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Research Article

An Application of Category Theory to a Class of the Systems of the Superquadratic Wave Equations

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We investigate the nontrivial solutions for a class of the systems of the superquadratic nonlinear wave equations with Dirichlet boundary condition and periodic condition with a superquadratic nonlinear terms at infinity which have continuous derivatives. We approach the variational method and use the critical point theory on the manifold, in terms of the limit relative category of the sublevel subsets of the corresponding functional.

1. Introduction

We investigate the nontrivial solutions for a class of the systems of the superquadratic nonlinear wave equations with Dirichlet boundary condition and periodic condition:

$$\begin{gathered}
u_{tt} - u_{xx} = av + F_u(x,t,u,v) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\
v_{tt} - v_{xx} = bu + F_v(x,t,u,v) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\
u\left(\pm\frac{\pi}{2}, t\right) = v\left(\pm\frac{\pi}{2}, t\right) = 0, \\
u(x,t+\pi) = u(x,t) = u(-x,t) = u(x,-t), \\
v(x,t+\pi) = v(x,t) = v(-x,t) = v(x,-t),
\end{gathered} \tag{1.1}$$

where $F : [-\pi/2, \pi/2] \times R \times R \times R \to R$ is a superquadratic function at infinity which has continuous derivatives $F_r(x,t,r,s)$, $F_s(x,t,r,s)$ with respect to r, s, for almost any

$(x,t) \in (-\pi/2, \pi/2) \times R$. Moreover we assume that F satisfies the following conditions:

(F1) $F(x,t,0,0) = F_x(x,t,0,0) = F_t(x,t,0,0) = 0$, $F_{xx}(x,t,0,0) = F_{tt}(x,t,0,0) = F_{xt}(x,t,0,0) = 0$, $F(x,t,r,s) > 0$ if $(r,s) \neq (0,0)$, $\inf_{(x,t)\in(-\pi/2,\pi/2)\times R, |r|^2+|s|^2=R^2} F(x,t,r,s) > 0$;

(F2) $|F_r(x,t,r,s)| + |F_s(x,t,r,s)| \leq c(|r|^\nu + |s|^\nu)$ for all x,t,r,s;

(F3) $rF_r(x,t,r,s) + sF_s(x,t,r,s) \geq \mu F(x,t,r,s)$ for all x,t,r,s;

(F4) $|F_r(x,t,r,s)| + |F_s(x,t,r,s)| \leq d(F(x,t,r,s)^{\delta_1} + F(x,t,r,s)^{\delta_2})$,

where $c > 0$, $d > 0$, $R > 0$, $\mu > 2$, $\nu > 1$ and $1/2 < \delta_1 \leq \delta_2 \leq 1/r$, for some $1 < r < 2$.

As the physical model for these systems we can find crossing two beams with travelling waves, which are suspended by cable under a load. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression.

Choi and Jung [1–3] investigate the existence and multiplicity of solutions of the single nonlinear wave equation with Dirichlet boundary condition. In [4] the authors show by critical point theory (Linking Theorem) that system (1.1) has at least one nontrivial solution (u,v). In this paper we show by the limit relative category theory that system (1.1) has at least two nontrivial solutions (u,v).

Let us set

$$\mathcal{L}(u,v) = (Lu, Lv), \quad Lu = u_{tt} - u_{xx}. \tag{1.2}$$

Then system (1.1) can be rewritten by

$$\begin{gathered} \mathcal{L}U = \nabla\left(\frac{1}{2}(AU,U) + F(x,t,u,v)\right), \\ U\left(\pm\frac{\pi}{2},t\right) = \begin{pmatrix}0\\0\end{pmatrix}, \\ U(x,t+\pi) = U(x,t) = U(-x,t) = U(x,-t), \end{gathered} \tag{1.3}$$

where ∇ is the gradient operator, $U = \begin{pmatrix}u\\v\end{pmatrix}$, $A = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix} \in M_{2\times 2}(R)$.

We note that $\sqrt{ab}$, $-\sqrt{ab}$ are two eigenvalues of the matrix $A = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix}$.

Let λ_{mn} be the eigenvalues of the eigenvalue problem $u_{tt} - u_{xx} = \lambda u$ in $(-\pi/2, \pi/2) \times R$, $u(\pm\pi/2, t) = 0$, $u(x,t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$.

Our main result is the following.

Theorem 1.1. *Assume that*

$$\lambda_{mn}^2 - ab \neq 0 \quad \forall m,n \text{ with } (m,n) \neq (0,0), \tag{1.4}$$

$$a > 0, \qquad b > 0, \tag{1.5}$$

$$\sqrt{ab} < 1. \tag{1.6}$$

Then, for any F with (F1), (F2), (F3) and (F4), system (1.3) *has at least two nontrivial solutions (u,v).*

In Section 2, we obtain some results on the nonlinear term F. In Section 3, we approach the variational method and recall the abstract results of the critical point theory on the manifold in terms of the limit relative category of the sublevel sets of the corresponding functional of (1.3), which plays a crucial role to prove the multiplicity result. In Section 4, we prove Theorem 1.1.

2. Some Results on the Nonlinear Term F

The eigenvalue problem for $u(x,t)$

$$\begin{gathered} u_{tt} - u_{xx} = \lambda u \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\ u\left(\pm\frac{\pi}{2}, t\right) = 0, \qquad u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t) \end{gathered} \tag{2.1}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \quad (m, n = 0, 1, 2, \ldots) \tag{2.2}$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\begin{gathered} \phi_{0n} = \frac{\sqrt{2}}{\pi} \cos(2n+1)x \quad \text{for } n \geq 0, \\ \phi_{mn} = \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x \quad \text{for } m > 0,\ n \geq 0. \end{gathered} \tag{2.3}$$

Let Q be the square $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ and H_0 the Hilbert space defined by

$$H_0 = \left\{ u \in L^2(Q) \mid u \text{ is even in } x \text{ and } t \text{ and } \int_Q u = 0 \right\}. \tag{2.4}$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, by

$$u = \sum h_{mn} \phi_{mn}. \tag{2.5}$$

We define a Hilbert space $\mathfrak{D}$ as follows:

$$\mathfrak{D} = \left\{ u \in \sum h_{mn} \phi_{mn} : \sum_{mn} \lambda_{mn}^2 h_{mn}^2 < +\infty \right\}. \tag{2.6}$$

