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Research Article

An Iterative Scheme with a Countable Family of Nonexpansive Mappings for Variational Inequality Problems in Hilbert Spaces

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We introduce a new iterative scheme with a countable family of nonexpansive mappings for the variational inequality problems in Hilbert spaces and prove some strong convergence theorems for the proposed schemes.

1. Introduction

Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $F : H \to H$ be a nonlinear mapping. The classical variational inequality problem (for short, $\mathrm{VI}(F, C)$) is to find a point $x \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x^* \in C. \tag{1.1}$$

This variational inequality was initially studied by Kinderlehrer and Stampacchia [1]. Since then, many authors have introduced and studied many kinds of the variational inequality problems (inclusions) and applied them to many fields.

It is well known that, if F is a strongly monotone and Lipschitzian mapping on C, then the $\mathrm{VI}(F, C)$ has a unique solution (see [2]).

Let $T : H \to H$ be a mapping. Recall that a mapping $T : H \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{1.2}$$

The set of fixed points of T is denoted by $F(T)$. Recently, the iterative methods for nonexpansive mappings and some kinds of nonlinear mappings have been applied to solve the convex minimization problems (see [3–7]).

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.3}$$

where C is the fixed point set of a nonexpansive mapping T on H, b is a given point in H and A is a strongly positive operator, that is, there is a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.4}$$

Recently, for solving the variational inequality on A, Marino and Xu [8] introduced the following general iterative scheme:

$$x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 0, \tag{1.5}$$

where A is a strongly positive linear bounded operator on H, f is a contraction on H and $\{\alpha_n\} \subset (0, 1)$.

More precisely, they gave the following result.

Theorem MX (see [8, Theorem 3.4]). *Let $\{x_n\}$ be generated by algorithm* (1.5) *with the sequence $\{\alpha_n\}$ satisfying the following conditions:*

(C1) $\lim_{n \to \infty} \alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C3) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n \to \infty} \alpha_{n+1} / \alpha_n = 1$.

Then the scheme $\{x_n\}$ defined by (1.5) *converges strongly to an element $x^* \in C = F(T)$ which is the unique solution of the variational inequality* (*for short,* $\mathrm{VI}(A - \gamma f, C)$)*:*

$$\langle (A - \gamma f) x^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.6}$$

Let $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$ and let $A, B : H \to H$ be two strongly positive linear bounded operators with coefficients $\overline{\gamma} \in (0, 1)$ and $\beta > 0$, respectively.

Motivated and inspired by the iterative sheme (1.5), Ceng et al. [9] introduced the following so-called *hybrid viscosity-like approximation algorithms* with variable parameters for nonexpansive mappings in Hilbert spaces.

Theorem CGY1 (see [9, Theorem 3.1]). *Let $0 < \gamma\alpha < \beta$ and $\overline{\gamma} \in (0,1)$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ and $\{\mu_n\}$ be a sequence in $(0, \min\{1, \|B\|^{-1}\})$. Starting with an arbitrary initial guess $x_0 \in H$, generate a sequence $\{x_n\}$ by the following iterative scheme:*

$$x_{n+1} = (I - \lambda_{n+1}A)Tx_n + \lambda_{n+1}\left[Tx_n - \mu_{n+1}\left(BTx_n - \gamma f(x_n)\right)\right], \quad \forall n \geq 0. \tag{1.7}$$

Assume that

(i) $\lim_{n\to\infty}\lambda_n = 0$,

(ii) $\sum_{n=1}^{\infty}\lambda_n = \infty$,

(iii) *either* $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$ *or* $\lim_{n\to\infty}\lambda_n/\lambda_{n+1} = 1$,

(iv) $(1-\overline{\gamma})/(\beta - \gamma\alpha) < \lim_{n\to\infty}\mu_n = \mu < (2-\overline{\gamma})/(\beta-\gamma\alpha)$.

Then the scheme $\{x_n\}$ defined by (1.7) converges strongly to an element $x^* \in C = F(T)$ which is the unique solution of the variational inequality (for short, $\mathrm{VI}(A - I + \mu(B - \gamma f), C)$*):*

$$\left\langle \left[A - I + \mu\left(B - \gamma f\right)\right]x^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C. \tag{1.8}$$

Theorem CGY2 (see [9, Theorem 3.2]). *Let $0 < \gamma\alpha < \beta$ and $\overline{\gamma} \in (0,1)$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ and $\{\mu_n\}$ be a sequence in $(0, \min\{1, \|B\|^{-1}\})$. Starting with an arbitrary initial guess $x_0 \in H$, generate a sequence $\{x_n\}$ by the following iterative scheme:*

$$x_{n+1} = (I - \lambda_{n+1}A)T_{[n+1]}x_n + \lambda_{n+1}\left[T_{[n+1]}x_n - \mu_{n+1}\left(BT_{[n+1]}x_n - \gamma f(x_n)\right)\right], \quad \forall n \geq 0. \tag{1.9}$$

