Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 912601, 8 pages
doi:10.1155/2010/912601

Research Article

Hardy-Hilbert-Type Inequalities with a Homogeneous Kernel in Discrete Case

Josip Pečarić[1] and Predrag Vuković[2]

[1] *Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia*
[2] *Faculty of Teacher Education, University of Zagreb, Ante Starčevića 55, 40000 Čakovec, Croatia*

Correspondence should be addressed to Predrag Vuković, predrag.vukovic@vus-ck.hr

Received 4 September 2009; Accepted 16 February 2010

Academic Editor: Ondřej Došlý

The main objective of this paper is a study of some new generalizations of Hilbert's and Hardy-Hilbert's type inequalities. We apply our general results to homogeneous functions. We shall obtain, in a similar way as Yang did in(2009), that the constant factors are the best possible when the parameters satisfy appropriate conditions.

1. Introduction

Hilbert and Hardy-Hilbert type inequalities (see [1]) are very significant weight inequalities which play an important role in many fields of mathematics. Although classical, such inequalities have attracted the interest of numerous mathematicians and have been generalized in many different ways. Also the numerous mathematicians reproved them using various techniques. Some possibilities of generalizing such inequalities are, for example, various choices of nonnegative measures, kernels, sets of integration, extension to multidimensional case, and so forth.

Similar inequalities, in operator form, appear in harmonic analysis where one investigates properties of boundedness of such operators. This is the reason why Hilbert's inequality is so popular and represents field of interest of numerous mathematicians: since Hilbert till nowadays.

We start with the following two discrete inequalities, which are the well-known Hilbert and Hardy-Hilbert type inequalities. More precisely, if $p > 1$, $(1/p) + (1/q) = 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then the following inequality holds (Hardy et al. [1]):

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=0}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=0}^{\infty} b_n^q \right\}^{1/q}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. The equivalent form of inequality (1.1) is (see Yang and Debnath [2])

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)^p<\left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=0}^{\infty}a_n^p, \tag{1.2}$$

where the constant factor $[\pi/\sin(\pi/p)]^p$ is still the best possible.

In this paper we refer to a recent paper of Yang (see [3]). In 2005, Yang [3] gave some extension of Hilbert's inequality with two pairs of conjugate exponents $(p,q),(r,s)$ $(p,r>1)$, and two parameters $\alpha,\lambda>0$ $(\alpha\lambda\le\min\{r,s\})$ as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(m^\alpha+n^\alpha)^\lambda}<k_{\alpha\lambda}(r)\left(\sum_{n=1}^{\infty}n^{p(1-\alpha\lambda/r)-1}a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}n^{q(1-\alpha\lambda/s)-1}b_n^q\right)^{1/q}, \tag{1.3}$$

where the constant factor $k_{\alpha\lambda}(r)=(1/\alpha)B(\lambda/r,\lambda/s)$ is the best possible.

Let $\phi(x)=(x+\alpha)^{p(1-\lambda/r)-1}$, $\varphi(x)=(x+\alpha)^{q(1-\lambda/s)-1}$, $\psi(x)=(x+\alpha)^{p\lambda/s-1}$ $x\in(0,\infty)$, and $l_\phi^p=\{a=\{a_n\}_{n=0}^{\infty};\|a\|_{p,\phi}:=\{\sum_{n=0}^{\infty}\phi(n)|a_n|^p\}^{1/p}<\infty\}$. Define a Hilbert-type linear operator T; for all $a\in l_\phi^p$, one has

$$(Ta)(n):=\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}a_m. \tag{1.4}$$

For $a\in l_\phi^p$, $b\in l_\varphi^q$, define the formal inner product of Ta and b as

$$(Ta,b):=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_mb_n}{(m+\alpha)^\lambda-(n+\alpha)^\lambda}. \tag{1.5}$$

Zhong (see [4]) proved the following theorem.

Theorem 1.1. *Suppose that (p,q) and (r,s) are two pairs of conjugate exponents, $r>1$, $p>1$, $1/2\le\alpha\le1$, $0<\lambda\le1$, $a_n,b_n\ge0$. If $\|a\|_{p,\phi}>0$, $\|b\|_{q,\varphi}>0$, then one has the equivalent inequalities as*

$$\begin{aligned}(Ta,b)&<k_\lambda(s)\|a\|_{p,\phi}\|b\|_{q,\varphi},\\ \|Ta\|_{p,\psi}&<k_\lambda(s)\|a\|_{p,\phi},\end{aligned} \tag{1.6}$$

where the constant factor $k_\lambda(s)=[(1/\lambda)B(1/s,1/r)]^2$ is the best possible.

