(\int_{\sigma B}\left(|u|\omega^{\alpha/p}\omega^{-\alpha/p}\right)^t dx\right)^{1/t}\\&\le\left(\int_{\sigma B}|u|^p\omega^\alpha dx\right)^{1/p}\left(\int_{\sigma B}\left(\omega^{-\alpha/p}\right)^{pt/(p-t)}dx\right)^{(p-t)/pt}\\&=\left(\int_{\sigma B}|u|^p\omega^\alpha dx\right)^{1/p}\left(\int_{\sigma B}\left(\frac{1}{\omega}\right)^{1/(r-1)}dx\right)^{\alpha(r-1)/p}\\&=\|u\|_{p,\sigma B,\omega^\alpha}\left(\int_{\sigma B}\left(\frac{1}{\omega}\right)^{1/(r-1)}dx\right)^{\alpha(r-1)/p}.\end{aligned}\tag{3.14}$$

Combining (3.13) with (3.14), we obtain the following estimate:

$$\begin{aligned}\|\nabla Q\|_{p,B,\omega^\alpha}+\|\nabla R\|_{p,B,\omega^\alpha}&\le C_5|B|^{(t-s)/st}\|u\|_{t,\sigma B}\left(\int_B \omega dx\right)^{\alpha/p}\\&\le C_5|B|^{(t-s)/st}\|u\|_{p,\sigma B,\omega^\alpha}\left(\int_B \omega dx\right)^{\alpha/p}\cdot\left(\int_{\sigma B}\left(\frac{1}{\omega}\right)^{1/(r-1)}dx\right)^{\alpha(r-1)/p}.\end{aligned}\tag{3.15}$$

Since $\omega \in A_r(\Omega)$, we have

$$\begin{aligned}
&\left(\int_B \omega dx\right)^{\alpha/p} \cdot \left(\int_{\sigma B} \left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/p} \\
&\quad \leq \left(\int_{\sigma B} \omega dx\right)^{\alpha/p} \cdot \left(\int_{\sigma B} \left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/p} \\
&\quad = \left\{ |\sigma B|^r \left(\frac{1}{|\sigma B|} \int_{\sigma B} \omega dx\right) \left(\frac{1}{|\sigma B|} \int_{\sigma B} \left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{r-1} \right\}^{\alpha/p} \\
&\quad \leq C_6 |B|^{\alpha r/p}.
\end{aligned} \tag{3.16}$$

Putting (3.16) into (3.15), we get

$$\|\nabla Q\|_{p,B,\omega^\alpha} + \|\nabla R\|_{p,B,\omega^\alpha} \leq C_7 |B|^{(t-s)/st} |B|^{\alpha r/p} \|u\|_{p,\sigma B,\omega^\alpha}. \tag{3.17}$$

Recall that $s = p/(1-\alpha)$, $t = p/(1+\alpha(r-1))$, then we have

$$\frac{t-s}{ts} = \frac{p(1-\alpha) - p[1+\alpha(r-1)]}{p^2} = -\frac{\alpha r}{p}. \tag{3.18}$$

So (3.17) can be rewritten as

$$\|\nabla Q\|_{p,B,\omega^\alpha} + \|\nabla R\|_{p,B,\omega^\alpha} \leq C_7 \|u\|_{p,\sigma B,\omega^\alpha}. \tag{3.19}$$

When $\alpha = 1$, by Lemma 3.3, we know that

$$\|\omega\|_{\beta,B} \leq C_8 |B|^{(1-\beta)/\beta} \|\omega\|_{1,B}, \tag{3.20}$$

where $\beta > 1$.

Let $s = \beta p/(\beta - 1)$, then $1 < p < s$ and $\beta = s/(s-p)$. By Lemma 2.3 with $1/p = 1/s + (s-p)/sp$ and (3.20), we have that

$$\begin{aligned}
\|\nabla Q\|_{p,B,\omega} &= \left(\int_B |\nabla Q|^p \omega dx\right)^{1/p} \\
&= \left(\int_B \left(|\nabla Q|\omega^{1/p}\right)^p dx\right)^{1/p} \\
&\leq \left(\int_B (|\nabla Q|)^s dx\right)^{1/s} \left(\int_B \left(\omega^{1/p}\right)^{sp/(s-p)} dx\right)^{(s-p)/sp} \\
&= \|\nabla Q\|_{s,B}\left(\int_B \omega^\beta dx\right)^{1/p\beta} \\
&\leq C_9 |B|^{(1-\beta)/p\beta} \|\omega\|_{1,B}^{1/p} \|\nabla Q\|_{s,B}.
\end{aligned} \tag{3.21}$$