Assume that

(i) $\lim_{n\to\infty}\lambda_n = 0$,

(ii) $\sum_{n=1}^{\infty}\lambda_n = \infty$,

(iii) *either* $\sum_{n=1}^{\infty}|\lambda_{n+N} - \lambda_n| < \infty$ *or* $\lim_{n\to\infty}\lambda_n/\lambda_{n+N} = 1$,

(iv) $(1-\overline{\gamma})/(\beta - \gamma\alpha) < \lim_{n\to\infty}\mu_n = \mu < (2-\overline{\gamma})/(\beta-\gamma\alpha)$.

In addition, assume that

$$\begin{aligned} C &= \bigcap_{i=1}^{N} F(T_i) = F(T_1T_2\cdots T_N) = F(T_NT_1\cdots T_2) \\ &= \cdots = F(T_2T_3\cdots T_NT_1). \end{aligned} \tag{1.10}$$

Then the scheme $\{x_n\}$ defined by (1.9) converges strongly to an element $x^* \in C = F(T)$ which is the unique solution of the variational inequality (for short, $\mathrm{VI}(A - I + \mu(B - \gamma f), C)$*):*

$$\left\langle \left[A - I + \mu\left(B - \gamma f\right)\right]x^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C. \tag{1.11}$$

In this paper, motivated and inspired by the above research results, we introduce a new iterative process with a countable family of nonexpansive mappings for the variational inequality problem in Hilbert spaces.

More precisely, let H be a Hilbert space and $\{T_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings from H to H such that $C = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$ and A, B be strongly positive linear bounded operators with coefficients $\eta \in (0,1)$ and $\beta > 0$, respectively. Let $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1]$ with $\alpha_0 = 1$. Take three fixed numbers γ, μ_1 and μ_2 such that $0 < \gamma\alpha < \beta$, $\mu_1 \in (0,1]$ and $\mu_2 \in ((1-\eta\mu_1)/(\beta-\gamma\alpha), \min\{1, \|B\|^{-1}, (2-\eta\mu_1)/(\beta-\gamma\alpha)\}]$. For any $x_1 \in H$, generate the iterative scheme $\{x_n\}$ by

$$x_{n+1} = \alpha_n\left[(I-\lambda_n\mu_1 A)x_n + \lambda_n\left[x_n - \mu_2(Bx_n - \gamma f(x_n))\right]\right] + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i x_n, \quad \forall n \geq 1. \tag{1.12}$$

We prove that the iterative scheme $\{x_n\}$ defined by (1.12) strongly converges to an element $x^* \in C$ which is the unique solution of the variational inequality (for short, $\mathrm{VI}(\mu_1 A - I + \mu_2(B-\gamma f), C)$):

$$\left\langle \left[\mu_1 A - I + \mu_2(B-\gamma f)\right]x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in C. \tag{1.13}$$

2. Preliminaries

Let H be a Hilbert space and T be a nonexpansive mapping of H into itself such that $F(T) \neq \emptyset$. For all $\hat{x} \in F(T)$ and $x \in H$, we have

$$\begin{aligned}\|x-\hat{x}\|^2 &\geq \|Tx - T\hat{x}\|^2 = \|Tx-\hat{x}\|^2 = \|Tx - x + (x-\hat{x})\|^2 \\ &= \|Tx-x\|^2 + \|x-\hat{x}\|^2 + 2\langle Tx - x, x-\hat{x}\rangle\end{aligned} \tag{2.1}$$

and hence

$$\|Tx-x\|^2 \leq 2\langle x - Tx, x-\hat{x}\rangle, \quad \forall \hat{x} \in F(T),\ x \in H. \tag{2.2}$$

Let $\{x_n\}$ be a sequence in a Hilbert space H and let $x \in H$. Throughout this paper, $x_n \to x$ and $x_n \rightharpoonup x$ denote that $\{x_n\}$ strongly converges to $x \in H$ and $\{x_n\}$ converges weakly to a point $x \in H$, respectively.