Results in this paper will be based on the following general form of Hilbert's and Hardy-Hilbert's inequality proven in [5]. All the measures are assumed to be σ-finite on some measure space Ω. Let $1/p + 1/q = 1$ with $p > 1$, $K(x,y)$, $f(x)$, $g(y)$, $\varphi(x)$, $\psi(y)$ be nonnegative functions. Then the following inequalities hold and are equivalent:

$$\int_{\Omega^2} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \leq \left(\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right)^{1/p}\left(\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right)^{1/q}, \tag{1.7}$$

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y) \leq \int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x), \tag{1.8}$$

where

$$F(x) = \int_\Omega \frac{K(x,y)}{\psi^p(y)}d\mu_2(y), \qquad G(y) = \int_\Omega \frac{K(x,y)}{\varphi^q(x)}d\mu_1(x). \tag{1.9}$$

It is of great importance to consider the case when the functions $F(x)$ and $G(y)$, defined by (1.9), are bounded. More precisely, Krnić and Pečarić in [5] proved the following result.

Theorem 1.2. *Let $1/p + 1/q = 1$ with $p > 1$, $K(x,y)$, $f(x)$, $g(y)$, $\varphi(x)$, $\psi(y)$ be nonnegative functions and $F(x) \leq F_1(x), G(y) \leq G_1(y)$, where $F(x)$ and $G(y)$ are defined by* (1.9). *Then the following inequalities hold and are equivalent:*

$$\int_{\Omega^2} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \leq \left(\int_\Omega \varphi^p(x)F_1(x)f^p(x)d\mu_1(x)\right)^{1/p}\left(\int_\Omega \psi^q(y)G_1(y)g^q(y)d\mu_2(y)\right)^{1/q}, \tag{1.10}$$

$$\int_\Omega G_1^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y) \leq \int_\Omega \varphi^p(x)F_1(x)f^p(x)d\mu_1(x).$$

In this paper a generalization of Theorem 1.1 for a general type of homogeneous kernels is obtained. Recall that for a homogeneous function $K(x,y)$ of degree $-\lambda$, $\lambda > 0$, equality $K(tx,ty) = t^{-\lambda}K(x,y)$ is satisfied for every $t > 0$. Further, we define $k(\alpha) := \int_0^\infty K(1,t)t^{-\alpha}dt$ and suppose that $k(\alpha) < \infty$ for $1 - \lambda < \alpha < 1$.

In what follows, without further explanation, we assume that all series and integrals exist on the respective domains of their definitions.

2. Main Results

We apply Theorem 1.2 to obtain the following theorem.

Theorem 2.1. *Let* $\lambda > 0$, $1/p + 1/q = 1$ *with* $p > 1$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two nonnegative real sequences. If* $K(x, y) \geq 0$ *is homogeneous function of degree* $-\lambda$ *strictly decreasing in both parameters* x *and* y, $\mu \geq 0$, *then the following inequalities hold and are equivalent:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m+\mu, n+\mu) a_m b_n \leq L\left(\sum_{m=1}^{\infty}(m+\mu)^{1-\lambda+p(A_1-A_2)} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}(n+\mu)^{1-\lambda+q(A_2-A_1)} b_n^q\right)^{1/q}, \quad (2.1)$$

$$\sum_{n=1}^{\infty}(n+\mu)^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\sum_{m=1}^{\infty} K(m+\mu, n+\mu) a_m\right)^p \leq L^p \sum_{m=1}^{\infty}(m+\mu)^{(1-\lambda)+p(A_1-A_2)} a_m^p, \quad (2.2)$$

where $A_1 \in (\max\{(1-\lambda)/q, 0\}, 1/q)$, $A_2 \in (\max\{(1-\lambda)/p, 0\}, 1/p)$ *and*

$$L = k(pA_2)^{1/p} k(2-\lambda-qA_1)^{1/q}. \quad (2.3)$$

Proof. We use the inequalities (1.7), (1.8), and Theorem 1.2 with counting measure. First, we prove the inequality (2.1). Put $\varphi(m+\mu) = (m+\mu)^{A_1}$ and $\psi(n+\mu) = (n+\mu)^{A_2}$ in the inequality (1.7). Then, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m+\mu, n+\mu) a_m b_n \leq \left(\sum_{m=1}^{\infty}(m+\mu)^{pA_1} F(m+\mu) a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}(n+\mu)^{qA_2} G(n+\mu) b_n^q\right)^{1/q}, \quad (2.4)$$

