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Research Article

An Iterative Algorithm of Solution for Quadratic Minimization Problem in Hilbert Spaces

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The purpose of this paper is to introduce an iterative algorithm for finding a solution of quadratic minimization problem in the set of fixed points of a nonexpansive mapping and to prove a strong convergence theorem of the solution for quadratic minimization problem. The result of this article improved and extended the result of G. Marino and H. K. Xu and some others.

1. Introduction and Preliminaries

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [1, 2] and the references therein. A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C}\frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.1}$$

where C is the fixed point set of a nonexpansive mapping T defined on H, and u is a given point in H. Let A be a strongly positive operator defined on H, that is, there is a constant $\gamma > 0$ with the property

$$\langle Ax, x\rangle \geq \gamma\|x\|^2, \quad \forall x \in H. \tag{1.2}$$

Then minimization (1.1) has a unique solution $x^* \in C$ which satisfies the optimality condition

$$\langle Ax^* - u, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

In [1, 2] it is proved that the sequence $\{x_n\}$ generated by the following algorithm

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0 \tag{1.4}$$

converges in norm to the solution x^* of (1.1) provided that the sequence $\{\alpha_n\}$ in $(0,1)$ satisfies conditions

$$\lim_{n\to\infty} \alpha_n = 0, \tag{C_1}$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \tag{C_2}$$

and additionally, either the condition

$$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \tag{C_3}$$

or the condition

$$\lim_{n\to\infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}} = 0. \tag{C_4}$$

The purpose of this paper is to introduce the following iterative algorithm:

$$\begin{aligned} x_{n+1} &= (I - \alpha_n A)y_n + \alpha_n u, \\ y_n &= \beta_n x_n + (1 - \beta_n)Tx_n, \end{aligned} \tag{1.5}$$

and to prove that the iterative sequence $\{x_n\}$ defined by (1.5) converges strongly to the solution x^* of (1.1) under the conditions (C_1), (C_2) and $0 < a \leq \beta_n \leq b < 1$ for some constants a, b.

Lemma 1.1 (see [3, 4]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)y_n, \quad n \geq 0, \tag{1.6}$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1. \tag{1.7}$$

Assume that

$$\limsup_{n\to\infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \right) \leq 0. \tag{1.8}$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.2 (see [1]). *Assume that A is a strongly positive linear bounded operator on a real Hilbert space H with coefficient $\gamma > 0$ and $0 < \alpha \le \|A\|^{-1}$. Then $\|I - \alpha A\| \le (1 - \alpha\gamma)$.*

Lemma 1.3 (see [5]). *Let H be a Hilbert space, K a closed convex subset of H, and $T : K \to K$ a nonexpansive mapping with nonempty fixed point set $F(T)$. If $\{x_n\}$ is a sequence in K weakly converging to x and if $x_n - Tx_n$ converges strongly to 0, then $x = Tx$.*

Lemma 1.4 (see [6]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \gamma_n \delta_n, \tag{1.9}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} \gamma_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

Theorem 2.1. *Suppose that A is strongly positive operator with coefficient $\gamma > 0$ as given in* (1.2). *Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions* (C_1), (C_2) *and $0 < a \le \beta_n \le b < 1$ for some constants a, b. Then the sequence $\{x_n\}$ generated by algorithm* (1.5) *converges strongly to the unique solution x^* of the minimization problem* (1.1).

Proof. First we show that $\{x_n\}$ is bounded. As a matter of fact, take $p \in F(T)$ and use Lemma 1.2 to obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|(I - \alpha_n A)(\beta_n x_n + (1-\beta_n)Tx_n - p) + \alpha_n(u - Ap)\| \\ &\le (1 - \gamma\alpha_n)\|x_n - p\| + \alpha_n \|u - Ap\|. \end{aligned} \tag{2.1}$$

By induction we can get

$$\|x_n - p\| \le \max\left\{ \|x_0 - p\|, \frac{1}{\gamma}\|u - Ap\| \right\}, \quad n \ge 0. \tag{2.2}$$

Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$. Next rewrite x_{n+1} in the form

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n z_n, \tag{2.3}$$

where

$$\lambda_n = 1 - (1 - \alpha_n)\beta_n, \tag{2.4}$$

$$z_n = \frac{\alpha_n \beta_n}{\lambda_n}(I - A)x_n + \frac{1 - \beta_n}{\lambda_n}(I - \alpha_n A)Tx_n + \frac{\alpha_n}{\lambda_n}u. \tag{2.5}$$