Lemma 2.1 (see [10]). *Let C be a closed convex subset of a Hilbert space H and T be a nonexpansive mapping from C into itself. Then $I - T$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \quad x_n - Tx_n \longrightarrow 0 \quad \text{implies } x = Tx. \tag{2.3}$$

The following lemma is an immediate consequence of the equality:

$$\|x+y\|^2 = \|x\|^2 + 2\langle y, x+y\rangle - \|y\|^2, \quad \forall x, y \in H. \tag{2.4}$$

Lemma 2.2. *Let H be a real Hilbert space. Then the following identity holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{2.5}$$

Lemma 2.3 (see [4, 11]). *Let $\{s_n\}$, $\{c_n\}$ be the sequences of nonnegative real numbers and let $\{a_n\} \subset (0,1)$. Suppose that $\{b_n\}$ is a sequence of real numbers such that*

$$s_{n+1} \le (1-a_n)s_n + b_n + c_n, \quad \forall n \ge 0. \tag{2.6}$$

Assume that $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold.

(1) *If $b_n \le \beta a_n$, where $(\beta \ge 0)$, then $\{s_n\}$ is a bounded sequence.*

(2) *If one has*

$$\sum_{n=0}^{\infty} a_n = \infty, \qquad \limsup_{n\to\infty} \frac{b_n}{a_n} \le 0, \tag{2.7}$$

then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 (see [8]). *Let H be a real Hilbert space, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$ and B be a strongly positive linear bounded operator with coefficient $\beta > 0$. Then, for any γ with $0 < \gamma < \beta/\alpha$,*

$$\langle x-y, (B-\gamma f)x - (B-\gamma f)y\rangle \ge (\beta - \gamma\alpha)\|x-y\|^2, \quad \forall x, y \in H, \tag{2.8}$$

that is, $B - \gamma f$ is strongly monotone with coefficient $\beta - \gamma\alpha$.

Lemma 2.5. *Assume A is a strongly monotone linear bounded operator on a Hilbert space H with coefficient $\alpha > 0$. Take a fixed number ρ such that $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\alpha$.*

Proof. The proof method is mainly from the idea of Marino and Xu [8, Lemma 2.5]. It is known that the norm of a linear bounded self-adjoint operator V on H is as follows:

$$\|V\| = \sup\{|\langle Vx, x\rangle| : x \in H, \ \|x\| = 1\}. \tag{2.9}$$

Now, for all $x \in H$ with $\|x\| = 1$, we see that (here 0 denotes zero point in H)

$$\begin{aligned}\|I - \rho A\| &= \sup\{\langle (I - \rho A)x, x\rangle : x \in H,\ \|x\| = 1\}\\&= \sup\{\langle (I - \rho A)x - (I - \rho A)0, x - 0\rangle : x \in H,\ \|x\| = 1\}\\&= \sup\{\langle (x - 0) - \rho(Ax - A0), x - 0\rangle : x \in H,\ \|x\| = 1\}\\&= \sup\left\{\|x\|^2 - \rho\langle (Ax - A0), x - 0\rangle : x \in H,\ \|x\| = 1\right\}\\&\le \sup\left\{1 - \rho\alpha\|x - 0\|^2 : x \in H,\ \|x\| = 1\right\}\\&= 1 - \rho\alpha.\end{aligned} \tag{2.10}$$

This completes the proof. □

Remark 2.6. Lemma 2.5 still holds if A is a strongly positive linear bounded operator (see [8, Lemma 2.5]). That is, Lemma 2.5 in this section and Lemma 2.5 in [8] both hold when A is a strongly monotone linear bounded operator or a strongly positive linear bounded one because an operator on a Hilbert space is strongly monotone linear if and only if it is strongly positive linear.

In fact, if A is a strongly monotone linear operator with coefficient $\alpha > 0$ on a Hilbert space H, then, for all $x \in H$,

$$\langle Ax, x\rangle = \langle Ax - A0, x - 0\rangle \ge \alpha\|x - 0\|^2 = \alpha\|x\|^2, \tag{2.11}$$

which shows that A is strongly positive linear. Assume that A is a strongly positive linear operator with coefficient $\alpha > 0$ on H. Then, for all $x, y \in H$,

$$\langle Ax - Ay, x - y\rangle = \langle A(x - y), x - y\rangle \ge \alpha\|x - y\|^2, \tag{2.12}$$

which shows that A is strongly monotone and linear.