Similarly, by Lemma 2.3 and (3.20), we have that

$$\begin{aligned}
\|\nabla R\|_{p,B,\omega} &= \left(\int_B |\nabla R|^p \omega dx\right)^{1/p} \\
&= \left(\int_B \left(|\nabla R|\omega^{1/p}\right)^p dx\right)^{1/p} \\
&\leq \left(\int_B (|\nabla R|)^s dx\right)^{1/s} \left(\int_B \left(\omega^{1/p}\right)^{sp/(s-p)} dx\right)^{(s-p)/sp} \\
&= \|\nabla R\|_{s,B}\left(\int_B \omega^\beta dx\right)^{1/p\beta} \\
&\leq C_{10} |B|^{(1-\beta)/p\beta} \|\omega\|_{1,B}^{1/p} \|\nabla R\|_{s,B}.
\end{aligned} \tag{3.22}$$

Putting (3.7) into (3.21), we have

$$\begin{aligned}
\|\nabla Q\|_{p,B,\omega} &\leq C_9 |B|^{(1-\beta)/p\beta} \|\omega\|_{1,B}^{1/p} \|\nabla Q\|_{s,B} \\
&\leq C_{11} |B|^{(1-\beta)/p\beta} \|\omega\|_{1,B}^{1/p} \|u\|_{s,B}.
\end{aligned} \tag{3.23}$$

Putting (3.8) into (3.22), we conclude that

$$\begin{aligned}
\|\nabla R\|_{p,B,\omega} &\leq C_{10} |B|^{(1-\beta)/p\beta} \|\omega\|_{1,B}^{1/p} \|\nabla R\|_{s,B} \\
&\leq C_{12} |B|^{(1-\beta)/p\beta} \|\omega\|_{1,B}^{1/p} \|u\|_{s,B}.
\end{aligned} \tag{3.24}$$

Combining (3.23) with (3.24), we obtain

$$\begin{aligned}
&\|\nabla Q\|_{p,B,\omega} + \|\nabla R\|_{p,B,\omega} \\
&\quad \le C_{11}|B|^{(1-\beta)/p\beta}\|\omega\|_{1,B}^{1/p}\|u\|_{s,B} + C_{12}|B|^{(1-\beta)/p\beta}\|\omega\|_{1,B}^{1/p}\|u\|_{s,B} \\
&\quad \le C_{13}|B|^{(1-\beta)/p\beta}\|\omega\|_{1,B}^{1/p}\|u\|_{s,B}.
\end{aligned} \tag{3.25}$$

Let $t = p/r$, then $t < p$. By Lemma 2.4, we have

$$\|u\|_{s,B} \le C_{14}|B|^{(t-s)/st}\|u\|_{t,\sigma B}, \tag{3.26}$$

where $\sigma > 1$. Putting (3.26) into (3.25), we get

$$\begin{aligned}
\|\nabla Q\|_{p,B,\omega} + \|\nabla R\|_{p,B,\omega} &\le C_{13}|B|^{(1-\beta)/p\beta}\|\omega\|_{1,B}^{1/p}\|u\|_{s,B} \\
&\le C_{15}|B|^{(1-\beta)/p\beta}|B|^{(t-s)/st}\|u\|_{t,\sigma B}\|\omega\|_{1,B}^{1/p}.
\end{aligned} \tag{3.27}$$

By Lemma 2.3, we get

$$\begin{aligned}
\|u\|_{t,\sigma B} &= \left(\int_{\sigma B} |u|^t dx\right)^{1/t} \\
&= \left(\int_{\sigma B} \left(|u|\omega^{1/p}\omega^{-1/p}\right)^t dx\right)^{1/t} \\
&\le \left(\int_{\sigma B} |u|^p \omega dx\right)^{1/p}\left(\int_{\sigma B}\left(\frac{1}{\omega}\right)^{pt/(p-t)} dx\right)^{(p-t)/pt} \\
&= \|u\|_{p,\sigma B,\omega}\left(\int_{\sigma B}\left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{(r-1)/p}.
\end{aligned} \tag{3.28}$$