where $F(m+\mu) = \sum_{n=1}^{\infty}(K(m+\mu, n+\mu)/(n+\mu)^{pA_2})$ and $G(n+\mu) = \sum_{m=1}^{\infty}(K(m+\mu, n+\mu)/(m+\mu)^{qA_1})$. Since $qA_1 > 0$ and $pA_2 > 0$, the functions $F(m+\mu)$ and $G(n+\mu)$ are strictly decreasing, where we have

$$F(m+\mu) < F_1(m+\mu) := \int_0^{\infty} \frac{K(m+\mu, y+\mu)}{(y+\mu)^{pA_2}} dy,$$
$$G(n+\mu) < G_1(n+\mu) := \int_0^{\infty} \frac{K(x+\mu, n+\mu)}{(x+\mu)^{qA_1}} dx. \quad (2.5)$$

Using homogeneity of the functions K and the substitution $u = (y+\mu)/(m+\mu)$ we get

$$F_1(m+\mu) \leq (m+\mu)^{1-\lambda-pA_2}\int_0^{\infty} K(1,t) t^{-pA_2} dt = (m+\mu)^{1-\lambda-pA_2} k(pA_2). \quad (2.6)$$

In a similar manner we obtain

$$G_1(n+\mu) \le (n+\mu)^{1-\lambda-qA_1} k(2-\lambda-qA_1). \tag{2.7}$$

Now, the result follows from Theorem 1.2. □

Remark 2.2. Equality in the previous theorem is possible only if

$$f(x)^p = K_1 \varphi(x)^{-(p+q)}, \qquad g(y)^q = K_2 \psi(y)^{-(p+q)}, \tag{2.8}$$

for arbitrary constants K_1 and K_2 (see [5]). Condition (2.8) immediately gives that nontrivial case of equality in (2.1) and (2.2) leads to divergent series.

Now, we consider some special choice of the parameters A_1 and A_2. More precisely, let the parameters A_1 and A_2 satisfy constraint

$$pA_2 + qA_1 = 2 - \lambda. \tag{2.9}$$

Then, the constant L from Theorem 2.1 becomes

$$L^* = k(pA_2). \tag{2.10}$$

Further, the inequalities (2.1) and (2.2) take form

$$\begin{aligned} &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m+\mu, n+\mu) a_m b_n \\ &\quad \le L^* \left(\sum_{m=1}^{\infty} (m+\mu)^{-1+pqA_1} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} (n+\mu)^{-1+pqA_2} b_n^q \right)^{1/q}, \end{aligned} \tag{2.11}$$

$$\sum_{n=1}^{\infty} (n+\mu)^{(p-1)(1-pqA_2)} \left(\sum_{m=1}^{\infty} K(m+\mu, n+\mu) a_m \right)^p \le (L^*)^p \sum_{m=1}^{\infty} (m+\mu)^{-1+pqA_1} a_m^p. \tag{2.12}$$

In the following theorem we show, in a similar way as Yang did in [6], that if the parameters A_1 and A_2 satisfy condition (2.9), then one obtains the best possible constant. To prove this result we need the next lemma (see [6]).

Lemma 2.3. *If $f(x)(\ge 0)$ is decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$, and $I_0 := \int_0^{\infty} f(x)dx < \infty$, then*

$$I_1 := \int_1^{\infty} f(x)dx \le \sum_{n=1}^{\infty} f(n) < I_0. \tag{2.13}$$

Theorem 2.4. *Let λ, μ, A_1, A_2, and $K(x,y)$ be defined as in Theorem 2.1. If the parameters A_1 and A_2 satisfy condition $pA_2 + qA_1 = 2 - \lambda$, then the constants $L^* = k(pA_2)$ and $(L^*)^p$ in the inequalities* (2.11) *and* (2.12) *are the best possible.*

Proof. For this purpose, with $\varepsilon > 0$, set $\tilde{a}_m = (m+\mu)^{-qA_1-\varepsilon/p}$ and $\tilde{b}_n = (n+\mu)^{-pA_2-\varepsilon/q}$. Now, let us suppose that there exists a smaller constant $0 < M < L^*$ such that the inequality (2.11) is valid. Let J denote the right-hand side of (2.11). Using Lemma 2.3, we have

$$J = M\left((1+\mu)^{-1-\varepsilon} + \sum_{n=2}^{\infty}\frac{1}{(n+\mu)^{1+\varepsilon}}\right) < M\left((1+\mu)^{-1-\varepsilon} + \int_1^{\infty}(x+\mu)^{-1-\varepsilon}dx\right)$$
$$= \frac{M}{\varepsilon(1+\mu)^{\varepsilon}}\left(\frac{\varepsilon}{1+\mu}+1\right). \tag{2.14}$$