3. Main Results

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$. Let $A, B : H \to H$ be strongly positive linear bounded operator with coefficient $\eta \in (0,1)$ and $\beta > 0$, respectively. Take a fixed number γ such that $0 \le \gamma\alpha < \beta$. Then, from Lemma 2.4, it follows that $B - \gamma f$ is strongly monotone with coefficient $\beta - \gamma\alpha > 0$. For any fixed numbers $\sigma_1 \in (0,1]$ and $\sigma_2 \in ((1-\eta\sigma_1)/(\beta-\gamma\alpha), (2-\eta\sigma_1)/(\beta-\gamma\alpha))$, we have $\theta = \eta\sigma_1 - 1 + \sigma_2(\beta-\gamma\alpha) \in (0,1)$, which can be seen easily from the following:

$$\sigma_2 < \frac{2-\eta\sigma_1}{\beta-\gamma\alpha} \Longleftrightarrow \sigma_2(\beta-\gamma\alpha) < 2-\eta\sigma_1 \\ \Longleftrightarrow \theta = \eta\sigma_1 - 1 + \sigma_2(\beta-\gamma\alpha) < 1, \tag{3.1}$$

$$\frac{1-\eta\sigma_1}{\beta-\gamma\alpha} < \sigma_2 \Longleftrightarrow \sigma_2(\beta-\gamma\alpha) + \eta\sigma_1 > 1 \\ \Longleftrightarrow \theta = \eta\sigma_1 - 1 + \sigma_2(\beta-\gamma\alpha) > 0. \tag{3.2}$$

Moreover, observe that

$$\begin{aligned} &\|(\sigma_1 A - I + \sigma_2(B-\gamma f))x - (\sigma_1 A - I + \sigma_2(B-\gamma f))y\| \\ &\quad = \|(\sigma_1 A - I)(x-y) + \sigma_2(B-\gamma f)(x-y)\| \\ &\quad \le \|\sigma_1 A - I\|\|x-y\| + \sigma_2[\|B(x-y)\| + \gamma\|fx - fy\|] \\ &\quad \le [\|\sigma_1 A - I\| + \sigma_2(\|B\| + \gamma\alpha)]\|x-y\|, \end{aligned} \tag{3.3}$$

which implies that $\sigma_1 A - I + \sigma_2(B-\gamma f)$ is Lipschitzian with coefficient $\|\sigma_1 A - I\| + \sigma_2(\|B\| + \gamma\alpha) > 0$.

On the other hand, from Lemma 2.4, it follows that

$$\begin{aligned} &\langle(\sigma_1 A - I + \sigma_2(B-\gamma f))x - (\sigma_1 A - I + \sigma_2(B-\gamma f))y, x-y\rangle \\ &\quad = \sigma_1\langle Ax - Ay, x-y\rangle + \sigma_2\langle(B-\gamma f)x - (B-\gamma f)y, x-y\rangle - \|x-y\|^2 \\ &\quad \ge \sigma_1\eta\|x-y\|^2 + \sigma_2(\beta-\gamma\alpha)\|x-y\|^2 - \|x-y\|^2 \\ &\quad = \theta\|x-y\|^2, \end{aligned} \tag{3.4}$$

which implies that $\sigma_1 A - I + \sigma_2(B-\gamma f)$ is strongly monotone with coefficient $\theta > 0$. Hence the variational inequality (for short, $\mathrm{VI}(\sigma_1 A - I + \sigma_2(B-\gamma f), C)$)

$$\langle \sigma_1 A - I + \sigma_2(B-\gamma f)x^*, x - x^*\rangle \ge 0, \quad \forall x \in C \tag{3.5}$$

has the unique solution.

Let $T : H \to H$ be a nonexpansive mapping. Take two fixed numbers μ_1 and μ_2 such that $\mu_1 \in (0,1]$ and $\mu_2 \in (0, \min\{1, \|B\|^{-1}\}]$ and, for all $\lambda \in (0, \min\{1, \|A\|^{-1}/\mu_1\})$, define a mapping $T^\lambda : H \to H$ by

$$T^\lambda x = (I - \lambda\mu_1 A)Tx + \lambda[Tx - \mu_2(BTx - \gamma f(x))], \quad \forall x \in H. \tag{3.6}$$

Then we have the following results.

Lemma 3.1. *If $\mu_2 \in ((1-\eta\mu_1)/(\beta-\gamma\lambda), (2-\eta\mu_1)/(\beta-\gamma\lambda))$, then T^{λ} is a contraction with coefficient $1-\lambda\tau$, where $\tau = \eta\mu_1 - 1 + \mu_2(\beta-\gamma\alpha) \in (0,1)$, that is,*

$$\left\|T^{\lambda}x - T^{\lambda}y\right\| \leq (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in H. \tag{3.7}$$