Then this space is a Banach space with norm

$$\|u\| = \left[\sum \lambda_{mn}^2 h_{mn}^2\right]^{1/2}. \tag{2.7}$$

Let us set $E = \mathfrak{D} \times \mathfrak{D}$. We endow the Hilbert space E with the norm

$$\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2. \tag{2.8}$$

We are looking for the weak solutions of (1.3) in $\mathfrak{D} \times \mathfrak{D}$, that is, (u,v) such that $u \in \mathfrak{D}$, $v \in \mathfrak{D}$, $Lu = av + F_u(x,t,u,v)$, $Lv = bu + F_v(x,t,u,v)$. Since $|\lambda_{mn}| \geq 1$ for all m, n, we have the following lemma.

We state the lemmas. For the proofs of Lemmas 2.1, 2.2, and 2.3, we refer [4].

Lemma 2.1. (i)$\|u\| \geq \|u\|_{L^2(Q)}$, *where* $\|u\|_{L^2(Q)}$ *denotes the* L^2 *norm of* u.

(ii)$\|u\| = 0$ *if and only if* $\|u\|_{L^2(Q)} = 0$.

(iii)$u_{tt} - u_{xx} \in \mathfrak{D}$ *implies* $u \in \mathfrak{D}$.

Lemma 2.2. *Suppose that c is not an eigenvalue of $L : \mathfrak{D} \to H_0$, $Lu = u_{tt} - u_{xx}$, and let $f \in H_0$. Then one has $(L-c)^{-1} f \in \mathfrak{D}$.*

By (F1) and (F3), we obtain the lower bound for $F(x,t,u,v)$ in the term of $|u|^\mu + |v|^\mu$.

Lemma 2.3. *Assume that F satisfies the conditions (F1) and (F3). Then there exist $a_0, b_0 \in R$ with $a_0 > 0$ such that*

$$F(x,t,r,s) \geq a_0\left(|r|^\mu + |s|^\mu\right) - b_0, \quad \forall x,t,r,s. \tag{2.9}$$

Lemma 2.4. *Assume that F satisfies the conditions (F1), (F2), and (F3). Then*

(i) $\int_Q F(x,t,0,0)dxdt = 0$, $\int_Q F(x,t,u,v)dxdt > 0$ *if* $(u,v) \neq (0,0)$, $\mathrm{grad}(\int_Q F(x,t,u,v))dxdt = o(\|(u,v)\|_E)$ *as* $(u,v) \to (0,0)$;

(ii) *there exist* $a_0 > 0$, $\mu > 2$ *and* $b_1 \in R$ *such that*

$$\int_Q F(x,t,u,v)dxdt \geq a_0\|(u,v)\|_{L^\mu}^\mu - b_1 \quad \forall (u,v) \in E; \tag{2.10}$$

(iii) $(u,v) \to \mathrm{grad}(\int_Q F(x,t,u\cdot v))dxdt$ *is a compact map;*

(iv) *if* $\int_Q [uF_u(x,t,u,v) + vF_v(x,t,u,v)]dxdt - 2\int_Q F(x,t,u,v)dxdt = 0$, *then* $\mathrm{grad}(\int_Q F(x,t,u,v)dxdt) = 0$;

(v) *if* $\|(u_n,v_n)\|_E \to +\infty$ *and*

$$\frac{\int_Q [u_n F_u(x,t,u_n,v_n) + v_n F_v(x,t,u_n,v_n)]dxdt - 2\int_Q F(x,t,u_n,v_n)dxdt}{\|(u,v)\|_E} \longrightarrow 0, \tag{2.11}$$

then there exists $((u_{h_n}, v_{h_n}))_n$ *and* $w \in E$ *such that*

$$\frac{\operatorname{grad}\left(\int_Q F(x,y,u_n,v_n)\,dxdt\right)}{\|(u_{h_n}, v_{h_n})\|_E} \longrightarrow w, \qquad \frac{(u_{h_n}, v_{h_n})}{\|(u_{h_n}, v_{h_n})\|_E} \rightharpoonup (0,0). \tag{2.12}$$

Proof. (i) Follows from (F1) and (F2), since $1 < \nu$.

(ii) By Lemma 2.3, for $U = (u,v) \in E$,

$$\int_Q F(x,t,U)\,dxdt \geq a_0 \|U\|_{L^\mu}^{\mu}\,dxdt - b_1, \tag{2.13}$$

where $b_1 \in R$. Thus (ii) holds.

(iii) Is easily obtained with standard arguments.

(iv) Is implied by (F3) and the fact that $F(x,t,u,v) > 0$ for $(u,v) \neq (0,0)$.

(v) By Lemma 2.3 and (F3), for $U = (u,v)$,

$$\begin{aligned}&\int_Q [uF_u(x,t,u,v) + vF_v(x,t,u,v)]\,dxdt - 2\int_Q F(x,t,u,v)\,dxdt\\ &\quad\geq (\mu-2)\int_Q F(x,t,u,v)\,dxd \geq (\mu-2)\left(a_0\|U\|_{L^\mu}^{\mu} - b_1\right).\end{aligned} \tag{2.14}$$

By (F2),

$$\left\|\operatorname{grad}\left(\int_Q F(x,t,u,v)\,dxdt\right)\right\|_E \leq C'\|F_U(x,t,U)\|_{L^r} \leq C''\||U|^\nu\|_{L^r}, \quad \text{for some } 1 < r < 2 \tag{2.15}$$

and suitable constants C', C''. To get the conclusion it suffices to estimate $\||U|^\nu/\|U\|_E\|_{L^r}$ in terms of $\|U\|_{L^\mu}^{\mu}/\|U\|_E$. If $\mu \geq r\nu$, then this is a consequence of *Hölder* inequality. If $\mu < r\nu$, by the standard interpolation arguments, it follows that $\||U|^\nu/\|U\|_E\|_{L^r} \leq C(\|U\|_{L^\mu}^{\mu}/\|U\|_E)^{\nu/\mu}\|U\|_E^{l}$, where l is such that $l = -1 + \nu/\mu$. Thus we prove (v). □