Further, let I denote the left-hand side of the inequality (2.11), for above choice of sequences $\tilde{a}_m$ and $\tilde{b}_n$. Applying, respectively, Lemma 2.3, Fubini's theorem, and substitution $t = (x+\mu)/(y+\mu)$, we have

$$I \geq \sum_{n=1}^{\infty}\left(\int_1^{\infty}K(x+\mu, n+\mu)(x+\mu)^{-qA_1-\varepsilon/p}dx\right)(n+\mu)^{-pA_2-\varepsilon/q}$$
$$\geq \int_1^{\infty}(x+\mu)^{-qA_1-\varepsilon/p}\left(\int_1^{\infty}K(x+\mu, y+\mu)(y+\mu)^{-pA_2-\varepsilon/q}dy\right)dx$$
$$= \int_1^{\infty}(x+\mu)^{-1-\varepsilon}\left(\int_0^{(x+\mu)/(1+\mu)}K(1,t)t^{-qA_1+\varepsilon/q}dt\right)dx$$
$$= \frac{1}{\varepsilon(1+\mu)^{\varepsilon}}\int_0^1 K(1,t)t^{-qA_1+\varepsilon/q}dt$$
$$+ \int_1^{\infty}(x+\mu)^{-1-\varepsilon}\left(\int_1^{(x+\mu)/(1+\mu)}K(1,t)t^{-qA_1+\varepsilon/q}dt\right)dx$$
$$= \frac{1}{\varepsilon(1+\mu)^{\varepsilon}}\left(\int_0^1 K(1,t)t^{-qA_1+\varepsilon/q}dt + \int_1^{\infty}K(1,t)t^{-qA_1-\varepsilon/p}dt\right). \tag{2.15}$$

From (2.11), (2.14), and (2.15) we get

$$M\left(\frac{\varepsilon}{1+\mu}+1\right) \geq \int_0^1 K(1,t)t^{-qA_1+\varepsilon/q}dt + \int_1^{\infty}K(1,t)t^{-qA_1-\varepsilon/p}dt. \tag{2.16}$$

By letting $\varepsilon \to 0^+$, we obtain

$$M \geq \int_0^1 K(1,t)t^{-qA_1}dt + \int_1^{\infty}K(1,t)t^{-qA_1}dt = k(qA_1). \tag{2.17}$$

Using symmetry of the function $K(x,y)$, we have $k(qA_1) = k(pA_2) = L^*$. Now, from (2.17) we obtain a contradiction with assumption $M < L^* = k(pA_2)$.

Finally, equivalence of the inequalities (2.11) and (2.12) means that the constant $(L^*)^p$ is the best possible in the inequality (2.12). This completes the proof. □

We proceed with some special homogeneous functions. Since the function $K(x,y) = 1/(x^\alpha + y^\alpha)^\lambda$ is homogeneous of degree $-\alpha\lambda$, by using Theorem 2.4 we obtain the following.

Corollary 2.5. *Let $\lambda > 0$, $\alpha > 0$, $\mu \geq 0$. Suppose that the parameters A_1, A_2 satisfy condition $pA_2 + gA_1 = 2 - \alpha\lambda$. Then the following inequalities hold and are equivalent:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{((m+\mu)^\alpha + (n+\mu)^\alpha)^\lambda} \leq L_1\left(\sum_{m=1}^{\infty}(m+\mu)^{-1+pqA_1}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}(n+\mu)^{-1+pqA_2}b_n^q\right)^{1/q},$$
$$\sum_{n=1}^{\infty}(n+\mu)^{(p-1)(1-pqA_2)}\left(\sum_{m=1}^{\infty}\frac{a_m}{((m+\mu)^\alpha + (n+\mu)^\alpha)^\lambda}\right)^p \leq L_1^p\sum_{m=1}^{\infty}(m+\mu)^{-1+pqA_1}a_m^p, \quad (2.18)$$

where the constant factors $L_1 = (1/\alpha)B((1-pA_2)/\alpha, (1-qA_1)/\alpha)$ and L_1^p are the best possible.