Proof. From Lemma 2.5 and Remark 2.6, it follows that, for all $x, y \in H$,

$$\begin{aligned}
\left\|T^{\lambda}x - T^{\lambda}y\right\| &= \|(I-\lambda\mu_1 A)Tx + \lambda[Tx - \mu_2(BTx - \gamma f(x))] \\
&\quad -(I-\lambda\mu_1 A)Ty - \lambda[Ty - \mu_2(BTy - \gamma f(y))]\| \\
&\leq \|(I-\lambda\mu_1 A)Tx - (I-\lambda\mu_1 A)Ty\| \\
&\quad + \lambda\|Tx - \mu_2(BTx - \gamma f(x)) - [Ty - \mu_2(BTy - \gamma f(y))]\| \\
&\leq \|(I-\lambda\mu_1 A)\|\|Tx - Ty\| \\
&\quad + \lambda[\|(I-\mu_2 B)Tx - (I-\mu_2 B)Ty\| + \mu_2\gamma\|f(x) - f(y)\|] \\
&\leq (1-\lambda\mu_1\eta)\|x-y\| + \lambda[\|I-\mu_2 B\|\|Tx - Ty\| + \mu_2\gamma\alpha\|x-y\|] \\
&\leq \{1-\lambda\mu_1\eta + \lambda[1-\mu_2(\beta-\gamma\alpha)]\}\|x-y\| \\
&= \{1-\lambda[\mu_1\eta - 1 + \mu_2(\beta-\gamma\alpha)]\}\|x-y\| \\
&= (1-\lambda\tau)\|x-y\|.
\end{aligned} \tag{3.8}$$

This completes the proof. □

Let $\{T_i\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings from H into itself such that $C = \bigcap_{n=1}^{\infty} F(T_i) \neq \emptyset$. Since each $F(T_i)$ is closed and convex, then C is closed and convex.

Throughout this paper, let $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$. Let $A, B : H \to H$ be strongly positive linear bounded mapping with coefficient $\eta \in (0,1)$ and $\beta > 0$, respectively. Take a fixed number γ such that $0 < \gamma\alpha < \beta$. Suppose that $\mu_1 \in (0,1]$, $\mu_2 \in ((1-\eta\mu_1)/(\beta-\gamma\alpha), \min\{1, \|B\|^{-1}, (2-\eta\mu_1)/(\beta-\gamma\alpha)\})$ (assuming that $(1-\eta\mu_1)/(\beta-\gamma\alpha) < \min\{1, \|B\|^{-1}\}$ such that $((1-\eta\mu_1)/(\beta-\gamma\lambda), \min\{1, \|B\|^{-1}, (2-\eta\mu_1)/(\beta-\gamma\lambda)\}]$ is nonempty), $\{\lambda_n\}_{n=1}^{\infty} \subset (0, \min\{1, \|A\|^{-1}/\mu_1\})$ with $\liminf_{n\to\infty}\lambda_n > 0$ and $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1]$ with $\alpha_0 = 1$.

Now, we can rewrite the iterative scheme (1.12) as follows:

$$x_{n+1} = \alpha_n T^{\lambda_n} x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i x_n, \quad \forall n \geq 1, \tag{3.9}$$

where $T^{\lambda_n}x_n = (I-\lambda_n\mu_1 A)x_n + \lambda_n[x_n - \mu_2(Bx_n - \gamma f(x_n))]$. Then, by Lemma 3.1, for all $x, y \in H$, we have

$$\left\|T_n^{\lambda_n}x - T_n^{\lambda_n}y\right\| \leq (1-\lambda_n\tau)\|x-y\|, \quad \forall n \geq 1, \tag{3.10}$$

where $\tau = \eta\mu_1 - 1 + \mu_2(\beta-\gamma\alpha) \in (0,1)$.

Lemma 3.2. *If $\{\alpha_n\}$ is strictly decreasing, then the scheme $\{x_n\}$ defined by* (3.9) *is bounded.*

Proof. Since $\|T^{\lambda_n}p-p\|=\lambda_n\|(\mu_1 A-I+\mu_2(B-\gamma f))p\|$, it follows from (3.10) that, for all $p\in C$,

$$\begin{aligned}\|x_{n+1}-p\| &= \left\|\alpha_n\left(T^{\lambda_n}x_n-p\right)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_ix_n-p)\right\| \\ &\le \alpha_n\left\|T^{\lambda_n}x_n-p\right\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_ix_n-p\| \\ &\le \alpha_n\left\|T^{\lambda_n}x_n-T^{\lambda_n}p\right\|+\alpha_n\left\|T^{\lambda_n}p-p\right\|+(1-\alpha_n)\|x_n-p\| \\ &\le \alpha_n(1-\lambda_n\tau)\|x_n-p\|+\alpha_n\lambda_n\|(\mu_1A-I+\mu_2(B-\gamma f))p\|+(1-\alpha_n)\|x_n-p\| \\ &= (1-\alpha_n\lambda_n\tau)\|x_n-p\|+\alpha_n\lambda_n\|(\mu_1A-I+\mu_2(B-\gamma f))p\|.\end{aligned} \tag{3.11}$$

