

Research Article

Existence of Solutions and Nonnegative Solutions for Prescribed Variable Exponent Mean Curvature Impulsive System Initialized Boundary Value Problems

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This paper investigates the existence of solutions and nonnegative solutions for prescribed variable exponent mean curvature impulsive system initialized boundary value problems. The proof of our main result is based upon Leray-Schauder's degree. The sufficient conditions for the existence of solutions and nonnegative solutions have been given.

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. On the Laplacian impulsive differential equations boundary value problems, there are many results (see [1–5]). Because of the nonlinearity of p -Laplacian, the results about p -Laplacian impulsive differential equations boundary value problems are rare (see [6]). In [7, 8], the authors discussed the existence of solutions of $p(r)$ -Laplacian system impulsive boundary value problems. Recently, the existence and asymptotic behavior of solutions of curvature equations have been studied extensively (see [9–15]). In [16], the authors generalized the usual mean curvature systems to variable exponent mean curvature systems. In this paper, we consider the existence of

solutions and nonnegative solutions for the prescribed variable exponent mean curvature system

$$-(\varphi(r, u'))' + f(r, u, u') = 0, \quad r \in (0, T), \quad r \neq r_i, \quad (1.1)$$

where $u : [0, T] \rightarrow \mathbb{R}^N$, with the following impulsive initialized boundary value conditions

$$\lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) = A_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right), \quad i = 1, \dots, k, \quad (1.2)$$

$$\lim_{r \rightarrow r_i^+} \varphi(r, u'(r)) - \lim_{r \rightarrow r_i^-} \varphi(r, u'(r)) = B_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right), \quad i = 1, \dots, k, \quad (1.3)$$

$$u'(0) = u(0) = 0, \quad (1.4)$$

where

$$\varphi(r, x) = \frac{|x|^{p(r)-1}x}{(1 + |x|^{q(r)p(r)})^{1/q(r)}}, \quad \forall r \in [0, T], \quad x \in \mathbb{R}^N, \quad (1.5)$$

$p, q \in C([0, T], \mathbb{R}^+)$ are absolutely continuous, where p, q satisfy $p(r) \geq 1, q(r) \geq 1$, $-(\varphi(r, u'))'$ is called the variable exponent mean curvature operator, $0 < r_1 < r_2 < \dots < r_k < T$ and $A_i, B_i \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$.

For any $v \in \mathbb{R}^N$, v^j will denote the j th component of v ; the inner product in \mathbb{R}^N will be denoted by $\langle \cdot, \cdot \rangle$; $|\cdot|$ will denote the absolute value and the Euclidean norm on \mathbb{R}^N . Denote that $J = [0, T]$, $J' = [0, T] \setminus \{r_0, r_1, \dots, r_{k+1}\}$, and $J_0 = [r_0, r_1]$, $J_i = (r_i, r_{i+1})$, $i = 1, \dots, k$, where $r_0 = 0, r_{k+1} = T$. Denote that J_i^o is the interior of J_i , $i = 0, 1, \dots, k$. Let

$$\begin{aligned} \text{PC}(J, \mathbb{R}^N) &= \left\{ x : J \rightarrow \mathbb{R}^N \left| \begin{array}{l} x \in C(J_i, \mathbb{R}^N), \quad i = 0, 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} x(r) \text{ exists for } i = 1, \dots, k \end{array} \right. \right\}, \\ \text{PC}^1(J, \mathbb{R}^N) &= \left\{ x \in \text{PC}(J, \mathbb{R}^N) \left| \begin{array}{l} x' \in C(J_i^o, \mathbb{R}^N), \\ \lim_{r \rightarrow r_i^+} x'(r) \text{ and } \lim_{r \rightarrow r_{i+1}^-} x'(r) \text{ exist for } i = 0, 1, \dots, k \end{array} \right. \right\}. \end{aligned} \quad (1.6)$$