Lemma 2.5. *Assume that F satisfies the conditions (F1), (F2), (F3), and (F4). Then there exist* ϕ, $\psi : [0,+\infty] \to R$ *continuous and such that*

$$\frac{\psi(s)}{s} \longrightarrow 0 \quad \text{as } s \longrightarrow 0, \qquad \phi(s) > 0 \quad \text{if } s > 0, \tag{2.16}$$

(i) $\|\operatorname{grad}\int_Q F(x,t,u,v)\,dxdt\|_E^2 \leq \psi(\int_Q F(x,t,u,v)\,dxdt)$, *for all* $(u,v) \in E$,

(ii) $\int_Q [uF_u(x,t,u,v) + vF_v(x,t\cdot u,v)]\,dxdt - 2\int_Q F(x,t,u,v)\,dxdt \geq \phi(u,v)$, *for all* $(u,v) \in E$.

Proof. (i) By (F4), for all $U = (u, v) \in E$,

$$\begin{aligned}
&\left\| \operatorname{grad}\left(\int_Q F(x,t,U)dxdt \right) \right\|_E \\
&\quad \le \| F_U(x,t,U) \|_{L^r} \\
&\quad \le C_1 \left\| F(x,t,U)^{\delta_1} + F(x,t,U)^{\delta_2} \right\|_{L^r} \\
&\quad \le C_2 \left(\left\| F(x,t,U)^{\delta_1} \right\|_{L^r} + \left\| F(x,t,U)^{\delta_2} \right\|_{L^r} \right) \\
&\quad \le C_3 \left(\left\| F(x,t,U)^{\delta_1} \right\|_{L^{1/\delta_1}} + \left\| F(x,t,U)^{\delta_2} \right\|_{L^{1/\delta_2}} \right) \\
&\quad \le C_4 \left(\| F(x,t,U) \|_{L^1}^{\delta_1} + \| F(x,t,U) \|_{L^1}^{\delta_2} \right) \\
&\quad = C_5 \left(\left(\int_Q F(x,t,U)dxdt \right)^{\delta_1} + \left(\int_Q F(x,t,U)dxdt \right)^{\delta_2} \right),
\end{aligned} \tag{2.17}$$

where $1 < r < 1/\delta_1, 1/\delta_2 < 2$, C_1, C_2, C_3, C_4, and C_5 are constants. Since $\delta_1, \delta_2 > 1/2$, (i) follows.

(ii) By (F3),

$$\begin{aligned}
&\int_Q [uF_u(x,t,u,v) + vF_v(x,t\cdot u,v)]dxdt - 2\int_Q F(x,t,u,v)dxdt \\
&\quad \ge (\mu - 2)\int_Q F(x,t,U)dxd \ge (\mu - 2)\left(a_0 \|U\|_{L^\mu}^{\mu} - b_1 \right).
\end{aligned} \tag{2.18}$$

Thus (ii) follows. □

3. Abstract Results of Critical Point Theory

Now we are looking for the weak solutions of system (1.3). We shall approach the variational method and recall the abstract results of the critical point theory on the manifold in terms of the limit relative category of the sublevel sets of the functional of (1.3). We observe that the weak solutions of (1.3) coincide with the critical points of the corresponding functional:

$$I : E \longrightarrow R \in C^{1,1}, \tag{3.1}$$

$$I(U) = \frac{1}{2}\int_Q \mathcal{L}U \cdot U\, dxdt - \frac{1}{2}\int_Q (AU, U)_{R^2} dxdt - \int_Q F(x,t,u,v)dxdt. \tag{3.2}$$

Now we recall the critical point theory for strongly indefinite functional. Since the functional I is strongly indefinite functional, it is convenient to use $(P.S.)_c^*$ condition and the limit relative category which is a suitable version of $(P.S.)_c$ condition and the relative category, respectively.

Now, we consider the critical point theory on the manifold with boundary. Let E be a Hilbert space and let M be the closure of an open subset of E such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for f on M. We recall the following notions: lower gradient of f on M, $(P.S.)_c^*$ condition, and the limit relative category (see [4]).

Definition 3.1. If $u \in M$, the lower gradient of f on M at u is defined by

$$\operatorname{grad}_M^- f(u) = \begin{cases} \nabla f(u) & \text{if } u \in \operatorname{int}(M), \\ \nabla f(u) + \left[\langle \nabla f(u), \nu(u)\rangle\right]^- \nu(u) & \text{if } u \in \partial M, \end{cases} \tag{3.3}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards. We say that u is a lower critical for f on M, if $\operatorname{grad}_M^- f(u) = 0$.

Since the functional $I(u)$ (which is introduced in Section 4) is strongly indefinite, the notion of the $(P.S.)_c^*$ condition and the limit relative category is a very useful tool for the proof of the main theorems.

Let E^-, E^0, E^+ be the subspace of E on which the functional $U \mapsto (1/2)\int_Q \mathcal{L}U \cdot U$ is positive definite, null, negative definite, and E^-, E^0, and E^+ are mutually orthogonal. Let P^+ be the projection for E onto E^+, P^0 the one from E onto E^0, and P^- the one from E onto E^-. Let $(E_n)_n$ be a sequence of closed subspaces of E with the conditions:

$$E_n = E_n^- \oplus E^0 \oplus E_n^+, \quad \text{where } E_n^+ \subset E^+,\ E_n^- \subset E^-\ \forall n \tag{3.4}$$

(E_n^+ and E_n^- are subspaces of E), $\dim E_n < +\infty$, $E_n \subset E_{n+1}$, $\bigcup_{n\in N} E_n$ is dense in E.

Let P_{E_n} be the orthogonal projections from E onto E_n. $M_n = M \cap E_n$, for any n, and let be the closure of an open subset of E_n and have the structure of a C^2 manifold with boundary in E_n. We assume that for any n there exists a retraction $r_n : M \to M_n$. For given $B \subset E$, we will write $B_n = B \cap E_n$.