By induction, we obtain

$$\|x_{n+1}-p\|\le\max\left\{\|x_1-p\|,\frac{1}{\tau}\|(\mu_1A-I+\mu_2(B-\gamma f))p\|\right\}. \tag{3.12}$$

Hence $\{x_n\}$ is bounded and so are $\{T^{\lambda_n}x_n\}$ and $\{T_ix_n\}$ for each $i\ge 1$. This completes the proof. □

Lemma 3.3. *If $\{\alpha_n\}$ is strictly decreasing and the following conditions hold:*

$$\sum_{n=1}^{\infty}\alpha_n=\infty,\qquad \sum_{n=1}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty, \tag{3.13}$$

then $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Proof. By the iterative scheme (3.9), we have

$$\begin{aligned}x_{n+1}-x_n &= \alpha_nT^{\lambda_n}x_n+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_ix_n-\left(\alpha_{n-1}T^{\lambda_{n-1}}x_{n-1}+\sum_{i=1}^{n-1}(\alpha_{i-1}-\alpha_i)T_ix_{n-1}\right) \\ &= \alpha_n\left(T^{\lambda_n}x_n-T^{\lambda_n}x_{n-1}\right)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_ix_n-T_ix_{n-1}) \\ &\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_ix_{n-1}-\sum_{i=1}^{n-1}(\alpha_{i-1}-\alpha_i)T_ix_{n-1}+\alpha_nT^{\lambda_n}x_{n-1}-\alpha_{n-1}T^{\lambda_{n-1}}x_{n-1} \\ &= \alpha_n\left(T^{\lambda_n}x_n-T^{\lambda_n}x_{n-1}\right)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_ix_n-T_ix_{n-1}) \\ &\quad+(\alpha_{n-1}-\alpha_n)T_nx_{n-1}+(\alpha_{n-1}\lambda_{n-1}-\alpha_n\lambda_n)[(\mu_1A-I+\mu_2(B-\gamma f))x_{n-1}] \\ &\quad+(\alpha_n-\alpha_{n-1})x_{n-1}\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\left(T^{\lambda_n}x_n - T^{\lambda_n}x_{n-1}\right) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_i x_n - T_i x_{n-1}) \\
&\quad + (\alpha_{n-1}-\alpha_n)T_n x_{n-1} + (\alpha_n - \alpha_{n-1})x_{n-1} \\
&\quad + [(\alpha_{n-1}-\alpha_n)\lambda_n + (\lambda_{n-1}-\lambda_n)\alpha_{n-1}]\left[\left(\mu_1 A - I + \mu_2\left(B-\gamma f\right)\right)x_{n-1}\right]
\end{aligned} \tag{3.14}
$$

and hence

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\le \alpha_n\left\|T^{\lambda_n}x_n - T^{\lambda_n}x_{n-1}\right\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_n)\|T_i x_n - T_i x_{n-1}\| \\
&\quad + (\alpha_{n-1}-\alpha_n)\|T_n x_{n-1}\| + (\alpha_{n-1}-\alpha_n)\|x_{n-1}\| \\
&\quad + [(\alpha_{n-1}-\alpha_n)\lambda_n + |\lambda_{n-1}-\lambda_n|\alpha_{n-1}]\left\|\left(\mu_1 A - I + \mu_2\left(B-\gamma f\right)\right)x_{n-1}\right\| \\
&\le \alpha_n(1-\lambda_n\tau)\|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + (\alpha_{n-1}-\alpha_n)\|T_n x_{n-1}\| + (\alpha_{n-1}-\alpha_n)\|x_{n-1}\| \\
&\quad + [(\alpha_{n-1}-\alpha_n) + |\lambda_{n-1}-\lambda_n|]\left\|\left(\mu_1 A - I + \mu_2\left(B-\gamma f\right)\right)x_{n-1}\right\| \\
&\le (1-\alpha_n\lambda_n\tau)\|x_n - x_{n-1}\| + (\alpha_{n-1}-\alpha_n)M + |\lambda_{n-1}-\lambda_n|M,
\end{aligned} \tag{3.15}
$$

where M is a constant. Since $\{\lambda_n\} \subset (0, \min\{1, \|A\|^{-1}/\mu_1\})$, there exists a constant $\lambda' > 0$ such that $\lambda_n \ge \lambda'$ for all $n \ge 1$. Therefore, we have

$$
\|x_{n+1}-x_n\| \le \left(1-\alpha_n\lambda'\tau\right)\|x_n - x_{n-1}\| + [(\alpha_{n-1}-\alpha_n) + |\lambda_{n-1}-\lambda_n|]M. \tag{3.16}
$$