Remark 2.6. If we put $\alpha = 1$, $A_1 = A_2 = (2-\lambda)/pq$ in Corollary 2.5, then the inequalities (2.18) become

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n+2\mu)^\lambda} \leq L_1\left(\sum_{m=1}^{\infty}(m+\mu)^{1-\lambda}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}(n+\mu)^{1-\lambda}b_n^q\right)^{1/q},$$
$$\sum_{n=1}^{\infty}(n+\mu)^{(p-1)(\lambda-1)}\left(\sum_{m=1}^{\infty}\frac{a_m}{(m+n+2\mu)^\lambda}\right)^p \leq L_1^p\sum_{m=1}^{\infty}(m+\mu)^{1-\lambda}a_m^p, \quad (2.19)$$

where the constant factors $L_1 = B(1/p+(\lambda-1)/q, 1/q+(\lambda-1)/p)$, and L_1^p are the best possible. For $\lambda = 1$ we obtain nonweighted case with the best possible constant $L_1 = B(1/p, 1/q)$. Setting $\mu = 1/2$ and $\lambda = 1$ in the inequalities (2.19) we obtain the inequalities (1.1) and (1.2) from Introduction.

Remark 2.7. It is easy to see that Theorem 2.4 is the generalization of Theorem 1.1. Namely, let us define $A_1 = 1/q - \lambda/qr$, $A_2 = 1/p - \lambda/ps$, and $K(m+\mu, n+\mu) = ((\ln((m+\mu)/(n+\mu)))/((m+\mu)^\lambda - (n+\mu)^\lambda))$. Note that the parameters A_1, A_2 satisfy condition $pA_2 + qA_1 = 2-\lambda$. Then, the best possible constant L^* from Theorem 2.4 becomes $k_\lambda(s)$ from Theorem 1.1 (see also [4]).

Remark 2.8. Similarly as in Corollary 2.5, for the homogeneous function of degree -1, $K(x,y) = (x^{\lambda-1} + y^{\lambda-1})/(x^\lambda + y^\lambda)$, nonnegative real sequences $a = \{a_m\}_{m=1}^\infty$, $b = \{b_m\}_{m=1}^\infty$, and the parameters $A_1 = A_2 = 1/pq$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(m+\mu)^{\lambda-1}+(n+\mu)^{\lambda-1}}{(m+\mu)^{\lambda}+(n+\mu)^{\lambda}}a_m b_n \le L_2\|a\|_p\|b\|_q,$$
$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{(m+\mu)^{\lambda-1}+(n+\mu)^{\lambda-1}}{(m+\mu)^{\lambda}+(n+\mu)^{\lambda}}a_m\right)^p \le L_2^p\|a\|_p^p, \tag{2.20}$$

where the constants $L_2 = (\pi/\lambda)(1/\sin(\pi/p) + (1/\sin(\pi/q)))$ and L_2^p are the best possible.

Remark 2.9. Let λ, A_1, A_2, and $K(x,y)$ be defined as in Theorem 2.1. Take $\mu = 0$ in the inequalities (2.11) and (2.12). By using Theorem 2.4 we get equivalent inequalities for general homogeneous kernel $K(x,y)$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)a_m b_n \le L^*\left(\sum_{m=1}^{\infty}m^{-1+pqA_1}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}n^{-1+pqA_2}b_n^q\right)^{1/q},$$
$$\sum_{n=1}^{\infty}n^{(p-1)(1-pqA_2)}\left(\sum_{m=1}^{\infty}K(m,n)a_m\right)^p \le (L^*)^p\sum_{m=1}^{\infty}m^{-1+pqA_1}a_m^p, \tag{2.21}$$

where the constant factors $L^* = k(pA_2)$ and $(L^*)^p$ are the best possible.

Setting $A_1 = 1/q - \lambda/qr$, $A_2 = 1/p - \lambda/ps$ in the inequalities (2.21) we obtain the result from [6]. Similarly, for above choice of the parameters A_1, A_2, and $K(x,y) = 1/(x^\alpha + y^\alpha)^\lambda$, we obtain Yang's result (1.3) from Introduction.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1967.

[2] B. Yang and L. Debnath, "On a new generalization of Hardy-Hilbert's inequality and its applications," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 484–497, 1999.

[3] B. Yang, "On best extensions of Hardy-Hilbert's inequality with two parameters," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 3, article 81, pp. 1–15, 2005.

[4] W. Zhong, "A Hilbert-type linear operator with the norm and its applications," *Journal of Inequalities and Applications*, vol. 2009, Article ID 494257, 18 pages, 2009.

[5] M. Krnić and J. Pečarić, "General Hilbert's and Hardy's inequalities," *Mathematical Inequalities & Applications*, vol. 8, no. 1, pp. 29–52, 2005.

[6] B. Yang, "On a Hilbert-type operator with a class of homogeneous kernels," *Journal of Inequalities and Applications*, vol. 2009, Article ID 572176, 9 pages, 2009.