Definition 3.2. Let $c \in R$. We say that f satisfies the $(P.S.)_c^*$ condition with respect to $(M_n)_n$, on the manifold with boundary M, if for any sequence $(k_n)_n$ in N and any sequence $(u_n)_n$ in M such that $k_n \to \infty$, $u_n \in M_{k_n}$, for all n, $f(u_n) \to c$, $\operatorname{grad}_{M_{k_n}}^- f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $\operatorname{grad}_M^- f(u) = 0$.

Let Y be a closed subspace of M.

Definition 3.3. Let B be a closed subset of M with $Y \subset B$. We define the relative category $\operatorname{cat}_{M,Y}(B)$ of B in (M, Y), as the least integer h such that there exist $h + 1$ closed subsets U_0, $U_1, \dots, U_h$ with the following properties:

$B \subset U_0 \cup U_1 \cup \cdots \cup U_h$;

$U_1, \dots, U_h$ are contractible in M;

$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$F(x, 0) = x \quad \forall x \in U_0,$$

$$\begin{aligned} &F(x,t) \in Y \quad \forall x \in Y,\ \forall t \in [0,1], \\ &F(x,1) \in Y \quad \forall x \in U_0. \end{aligned} \tag{3.5}$$

If such an h does not exist, we say that $\mathrm{cat}_{M,Y}(B) = +\infty$.

Definition 3.4. Let (X,Y) be a topological pair and let $(X_n)_n$ be a sequence of subsets of X. For any subset B of X we define the limit relative category of B in (X,Y), with respect to $(X_n)_n$, by

$$\mathrm{cat}^*_{(X,Y)}(B) = \limsup_{n\to\infty} \mathrm{cat}_{(X_n,Y_n)}(B_n). \tag{3.6}$$

Now we consider a theorem which gives an estimate of the number of critical points of a functional, in terms of the limit relative category of its sublevels. The theorem is proved repeating the classical arguments, using the nonsmooth version of the classical Deformation Lemma for functions on manifolds with boundary.

Let Y be a fixed subset of M. We set

$$\begin{aligned} \mathcal{B}_i &= \left\{B \subset M \mid \mathrm{cat}^*_{(M,Y)}(B) \geq i\right\}, \\ c_i &= \inf_{B\in\mathcal{B}_i} \sup_{x\in B} f(x). \end{aligned} \tag{3.7}$$

We have the following multiplicity theorem.

Theorem 3.5. *Let $i \in N$ and assume that*

(1) $c_i < +\infty$,

(2) $\sup_{x\in Y} f(x) < c_i$,

(3) *the $(P.S.)^*_{c_i}$ condition with respect to $(M_n)_n$ holds.*

Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \cdots = c_{i+k-1} = c, \tag{3.8}$$

then

$$\mathrm{cat}_M\left(\left\{x \in M \mid f(x) = c, \mathrm{grad}^-_M f(x) = 0\right\}\right) \geq k. \tag{3.9}$$

Proof. Let $c = c_i$; using the $(P.S.)^*_c$ condition, with respect to $(M_n)_n$, one can prove that, for any neighbourhood N of

$$K_c = \left\{x \mid f(x) = c, \mathrm{grad}^-_M f(x) = 0\right\}, \tag{3.10}$$

there exist n_0 in N and $\delta > 0$ such that $\|\mathrm{grad}_{\overline{M}}^{-}\| \geq \delta$ for all $n \geq n_0$ and all $x \in E_n \setminus N$ with $c - \delta \leq f(x) \leq c + \delta$. Moreover it is not difficult to see that, for all n, the function $\widetilde{f}_n : E_n \to R \cup \{+\infty\}$ defined by $\widetilde{f}_n = f(x)$, if $x \in M_n$, $\widetilde{f}_n(x) = +\infty$, otherwise, is ϕ-convex of order two, according to the definitions of [3]. Then the conclusion follows using the same arguments of [4, 5] and the nonsmooth version of the classical Deformation Lemma. □

Lemma 3.6 (Deformation Lemma). *Let $h : H \to R \cup \{+\infty\}$ be a lower semicontinuous function and assume h to be ϕ-convex of order 2 (see [3]). Let $c \in R$, $\delta > 0$, and D be a closed set in H such that*

$$\inf\left\{ \left\|\mathrm{grad}_{\overline{M}} h(x)\right\| \mid c - \delta \leq h(x) \leq c + \delta,\ \mathrm{dist}(x, D) < \delta \right\} > 0. \tag{3.11}$$

Then there exists $\epsilon > 0$ and a continuous deformation $\eta : h^{c+\epsilon} \cap D \times [0,1] \to h^{c+\epsilon} \cap D_\delta$ (D_δ is the δ-neighborhood of D and $h^c = \{x \mid h(x) \leq 0\}$) such that

(i) $\eta(x,0) = x$ *for all* $x \in h^{c+\epsilon} \cap D$,

(ii) $\eta(x,t) = x$ *for all* $x \in h^{c-\epsilon} \cap D$, *for all* $t \in [0,1]$,

(iii) $\eta(x,1) \in h^{c-\epsilon}$ *for all* $x \in h^{c+\epsilon} \cap D$, *for all* $t \in [0,1]$.

Proof. See [6, Lemmas 4.5 and 4.6]. □

Now we state the following multiplicity result (for the proof see [7, Theorem 4.6]) which will be used in the proofs of our main theorems.