For any $u(r) = (u^1(r), \dots, u^N(r)) \in \text{PC}(J, \mathbb{R}^N)$, denote that $|u^i|_0 = \sup\{|u^i(r)| \mid r \in J'\}$. Obviously, $\text{PC}(J, \mathbb{R}^N)$ is a Banach space with the norm $\|u\|_0 = (\sum_{i=1}^N |u^i|_0^2)^{1/2}$, and $\text{PC}^1(J, \mathbb{R}^N)$ is a Banach space with the norm $\|u\|_1 = \|u\|_0 + \|u'\|_0$. In the following, $\text{PC}(J, \mathbb{R}^N)$ and $\text{PC}^1(J, \mathbb{R}^N)$ will be simply denoted by PC and PC^1 , respectively. Denote that $L^1 = L^1(J, \mathbb{R}^N)$, and the norm in L^1 is $\|u\|_{L^1} = [\sum_{i=1}^N (\int_0^T |u^i(r)| dr)^2]^{1/2}$.

The study of differential equations and variational problems with variable exponent conditions is a new and interesting topic. For the applied background on this kind of problems we refer to [17–19]. Many results have been obtained on this kind of problems,

for example, [20–35]. If $p(r) \equiv p$ (a constant) and $q(r) \equiv q$ (a constant), then (1.1) is the well-known mean curvature system. Since problems with variable exponent growth conditions are more complex than those with constant exponent growth conditions, many methods and results for the latter are invalid for the former; for example, if $\Omega \subset \mathbb{R}^n$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (1/p(x)) |\nabla u|^{p(x)} dx}{\int_{\Omega} (1/p(x)) |u|^{p(x)} dx} \quad (1.7)$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [25]), but the fact that $\lambda_p > 0$ is very important in the study of p -Laplacian problems.

In this paper, we investigate the existence of solutions for the prescribed variable exponent mean curvature impulsive differential system initialized boundary value problems; the proof of our main result is based upon Leray-Schauder's degree. This paper was motivated by [6, 13, 36].

Let $N \geq 1$, then the function $f : J \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is assumed to be Caratheodory; by this we mean that

- (i) for almost every $t \in J$ the function $f(t, \cdot, \cdot)$ is continuous,
- (ii) for each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y)$ is measurable on J ,
- (iii) for each $R > 0$ there is a $\beta_R \in L^1(J, \mathbb{R})$ such that, for almost every $t \in J$ and every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R, |y| \leq R$, one has

$$|f(t, x, y)| \leq \beta_R(t). \quad (1.8)$$

We say a function $u : J \rightarrow \mathbb{R}^N$ is a solution of (1.1) if $u \in PC^1$ with $\varphi(r, u')$ absolutely continuous on J_i^o , $i = 0, 1, \dots, k$, which satisfies (1.1) a.e. on J .

This paper is divided into three sections; in the second section, we present some preliminary. Finally, in the third section, we give the existence of solutions and nonnegative solutions for system (1.1)–(1.4).

2. Preliminary

In this section, we will do some preparation.

Lemma 2.1 (see [16]). *φ is a continuous function and satisfies the following.*

- (i) For any $r \in J$, $\varphi(r, \cdot)$ is strictly monotone, that is,

$$\langle \varphi(r, x_1) - \varphi(r, x_2), x_1 - x_2 \rangle > 0, \quad \text{for any } x_1, x_2 \in \mathbb{R}^N, x_1 \neq x_2. \quad (2.1)$$

- (ii) For any fixed $r \in J$, $\varphi(r, \cdot)$ is a homeomorphism from \mathbb{R}^N to

$$E = \{x \in \mathbb{R}^N \mid |x| < 1\}. \quad (2.2)$$

For any $r \in J$, denote by $\varphi^{-1}(r, \cdot)$ the inverse operator of $\varphi(r, \cdot)$, then

$$\varphi^{-1}(r, x) = \left(1 - |x|^{q(r)}\right)^{-1/p(r)q(r)} |x|^{1/p(r)-1} x, \quad \text{for } x \in E \setminus \{0\}, \quad \varphi^{-1}(r, 0) = 0. \quad (2.3)$$