Putting (3.28) into (3.27), it follows that

$$\begin{aligned}
&\|\nabla Q\|_{p,B,\omega} + \|\nabla R\|_{p,B,\omega} \\
&\quad \le C_{15}|B|^{(1-\beta)/p\beta}|B|^{(t-s)/st}\|u\|_{t,\sigma B}\|\omega\|_{1,B}^{1/p} \\
&\quad \le C_{15}|B|^{(1-\beta)/p\beta}|B|^{(t-s)/st}\|u\|_{p,\sigma B,\omega}\left(\int_{\sigma B}\left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{(r-1)/p}\left(\int_B \omega dx\right)^{1/p}.
\end{aligned} \tag{3.29}$$

Since $\omega \in A_r(\Omega)$, we have

$$\begin{aligned}
&\left(\int_B \omega dx\right)^{1/p} \cdot \left(\int_{\sigma B} \left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{(r-1)/p} \\
&\quad \le \left(\int_{\sigma B} \omega dx\right)^{1/p} \cdot \left(\int_{\sigma B} \left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{(r-1)/p} \\
&\quad = \left\{ |\sigma B|^r \left(\frac{1}{|\sigma B|}\int_{\sigma B} \omega dx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B}\left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{r-1}\right\}^{1/p} \\
&\quad \le C_{16}|B|^{r/p}.
\end{aligned} \tag{3.30}$$

Putting (3.30) into (3.29), we get the following inequality:

$$\|\nabla Q\|_{p,B,\omega} + \|\nabla R\|_{p,B,\omega} \le C_{17}|B|^{(1-\beta)/p\beta}|B|^{(t-s)/st}|B|^{r/p}\|u\|_{p,\sigma B,\omega}. \tag{3.31}$$

Since $(1-\beta)/p\beta + (t-s)/st + r/p = -1/s + (t-s)/st + 1/t = (t-s)/st + (s-t)/st = 0$, (3.31) can be rewritten as

$$\|\nabla Q\|_{p,B,\omega} + \|\nabla R\|_{p,B,\omega} \le C_{17}\|u\|_{p,\sigma B,\omega}. \tag{3.32}$$

So inequality (3.4) is true for $0 < \alpha \le 1$. □

4. The Global Weighted Estimates

Based on the local weighted estimate for the decomposition and the Whitney covering lemma, we get the global weighted estimate on domain $\Omega \subset \mathbb{R}^n$.

Theorem 4.1. *Let $u \in L^p(\Omega, \wedge^l)$ be a differential form satisfying the nonhomogeneous A-harmonic equation* (1.2) *in a bounded domain $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, $l = 1, 2, \ldots, n-1$, and $\omega \in A_r(\Omega)$, $r > 1$. Then, there exist differential forms $Q \in L_1^P(\Omega, \wedge^{l-1})$ and $R \in L_1^P(\Omega, \wedge^{l+1})$, such that*

$$\begin{gathered}
u = dQ + d^\star R, \qquad d^\star Q = dR = 0, \\
\|\nabla Q\|_{p,\Omega,\omega^\alpha} + \|\nabla R\|_{p,\Omega,\omega^\alpha} \le C\|u\|_{p,\Omega,\omega^\alpha},
\end{gathered} \tag{4.1}$$

where $0 < \alpha \le 1$ and $C > 0$ is a constant.

Proof. By Lemma 2.5, $\Omega \subset \mathbb{R}^n$ has a modified Whitney cover of cubes $\nu = \{B_i\}$. Let $\sigma = \sqrt{5/4}$.

First, we assume that all cubes B_i satisfy $\sigma B_i \subset \Omega$. By Lemma 2.5 and (3.6), we know that

$$\begin{aligned}
\|\nabla Q\|_{p,\Omega,\omega^\alpha} + \|\nabla R\|_{p,\Omega,\omega^\alpha} &= \left(\int_\Omega |\nabla Q|^p \omega^\alpha dx\right)^{1/p} + \left(\int_\Omega |\nabla R|^p \omega^\alpha dx\right)^{1/p} \\
&\le \sum_{B\in\nu} \left\{ \left(\int_B |\nabla Q|^p \omega^\alpha dx\right)^{1/p} + \left(\int_B |\nabla R|^p \omega^\alpha dx\right)^{1/p} \right\} \\
&\le \sum_{B\in\nu} C_1 \left(\int_{\sigma B} |u|^p \omega^\alpha dx\right)^{1/p} \\
&\le \sum_{B\in\nu} C_2 \left(\int_{\sigma B} |u|^p \omega^\alpha dx\right)^{1/p} \chi_{\sigma B}(x) \\
&\le \sum_{B\in\nu} C_2 \left(\int_{\Omega} |u|^p \omega^\alpha dx\right)^{1/p} \chi_{\sigma B}(x) \\
&\le \left(\int_{\Omega} |u|^p \omega^\alpha dx\right)^{1/p} \sum_{B\in\nu} C_2 \chi_{\sigma B}(x) \\
&\le C\left(\int_{\Omega} |u|^p \omega^\alpha dx\right)^{1/p}.
\end{aligned} \tag{4.2}$$