Theorem 3.7. *Let H be a Hilbert space and let $H = X_1 \oplus X_2 \oplus X_3$, where X_1, X_2, X_3 are three closed subspaces of H with X_1, X_2 of finite dimension. For a given subspace X of H, let P_X be the orthogonal projection from H onto X. Set*

$$C = \{x \in H \mid \|P_{X_2} x\| \geq 1\}, \tag{3.12}$$

and let $f : W \to R$ be a $C^{1,1}$ function defined on a neighborhood W of C. Let $1 < \rho < R$, $R_1 > 0$. We define

$$\begin{aligned}
\Delta &= \{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \|x_1\| \leq R_1,\ 1 \leq \|x_2\| \leq R\},\\
\Sigma &= \{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \|x_1\| \leq R_1,\ \|x_2\| = 1\}\\
&\quad \cup \{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \|x_1\| \leq R_1,\ \|x_2\| = R\}\\
&\quad \cup \{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2,\ \|x_1\| = R_1,\ 1 \leq \|x_2\| \leq R\},\\
S &= \left\{x \in X_2 \oplus X_3 \mid \|x\| = \rho\right\},\\
B &= \left\{x \in X_2 \oplus X_3 \mid \|x\| \leq \rho\right\}.
\end{aligned} \tag{3.13}$$

Assume that

$$\sup f(\Sigma) < \inf f(S) \tag{3.14}$$

and that the $(P.S.)_c$ *condition holds for* f *on* C*, with respect to the sequence* $(C_n)_n$*, for all* $c \in [\alpha, \beta]$*, where*

$$\alpha = \inf f(S), \qquad \beta = \sup f(\Delta). \tag{3.15}$$

Moreover one assumes $\beta < +\infty$ *and* $f|_{X_1 \oplus X_3}$ *has no critical points* z *in* $X_1 \oplus X_3$ *with* $\alpha \leq f(z) \leq \beta$*. Then there exist two lower critical points* z_1*,* z_2 *for* f *on* C *such that* $\alpha \leq f(z_i) \leq \beta$*,* $i = 1.2$*.*

4. Proof of Theorem 1.1

Let $I^{1,1}_{\mathrm{loc}}(E, R)$ be the functional defined in (3.2). Let Y be a closed subspace of E^+ with finite dimension. Let us set

$$X_1 = E^- \oplus E^0, \qquad X_2 = Y, \qquad X_3 = (X_1 \oplus X_2)^{\perp} (\subset E^+). \tag{4.1}$$

Then E is the topological direct sum of the subspaces X_1, X_2, and X_3. Let P_X be the orthogonal projection from E onto X. Let us set

$$C = \{U \in H \mid \|P_{X_2} U\| \geq 1\}. \tag{4.2}$$

Then C is the smooth manifold with boundary. Let $C_n = C \cap E_n$. Let us define a functional $\Psi : E \setminus \{X_1 \oplus X_3\} \to E$ by

$$\Psi(U) = U - \frac{P_{X_2} U}{\|P_{X_2} U\|} = P_{X_1 \oplus X_3} U + \left(1 - \frac{1}{\|P_{X_2} U\|}\right) P_{X_2} U. \tag{4.3}$$

We have

$$\nabla \Psi(U)(V) = V - \frac{1}{\|P_{X_2} U\|} \left(P_{X_2} V - \left\langle \frac{P_{X_2} U}{\|P_{X_2} U\|}, V \right\rangle \frac{P_{X_2} U}{\|P_{X_2} U\|} \right). \tag{4.4}$$

Let us define the constrained functional $\widetilde{I} : C \to R$ by

$$\widetilde{I} = I \circ \Psi. \tag{4.5}$$

Then $\widetilde{I} \in C^{1,1}_{\mathrm{loc}}$. It turns out that

$$\operatorname{grad}_C^- \widetilde{I}\left(\widetilde{Z}\right) = \begin{cases} P_{X_1 \oplus X_3} \nabla I(Z) + \left(1 - \dfrac{1}{\left\|P_{X_2} \widetilde{Z}\right\|_E}\right) P_{X_2} \nabla I(Z) & \text{if } Z \in \operatorname{int}(C), \\ P_{X_1 \oplus X_3} \nabla I(Z) - \left\langle \nabla I(Z), \dfrac{P_{X_2} \widetilde{Z}}{\left\|P_{X_2} \widetilde{Z}\right\|_E} \right\rangle^+ \dfrac{P_{X_2} \widetilde{Z}}{\left\|P_{X_2} \widetilde{Z}\right\|_E} & \text{if } Z \in \partial C. \end{cases} \tag{4.6}$$

We note that if $\widetilde{U}$ is the critical point of $\widetilde{I}$ and lies in the interior of C, then $U = \Psi(\widetilde{U})$ is the critical point of I. Thus it suffices to find the critical points, which lies in the interior of C, for $\widetilde{I}$. We also note that

$$\left\|\operatorname{grad}_C^- \widetilde{I}\left(\widetilde{U}\right)\right\|_E \geq \left\|P_{X_1 \oplus X_3} \nabla I\left(\Psi\left(\widetilde{U}\right)\right)\right\|_E, \quad \forall \widetilde{U} \in \partial C. \tag{4.7}$$

Let us set

$$\begin{aligned}
S_{23}(\rho) &= \left\{U \in X_2 \oplus X_3 \mid \|U\|_E = \rho\right\}, \quad \rho > 0, \\
\widetilde{S_{23}(\rho)} &= \Psi^{-1}(S_{23}(\rho)), \\
\Delta_{12}(R, R_1) &= \{U_1 + U_2 \mid U_1 \in X_1, U_2 \in X_2, \|U_1\|_E \leq R_1, 1 \leq \|U_2\|_E \leq R\}, \\
\Delta_{12}\widetilde{(R, R_1)} &= \Psi^{-1}(\Delta_{12}(R, R_1)), \\
\Sigma_{12}(R, R_1) &= \{U_1 + U_2 \mid U_1 \in X_1, U_2 \in X_2, \|U_1\|_E \leq R_1, \|U_2\|_E = 1\} \\
&\quad \cup \{U_1 + U_2 \mid U_1 \in X_1, U_2 \in X_2, \|U_1\|_E \leq R_1, \|U_2\|_E = R\} \\
&\quad \cup \{U_1 + U_2 \mid U_1 \in X_1, U_2 \in X_2, \|U_1\|_E = R_1, 1 \leq \|U_2\|_E \leq R\}, \\
\Sigma_{12}\widetilde{(R, R_1)} &= \Psi^{-1}(\Sigma_{12}(R, R_1)).
\end{aligned} \tag{4.8}$$

We will prove the multiplicity result by using Theorem 3.7 for $\widetilde{I}$, C, $\widetilde{S_{23}(\rho)}$, $\Delta_{12}\widetilde{(R, R_1)}$, and $\Sigma_{12}\widetilde{(R, R_1)}$. Now we have the following linking geometry for $\widetilde{I}$.