Put $c_n = [(\alpha_{n-1}-\alpha_n)+|\lambda_{n-1}-\lambda_n|]M$. Since $\{\alpha_n\}$ is a strictly decreasing sequence and $\sum_{n=2}^{\infty}|\lambda_{n-1}-\lambda_n| < \infty$, we have $\sum_{n=2}^{\infty} c_n < \infty$. By Lemma 2.3, it follows that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. This completes the proof. □

Lemma 3.4. *If $\{\alpha_n\}$ is strictly decreasing and the following conditions hold:*

$$
\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty \tag{3.17}
$$

then $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$, *for all* $i \ge 1$.

Proof. By the iterative scheme (3.9), we have

$$
x_{n+1} + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(x_n - T_i x_n) - (1-\alpha_n)x_n = \alpha_n T^{\lambda_n}x_n, \tag{3.18}
$$

that is,

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(x_n-T_ix_n)=\alpha_n\left(T^{\lambda_n}x_n-x_n\right)+(x_n-x_{n+1}). \tag{3.19}$$

Hence, for any $p \in C$, we get

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-T_ix_n,x_n-p\rangle=\alpha_n\left\langle T^{\lambda_n}x_n-x_n,x_n-p\right\rangle+\langle x_n-x_{n+1},x_n-p\rangle. \tag{3.20}$$

Since each T_i is nonexpansive, it follows from (2.2) that

$$\|T_ix_n-x_n\|^2\le 2\langle x_n-T_ix_n,x_n-p\rangle. \tag{3.21}$$

Hence, combining (3.21) with (3.20), it follows that

$$\begin{aligned}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_ix_n-x_n\|^2&\le 2\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-T_ix_n,x_n-p\rangle\\&=2\alpha_n\left\langle T^{\lambda_n}x_n-x_n,x_n-p\right\rangle+2\langle x_n-x_{n+1},x_n-p\rangle,\end{aligned} \tag{3.22}$$

which implies that

$$\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_ix_n-x_n\|^2\le\alpha_n\left\langle T^{\lambda_n}x_n-x_n,x_n-p\right\rangle+\langle x_n-x_{n+1},x_n-p\rangle. \tag{3.23}$$

Since each $(\alpha_{i-1}-\alpha_i)\|T_ix_n-x_n\|^2\ge 0$ and $\alpha_{i-1}-\alpha_i>0$, then we have

$$\frac{1}{2}(\alpha_{i-1}-\alpha_i)\|T_ix_n-x_n\|^2\le\alpha_n\left\langle T^{\lambda_n}x_n-x_n,x_n-p\right\rangle+\langle x_n-x_{n+1},x_n-p\rangle, \tag{3.24}$$

that is,

$$\|T_ix_n-x_n\|^2\le\frac{2\alpha_n}{\alpha_{i-1}-\alpha_i}\left\langle T^{\lambda_n}x_n-x_n,x_n-p\right\rangle+\frac{2}{\alpha_{i-1}-\alpha_i}\langle x_n-x_{n+1},x_n-p\rangle. \tag{3.25}$$

Since $\{x_n\}$ and $\{T^{\lambda_n}x_n\}$ are both bounded, there exists a constant $M'>0$ such that

$$\|T_ix_n-x_n\|^2\le M'\left(\frac{\alpha_n}{\alpha_{i-1}-\alpha_i}+\frac{1}{\alpha_{i-1}-\alpha_i}\|x_n-x_{n+1}\|\right). \tag{3.26}$$

By Lemma 3.3 and the assumption condition $\lim_{n\to\infty}\alpha_n=0$, it follows that

$$\lim_{n\to\infty}\|T_ix_n-x_n\|=0,\quad \forall i\ge 1. \tag{3.27}$$

This completes the proof. □

Finally, we give the main result in this paper.

Theorem 3.5. *If $\{\alpha_n\}$ is strictly decreasing and the following conditions hold:*

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| < \infty, \tag{3.28}$$

then the scheme $\{x_n\}$ defined by (3.9) *converges strongly to an element $x^* \in C$ which is the unique solution of the variational inequality* ($\mathrm{VI}(\mu_1 A - I + \mu_2(B-\gamma f), C)$)*:*

$$\left\langle [\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x - x^*\right\rangle \geq 0, \quad \forall x \in C. \tag{3.29}$$

Proof. First, we prove that $\limsup_{n\to\infty}\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x_{n+1} - x^*\rangle \leq 0$.