In the above proof, if there exists one cube $B_0 \in \nu$, such that σB_0 can not be contained in Ω completely, the proof is as follows. Set $\Omega_1 = \bigcup_{B\in\nu} \sigma B$, and define $U(x)$ on Ω_1 as follows:

$$U(x) = \begin{cases} u(x), & x \in \Omega, \\ 0 & x \in \Omega_1 - \Omega. \end{cases} \tag{4.3}$$

Then, we know that the following formulas are true:

$$\begin{aligned}
\left(\int_{\sigma B} |u|^p \omega^\alpha dx\right)^{1/p} &= \left(\int_{\sigma B} |U|^p \omega^\alpha dx\right)^{1/p}, \\
\left(\int_{\Omega_1} |U|^p \omega^\alpha dx\right)^{1/p} &= \left(\int_{\Omega} |u|^p \omega^\alpha dx\right)^{1/p}.
\end{aligned} \tag{4.4}$$

Note that (4.4) hold, and then the following formula is true:

$$\begin{aligned}
\|\nabla Q\|_{p,\Omega,\omega^{\alpha}} + \|\nabla R\|_{p,\Omega,\omega^{\alpha}} &= \left(\int_{\Omega} |\nabla Q|^{p} \omega^{\alpha} dx\right)^{1/p} + \left(\int_{\Omega} |\nabla R|^{p} \omega^{\alpha} dx\right)^{1/p} \\
&\le \sum_{B\in\mathcal{V}} \left\{ \left(\int_{B} |\nabla Q|^{p} \omega^{\alpha} dx\right)^{1/p} + \left(\int_{B} |\nabla R|^{p} \omega^{\alpha} dx\right)^{1/p} \right\} \\
&\le \sum_{B\in\mathcal{V}} C_3 \left(\int_{\sigma B} |u|^{p} \omega^{\alpha} dx\right)^{1/p} \\
&= \sum_{B\in\mathcal{V}} C_3 \left(\int_{\sigma B} |U|^{p} \omega^{\alpha} dx\right)^{1/p} \\
&\le \sum_{B\in\mathcal{V}} C_4 \left(\int_{\sigma B} |U|^{p} \omega^{\alpha} dx\right)^{1/p} \chi_{\sigma B}(x) \\
&\le \sum_{B\in\mathcal{V}} C_5 \left(\int_{\Omega_1} |U|^{p} \omega^{\alpha} dx\right)^{1/p} \chi_{\sigma B}(x) \\
&\le \left(\int_{\Omega_1} |U|^{p} \omega^{\alpha} dx\right)^{1/p} \sum_{B\in\mathcal{V}} C_5 \chi_{\sigma B}(x) \\
&\le C \left(\int_{\Omega_1} |U|^{p} \omega^{\alpha} dx\right)^{1/p} \\
&= C \left(\int_{\Omega} |u|^{p} \omega^{\alpha} dx\right)^{1/p}.
\end{aligned} \tag{4.5}$$

The proof of Theorem 4.1 is completed. □

In Theorems 3.4 and 4.1, there is a real parameter α, which makes the results more flexible. By choosing different value of the parameter α, we get the estimates in different forms. For example, if we take $\alpha = 1$, we have the following corollary.

Corollary 4.2. *Let $u \in L^{p}(\Omega, \wedge^{l})$ be the solution to the nonhomogeneous A-harmonic equation* (1.2)*, $1 < p < \infty$, $l = 1, 2, \ldots, n-1$, and $\omega \in A_r(\Omega)$, $r > 1$. Then, there exist differential forms $Q \in L_1^{p}(\Omega, \wedge^{l-1})$ and $R \in L_1^{p}(\Omega, \wedge^{l+1})$, such that*

$$\begin{gathered}
u = dQ + d^{\star}R, \qquad d^{\star}Q = dR = 0, \\
\|\nabla Q\|_{p,\Omega,\omega} + \|\nabla R\|_{p,\Omega,\omega} \le C\|u\|_{p,\Omega,\omega},
\end{gathered} \tag{4.6}$$

where $C > 0$ is a constant.

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