Lemma 4.1. *Assume that the conditions* (1.4), (1.5), *and* (1.6) *hold. Then there exist* $R > \rho > 0$, $R_1 > 0$, *and* $R > 1$ *such that*

$$\sup_{\widetilde{V} \in \Sigma_{12}\widetilde{(R, R_1)}} \widetilde{I}\left(\widetilde{V}\right) < \inf_{\widetilde{W} \in \widetilde{S_{23}(\rho)}} \widetilde{I}\left(\widetilde{W}\right) \tag{4.9}$$

Proof. It suffices to show that there exist $R > \rho > 0$, $R_1 > 0$ and $R > 1$ such that for $V = \psi(\widetilde{V})$, $W = \psi(\widetilde{W})$,

$$\sup_{V \in \Sigma_{12}(R, R_1)} I(V) < \inf_{W \in S_{23}(\rho)} I(W), \tag{4.10}$$

because

$$\sup_{\widetilde{V} \in \Sigma_{12}\widetilde{(R, R_1)}} \widetilde{I}\left(\widetilde{V}\right) = \sup_{V \in \Sigma_{12}(R, R_1)} I(V), \qquad \inf_{\widetilde{W} \in \widetilde{S_{23}(\rho)}} \widetilde{I}\left(\widetilde{W}\right) = \inf_{W \in S_{23}(\rho)} I(W). \tag{4.11}$$

By (1.6) and (i) of Lemma 2.4, we can find a small number ρ such that, for $U \in E^+$,

$$I(U) = \frac{1}{2}\int_Q \mathcal{L}U \cdot U - \frac{1}{2}\int_Q (AU,U)_{R^2} - \int_Q F(x,t,u,v)dxdt \geq \frac{1}{2}\left(1 - \frac{\sqrt{ab}}{\lambda_{00}}\right)\|U\|_E^2 - 0(\|U\|_E). \tag{4.12}$$

Since $\sqrt{ab} < 1 = \lambda_{00}$, there exist a small number $\rho > 0$ and a small sphere $S_{23}(\rho) \subset E^+$ with radius ρ such that if $U \in S_{23}(\rho) \subset E^+$, then $\inf I(U) > 0$.

Next we will show that there exist $R > \rho$, $R_1 > 0$, and $R > 1$ such that $\sup_{V\in\Sigma_{12}(R,R_1)} I(V) < 0$. Let $U(\neq (0,0)) \in E^0 \oplus E^- \oplus Y$. We note that

$$\begin{aligned} &\text{if } U \in E^+, \text{ then } \int_Q (\mathcal{L}U \cdot U - (AU,U)_{R^2})dxdt \geq \tau_1\|U\|_E^2, \\ &\text{if } U \in E^-, \text{ then } \int_Q (\mathcal{L}U \cdot U - (AU,U)_{R^2})dxdt \leq -\tau_2\|U\|_E^2 \end{aligned} \tag{4.13}$$

for some $\tau_1 > 0$, $\tau_2 > 0$. Let us choose a sequence $(U_n)_n$, $U_n = (u_n, v_n)$ such that $\|U_n\|_E \to \infty$. Let us set $\check{U}_n = U_n/\|U_n\|_E$. By Lemma 2.3, we have that

$$\frac{I(U_n)}{\|U_n\|_E^2} \leq \|\mathcal{L} - A\| \left\|P^+\check{U}_n\right\|_E^2 - a_0\left\|\check{U}_n\right\|_{L^\mu}^\mu \|U_n\|_E^{\mu-2} + \frac{b_0}{\|U_n\|^2} - \tau_2\left\|P^-\check{U}_n\right\|_E^2. \tag{4.14}$$

Since $\|U_n\|_E \to \infty$, two possible cases arise. For the case $\|\check{U}_n\|_{L^\mu} \to 0$ it follows that $\check{U}_n \rightharpoonup 0$, and hence $P^+\check{U}_n \to 0$ and $P^0\check{U}_n \to 0$. Then $\|P^-\check{U}_n\|_E \to 1$. Then

$$\limsup_{n\to\infty} \frac{I(U_n)}{\|U_n\|_E^2} \leq -\tau_2. \tag{4.15}$$

For the case $\|\check{U}_n\|_{L^\mu} \geq \epsilon > 0$ (4.14) implies

$$\lim_{n\to\infty} \frac{I(U_n)}{\|U_n\|_E^2} = -\infty. \tag{4.16}$$

In any case

$$\lim \sup_{\|U\|\to\infty \top U\in E^0\oplus E^-\oplus Y} \frac{I(U)}{\|U\|^2} < 0. \tag{4.17}$$

Thus we can choose a large number $R > \rho > 0$, $R_1 > 0$, and $R > 1$ such that if $U \in \Sigma_{12}(R,R_1)$, then $\sup I(U) \leq 0$. Thus $\sup_{V\in\Sigma_{12}(R,R_1)} I(V) < \inf_{W\in S_{23}(\rho)} I(W)$. So the lemma is proved. □

Lemma 4.2. *Assume that the conditions* (1.4), (1.5), *and* (1.6) *hold. Then $\tilde{I}$ has no critical point $\tilde{U}$ such that $\tilde{I}(\tilde{U}) = c > 0$ and $\tilde{U} \in \partial C$.*

Proof. It suffices to prove that I has no critical point $U = \psi(\widetilde{U})$ such that $I(U) = c$ and $U \in X_1 \oplus X_3$. We notice that from Lemma 4.1, for fixed $U_1 \in X_1$, the functional $U_3 \mapsto I(U_1 + U_3)$ is weakly convex in X_3, while, for fixed $U_3 \in X_3$, the functional $U_1 \mapsto I(U_1 + U_3)$ is strictly concave in X_1. Moreover $(0,0)$ is a critical point in $X_1 \oplus X_3$ with $I(0,0) = 0$. So if $U = U_1 + U_3$ is another critical point for $I|_{X_1 \oplus X_3}$, then we have

$$0 = I(0,0) \leq I(U_3) \leq I(U_1 + U_3) \leq I(U_1) \leq I(0,0) = 0. \tag{4.18}$$

So $I(U_1 + U_3) = I(0,0) = 0$. □

We shall prove that the functional $\widetilde{I}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$ for any $c \in [\alpha, \beta]$, where $\alpha = \inf_{\widetilde{W} \in \widetilde{S_{23}(\rho)}} \widetilde{I}(\widetilde{W})$ and $\beta = \sup_{\widetilde{V} \in \widetilde{\Delta_{12}(R,R_1)}} \widetilde{I}(\widetilde{V})$.