Since $\alpha_n \to 0$ and $0 < a \le \beta_n \le b < 1$, then

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1. \tag{2.6}$$

Next some manipulations give us that

$$\begin{aligned} z_{n+1} - z_n = {}& \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}(I-A)x_{n+1} - \frac{\beta_n\alpha_n}{\lambda_n}(I-A)x_n + \left(\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\right)u \\ & + \frac{1-\beta_{n+1}}{\lambda_{n+1}}(Tx_{n+1} - Tx_n) - \frac{(1-\beta_{n+1})\alpha_{n+1}}{\lambda_{n+1}}A(Tx_{n+1} - Tx_n) \\ & + \left(\frac{1-\beta_{n+1}}{\lambda_{n+1}} - \frac{1-\beta_n}{\lambda_n}\right)Tx_n - \left(\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\right)(1-\beta_n)ATx_n \\ & - \frac{\alpha_{n+1}}{\lambda_{n+1}}(\beta_n - \beta_{n+1})ATx_n. \end{aligned} \tag{2.7}$$

Therefore,

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le {}& \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}\|(I-A)x_{n+1}\| + \frac{\beta_n\alpha_n}{\lambda_n}\|(I-A)x_n\| + \left|\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\right|\|u\| \\ & + \left(\frac{1-\beta_{n+1}}{\lambda_{n+1}} - 1\right)\|x_{n+1} - x_n\| + \frac{(1-\beta_{n+1})\alpha_{n+1}}{\lambda_{n+1}}\|A\|\|x_{n+1} - x_n\| \\ & + \left|\frac{1-\beta_{n+1}}{\lambda_{n+1}} - \frac{1-\beta_n}{\lambda_n}\right|\|Tx_n\| + \left|\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\right|\|(1-\beta_n)ATx_n\| \\ & + \frac{\alpha_{n+1}}{\lambda_{n+1}}|\beta_n - \beta_{n+1}|\|ATx_n\|. \end{aligned} \tag{2.8}$$

Since $\lambda_n = 1 - (1-\alpha_n)\beta_n$ and $\alpha_n \to 0$, then

$$\lim_{n\to\infty} \frac{1-\beta_n}{\lambda_n} = \lim_{n\to\infty}\left(1 - \frac{\alpha_n\beta_n}{\lambda_n}\right) = 1. \tag{2.9}$$

Then last inequality implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0, \tag{2.10}$$

and so an application of Lemma 1.1 asserts that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{2.11}$$

By (2.5) we have that

$$z_n - Tx_n = \frac{\alpha_n \beta_n}{\lambda_n}(I - A)x_n + \left(\frac{1-\beta_n}{\lambda_n} - 1\right)Tx_n - \frac{(1-\beta_n)\alpha_n}{\lambda_n}ATx_n + \frac{\alpha_n}{\lambda_n}u. \tag{2.12}$$

Again since $\alpha_n \to 0$, $\{x_n\}$ is bounded, and $\lambda_n = 1 - (1 - \alpha_n)\beta_n$, then we deduce from (2.12) that

$$\lim_{n\to\infty} \|z_n - Tx_n\| = 0. \tag{2.13}$$

This together with (2.11) yields

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.14}$$

By using Lemma 1.3, we obtain $\omega_w(x_n) \subset F(T)$, where $\omega_w(x_n) = \{z : \exists \;\; x_{n_k} \rightharpoonup z\}$ is the set of weak ω-limit points of sequence $\{x_n\}$.

Let x^* be the unique solution to the minimization (1.1). Then by the definition of algorithm (1.5), we can write

$$x_{n+1} - x^* = (I - \alpha_n A)\left(\beta_n x_n + \left(1-\beta_n\right)Tx_n - x^*\right) + \alpha_n(u - Ax^*). \tag{2.15}$$

Since H is a Hilbert space, then we have that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \left\|(I - \alpha_n A)(\beta_n x_n + \left(1-\beta_n\right)Tx_n - x^*)\right\|^2 + 2\alpha_n\langle u - Ax^*, x_{n+1} - x^*\rangle \\ &\le \left(1 - \gamma\alpha_n\right)\|x_n - x^*\| + 2\alpha_n\langle u - Ax^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{2.16}$$

However, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - Ax^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle u - Ax^*, x_{n_k} - x^*\rangle, \tag{2.17}$$

and also $\{x_{n_k}\}$ converges weakly to a fixed point $p \in F(T)$. It follows from optimality condition (1.3) that

$$\limsup_{n\to\infty}\langle u - Ax^*, x_n - x^*\rangle = \left\langle u - Ax^*, p - x^*\right\rangle \le 0. \tag{2.18}$$

Therefore, by using Lemma 1.4 and noticing (2.18), we conclude that $x_n \to x^*$. This completes the proof. □

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