To prove this, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x_n - x^*\right\rangle = \lim_{i\to\infty}\left\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x_{n_i} - x^*\right\rangle. \tag{3.30}$$

Without loss of generality, we may, further, assume that $x_{n_i} \rightharpoonup \hat{x}$ for some $\hat{x} \in H$. From Lemmas 2.1 and 3.3, it follows that $\hat{x} \in F(T_i)$ for each $i \geq 1$ and so $\hat{x} \in C = \cap_{i=1}^{\infty} F(T_i)$. Since x^* is the unique solution of the problem $\mathrm{VI}(\mu_1 A - I + \mu_2(B-\gamma f), C)$, we obtain

$$\limsup_{n\to\infty}\left\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x_n - x^*\right\rangle = \left\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, \hat{x} - x^*\right\rangle \leq 0. \tag{3.31}$$

It follows from Lemma 2.2 and (3.10) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\| \left[\alpha_n\left(T^{\lambda_n}x_n - T^{\lambda_n}x^*\right) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_n)(T_i x_n - x^*)\right] + \alpha_n\left(T^{\lambda_n}x^* - x^*\right)\right\|^2 \\
&\leq \left\|\alpha_n\left(T^{\lambda_n}x_n - T^{\lambda_n}x^*\right) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_n)(T_i x_n - x^*)\right\|^2 \\
&\quad + 2\alpha_n\left\langle T^{\lambda_n}x^* - x^*, x_{n+1} - x^*\right\rangle \\
&\leq [\alpha_n(1-\lambda_n\tau)\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|]^2 \\
&\quad + 2\alpha_n\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n\lambda_n\tau)\|x_n - x^*\|^2 + 2\alpha_n\left\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x_{n+1} - x^*\right\rangle \\
&\leq (1-\alpha_n\lambda'\tau)\|x_n - x^*\|^2 + 2\alpha_n\left\langle -[\mu_1 A - I + \mu_2(B-\gamma f)]x^*, x_{n+1} - x^*\right\rangle,
\end{aligned} \tag{3.32}$$

where $\lambda' > 0$ is a constant such that $\lambda_n \geq \lambda'$ for all $n \geq 1$. Since $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \langle -[\mu_1 A - I + \mu_2(B - \gamma f)]x^*, x_{n+1} - x^* \rangle \leq 0$, by Lemma 2.3, we conclude that the scheme $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Remark 3.6. (1) For each $n \geq 1$, a simple example on control parameters is $\alpha_n = 1/n$ and $\lambda_n = \lambda$, where λ is a constant in $(0, \min\{1, \|A\|^{-1}/\mu_1\})$.

(2) We obtain the desired results without any assumptions on the family $\{T_i\}_{i=1}^{\infty}$. Foe example, in Theorem CGY2, the authors gave the strong condition (1.10).

Remark 3.7. (1) If $T_1 = T_2 = \cdots = T_n = \cdots = T$ in (3.9), then we have the following iterative scheme:

$$x_{n+1} = \alpha_n \left[(I - \lambda_n \mu_1 A)x_n + \lambda_n \left(x_n - \mu_2 \left(Bx_n - \gamma f(x_n) \right) \right) \right] + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \tag{3.33}$$

and the scheme $\{x_n\}$ defined by (3.33) converges strongly to an element $x^* \in C$ which is the unique solution of the variational inequality $(\mathrm{VI}(\mu_1 A - I + \mu_2(B - \gamma f), C))$:

$$\left\langle \left[\mu_1 A - I + \mu_2 (B - \gamma f) \right] x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in C. \tag{3.34}$$

(2) If $A = I$ and $\mu_1 = 1$ in (3.33), then we have the following iterative scheme:

$$x_{n+1} = \alpha_n \left(I - \lambda_n \mu_2 (B - \gamma f) \right) x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i x_n, \quad \forall n \geq 1, \tag{3.35}$$

and the scheme $\{x_n\}$ defined by (3.35) converges strongly to an element $x^* \in C$ which is the unique solution of the variational inequality $(\mathrm{VI}(B - \gamma f, C))$:

$$\left\langle \left(\mu_2 (B - \gamma f) \right) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in C. \tag{3.36}$$

(3) Furthermore, if $\mu_2 = 1$ and $\gamma = 0$ in (3.35), then we have the following iterative scheme:

$$x_{n+1} = \alpha_n (I - \lambda_n B) x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i x_n, \quad \forall n \geq 1, \tag{3.37}$$

and the scheme $\{x_n\}$ defined by (3.37) converges strongly to an element $x^* \in C$ which is the unique solution of the variational inequality $(\mathrm{VI}(B, C))$, which is Stampacchia's variational inequality:

$$\langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3.38}$$

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