To prove that $\widetilde{I}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$ for any $c \in [\alpha, \beta]$, we first shall prove that I satisfies the $(P.S.)^*_c$ condition with respect to $(E_n)_n$ for any real number $c \in R$.

Lemma 4.3. *Assume that the conditions* (1.4), (1.5), *and* (1.6) *hold. Then, for any F with (F1), (F2), (F3), and (F4), the functional I satisfies the $(P.S.)^*_c$ condition with respect to $(E_n)_n$ for any real number $c \in R$.*

Proof. Let $c \in R$ and (h_n) be a sequence in N such that $h_n \to +\infty$, and let $(U_n)_n$ be a sequence such that

$$U_n = (u_n, v_n) \in E_{h_n}, \quad \forall n, \quad I(U_n) \longrightarrow c, \quad P_{E_{h_n}} \nabla I(U_n) \longrightarrow 0. \tag{4.19}$$

We claim that $(U_n)_n$ is bounded. By contradiction we suppose that $\|U_n\|_E \to +\infty$ and set $\widehat{U}_n = U_n / \|U_n\|_E$. Then

$$\begin{aligned}\left\langle P_{E_{h_n}} \nabla I(U_n), \widehat{U}_n \right\rangle &= \left\langle \nabla I(U_n), \widehat{U}_n \right\rangle \\ &= 2\frac{I(U_n)}{\|U_n\|_E} - \frac{\int_Q \nabla F(x,t,U_n) \cdot U_n dxdt - 2\int_Q F(x,t,U_n) dxdt}{\|U_n\|_E} \longrightarrow 0.\end{aligned} \tag{4.20}$$

Hence

$$\frac{\int_Q \nabla F(x,t,U_n) \cdot U_n dxdt - 2\int_Q F(x,t,U_n) dxdt}{\|U_n\|_E} \longrightarrow 0. \tag{4.21}$$

By (v) of Lemma 2.4,

$$\frac{\operatorname{grad} \int_Q F(x,t,U_n) dxdt}{\|U_n\|_E} \text{ converges} \tag{4.22}$$

and $\widehat{U}_n \rightharpoonup 0$. We get

$$\frac{P_{E_{h_n}} \nabla I(U_n)}{\|U_n\|_E} = P_{E_{h_n}} \mathcal{L}\widehat{U}_n - A\widehat{U}_n - \frac{P_{E_{h_n}} \operatorname{grad}\left(\int_Q F(x,t,U_n)dxdt\right)}{\|U_n\|_E} \longrightarrow 0, \tag{4.23}$$

and so $(P_{E_{h_n}} \mathcal{L}\widehat{U}_n - A\widehat{U}_n)_n$ converges. Since $(\widehat{U}_n)_n$ is bounded and $\mathcal{L} - A$ is a compact mapping, up to subsequence, $(\widehat{U}_n)_n$ has a limit. Since $\widehat{U}_n \rightharpoonup (0,0)$, we get $\widehat{U}_n \to (0,0)$, which is a contradiction to the fact that $\|\widehat{U}_n\|_E = 1$. Thus $(U_n)_n$ is bounded. We can now suppose that $U_n \rightharpoonup U$ for some $U \in E$. Since the mapping $U \mapsto \operatorname{grad}(\int_Q F(x,t,U)dxdt)$ is a compact mapping, $\operatorname{grad}(\int_Q F(x,t,U_n)dxdt) \to \operatorname{grad}(\int_Q F(x,t,u,v)dxdt)$. Thus $(P_{E_{h_n}}(\mathcal{L}U_n - AU_n))_n$ converges. Since $\mathcal{L} - A$ is a compact operator and $(U_n)_n$ is bounded, we deduce that, up to a subsequence, $(U_n)_n$ converges to some U strongly with $\nabla I(U) = \lim \nabla I(U_n) = 0$. Thus we prove the lemma. □

Lemma 4.4. *Assume that the conditions* (1.4), (1.5), *and* (1.6) *hold. Then the functional $\widetilde{I}$ satisfies the $(P.S.)_c^*$ condition with respect to $(C_n)_n$ for any $c \in [\alpha, \beta]$.*

Proof. Let $(h_n)_n$ be a sequence in N with $h_n \to +\infty$ and let $(\widetilde{Z_n})_n$ be a sequence in C with $\widetilde{Z_n} \in C_{h_n}$ for all n, $\widetilde{I}(\widetilde{Z_n}) \to c$ and $\operatorname{grad}^-_{C_{h_n}} \widetilde{I}|_{E_{h_n}}(\widetilde{Z_n}) \to 0$. Set $Z_n = \Psi(\widetilde{Z_n})$. Then $I(Z_n) \to c$. We first consider the case $\widetilde{Z_n} \notin \partial C_{h_n}$ for large n. Since for large n $P_{E_n} \circ P_{X_2} = P_{X_2} \circ P_{E_n} = P_{X_2}$, we have, by (4.6),

$$\begin{aligned}
\operatorname{grad}^-_{C_{h_n}} \widetilde{I}\left(\widetilde{Z_n}\right) &= P_{E_{h_n}} \Psi'\left(\widetilde{Z_n}\right) \nabla I(Z_n) = \Psi'\left(\widetilde{Z_n}\right)(P_{E_{h_n}} \nabla I(Z_n)) \\
&= P_{E_{h_n}} P_{X_1 \oplus X_3} \nabla I(Z_n) + P_{E_{h_n}} \left(1 - \frac{1}{\left\|P_{X_2}\widetilde{Z_n}\right\|_E}\right) P_{X_2} \nabla I(Z_n) \longrightarrow 0,
\end{aligned} \tag{4.24}$$

thus

$$P_{X_1 \oplus X_3} P_{E_{h_n}} \nabla I(Z_n) \longrightarrow 0, \qquad \left(1 - \frac{1}{\left\|P_{X_2}\widetilde{Z_n}\right\|_E}\right) P_{X_2} \nabla I(Z_n) \longrightarrow 0. \tag{4.25}$$

It is impossible that $\|P_{X_2}\widetilde{Z_n}\|_E \to 1$ because $\operatorname{dist}(Z_n, X_2) \to 0$. Thus $P_{E_{h_n}} \nabla I(Z_n) \to 0$. Using $(P.S.)_c^*$ for I of Lemma 4.3 it follows that $(Z_n)_n$ has a subsequence $(Z_{k_n})_n$ such that $\widetilde{Z_{k_n}} \to Z$ for some Z in X_2. Since Ψ is invertible in $\operatorname{int}(C)$, $\widetilde{Z_{k_n}} \to \Psi^{-1}(Z)$. Next we consider the case $\widetilde{Z_n} \in \partial C_{h_n}$ for infinitely many n. We claim that this case cannot occur. If $\widetilde{Z_n} \in \partial C_{h_n}$, then $\|P_{X_2}\widetilde{Z_n}\|_E = 1$. Thus we have

$$\operatorname{grad}^-_{C_{h_n}} \widetilde{I}\left(\widetilde{Z}_n\right) = P_{E_{h_n}}\left(P_{X_1 \oplus X_3} \nabla I(Z_n) - \left\langle \nabla I(Z_n), P_{X_2}\widetilde{Z}_n \right\rangle^+ P_{X_2}\widetilde{Z}_n\right) \longrightarrow 0. \tag{4.26}$$

Using the properties of the projections we get

$$P_{E_{h_n}} P_{X_1 \oplus X_3} \nabla I(Z_n) \longrightarrow 0, \tag{4.27}$$

which contradicts to Lemma 4.2. In fact, let $\tilde{Z}$ be the limit point of the subsequence $\widetilde{Z_{k_n}}$ of $\widetilde{Z_n}$, then $\tilde{Z} \in \partial C$ and

$$\operatorname{grad}_C^- \tilde{I}\left(\tilde{Z}\right) = P_{X_1 \oplus X_3} \operatorname{grad} I(Z) - \left\langle \operatorname{grad} I(Z), P_{X_2 \tilde{Z}} \right\rangle P_{X_2} \tilde{Z}. \tag{4.28}$$

□

Proof of Theorem 1.1. We assume that the conditions (1.4), (1.5), and (1.6) hold and F satisfies (F1), (F2), (F3), and (F4). We note that $\tilde{I} : C \rightarrow R \in C^{1,1}_{\mathrm{loc}}$ and by Lemma 4.1, there exist $R > \rho > 0$, $R_1 > 0$ and $R > 1$ such that

$$\sup_{\tilde{V} \in \widetilde{\Sigma_{12}(R, R_1)}} \tilde{I}\left(\tilde{V}\right) < \inf_{\widetilde{W} \in \widetilde{S_{23}(\rho)}} \tilde{I}\left(\widetilde{W}\right). \tag{4.29}$$

By Lemma 4.2, $\tilde{I}$ has no critical point $\widetilde{U}$ in $X_1 \oplus X_3$ whose critical value is $c > 0$. By Lemma 4.4, $\tilde{I}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$ for any $c \in [\alpha, \beta]$, $\alpha > 0$. Thus by Theorem 3.7, $\tilde{I}$ has at least two critical points $\widetilde{U}_i$, $i = 1, 2$, in $\operatorname{int} C$ with

$$0 < \inf_{\widetilde{W} \in \widetilde{S_{23}(\rho)}} \tilde{I}\left(\widetilde{W}\right) \leq \tilde{I}\left(\tilde{U}_i\right) \leq \sup_{\tilde{V} \in \widetilde{\Delta_{12}(R, R_1)}} \tilde{I}\left(\tilde{V}\right). \tag{4.30}$$

Since $\tilde{I}(0,0) = 0$ and (0,0) is the isolate point, $\widetilde{U}_i$, $i = 1, 2$ are nontrivial. Thus I has at least two nontrivial critical points U_i, $i = 1, 2$, in X_2 with

$$\inf_{W \in S_{23}(\rho)} I(W) \leq I(U_i) \leq \sup_{V \in \Delta_{12}(R, R_1)} I(V). \tag{4.31}$$

Thus system (1.3) has at least two nontrivial solutions. Thus Theorem 1.1 is proved. □

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References

[1] Q.-H. Choi and T. Jung, “An application of a variational reduction method to a nonlinear wave equation,” *Journal of Differential Equations*, vol. 117, no. 2, pp. 390–410, 1995.
[2] Q.-H. Choi and T. Jung, “Multiplicity results for nonlinear wave equations with nonlinearities crossing eigenvalues,” *Hokkaido Mathematical Journal*, vol. 24, no. 1, pp. 53–62, 1995.

[3] T. S. Jung and Q.-H. Choi, "An application of category theory to the nonlinear wave equation with jumping nonlinearity," *Honam Mathematical Journal*, vol. 26, no. 4, pp. 589–608, 2004.

[4] T. Jung and Q.-H. Choi, "Critical point theory applied to a class of the systems of the superquadratic wave equations," *Boundary Value Problems*, vol. 2008, Article ID 742030, 11 pages, 2008.

[5] T. Bartsch and M. Clapp, "Critical point theory for indefinite functionals with symmetries," *Journal of Functional Analysis*, vol. 138, no. 1, pp. 107–136, 1996.

[6] G. Fournier, D. Lupo, M. Ramos, and M. Willem, "Limit relative category and critical point theory," *Dynamics Reported*, vol. 3, pp. 1–23, 1993.

[7] A. M. Micheletti and C. Saccon, "Multiple nontrivial solutions for a floating beam equation via critical point theory," *Journal of Differential Equations*, vol. 170, no. 1, pp. 157–179, 2001.