Let one now consider the following simple problem:

$$(\varphi(r, u'(r)))' = h(r), \quad r \in (0, T), \quad r \neq r_i, \quad (2.4)$$

with the following impulsive boundary value conditions:

$$\begin{aligned} \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= a_i, \quad i = 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} \varphi(r, u'(r)) - \lim_{r \rightarrow r_i^-} \varphi(r, u'(r)) &= b_i, \quad i = 1, \dots, k, \\ u'(0) &= u(0) = 0, \end{aligned} \quad (2.5)$$

where $a_i, b_i \in \mathbb{R}^N$, $\sum_{i=1}^k |b_i| < 1$; $h \in L^1$.

Obviously, $u'(0) = 0$ implies that $\varphi(0, u'(0)) = 0$. If u is a solution of (2.4) with (2.5), by integrating (2.4) from 0 to r , then one finds that

$$\varphi(r, u'(r)) = \sum_{r_i < r} b_i + \int_0^r h(t) dt, \quad \forall r \in J'. \quad (2.6)$$

Define $a = (a_1, \dots, a_k) \in \mathbb{R}^{kN}$, $b = (b_1, \dots, b_k) \in \mathbb{R}^{kN}$, and operator $F : L^1 \rightarrow PC$ as

$$F(h)(r) = \int_0^r h(t) dt, \quad \forall h \in L^1. \quad (2.7)$$

From the definition of φ , one can see that

$$\sup_{r \in J} \left| \sum_{r_i < r} b_i + F(h)(r) \right| < 1. \quad (2.8)$$

Denote that

$$D_b = \left\{ h \in L^1 \left| \sup_{r \in J} \left| \sum_{r_i < r} b_i + F(h)(r) \right| < 1 \right. \right\}. \quad (2.9)$$

By (2.6), one has

$$u(r) = \sum_{r_i < r} a_i + F \left\{ \varphi^{-1} \left[r, \left(\sum_{r_i < r} b_i + F(h) \right) \right] \right\} (r), \quad \forall r \in J. \quad (2.10)$$

Denote that $W = \mathbb{R}^{2kN} \times L^1$ with the norm

$$\|w\|_W = \sum_{i=1}^k |a_i| + \sum_{i=1}^k |b_i| + \|h\|_{L^1}, \quad \forall w = (a, b, h) \in W, \quad (2.11)$$

then W is a Banach space.

Let one define the nonlinear operator $K_b : D_b \rightarrow PC^1$ as

$$K_b(h)(r) = F \left\{ \varphi^{-1} \left[r, \left(\sum_{r_i < r} b_i + F(h) \right) \right] \right\} (r), \quad \forall r \in J. \quad (2.12)$$

Lemma 2.2. *The operator K_b is continuous and sends closed equiintegrable subsets of D_b into relatively compact sets in PC^1 .*

Proof. It is easy to check that $K_b(h)(\cdot) \in PC^1$, for all $h \in D_b$. Since

$$K_b(h)'(t) = \varphi^{-1} \left\{ t, \left[\sum_{r_i < t} b_i + F(h) \right] \right\}, \quad \forall t \in J, \quad (2.13)$$

it is easy to check that K_b is a continuous operator from D_b to PC^1 .

Let now \bar{U} be a closed equiintegrable set in D_b , then there exists $\eta \in L^1$, such that, for any $u \in \bar{U}$,

$$|u(t)| \leq \eta(t) \quad \text{a.e. on } J. \quad (2.14)$$

We want to show that $\overline{K_b(\bar{U})} \subset PC^1$ is a compact set.

Let $\{u_n\}$ is a sequence in $K_b(\bar{U})$, then there exists a sequence $\{h_n\} \in \bar{U}$ such that $u_n = K_b(h_n)$. For any $t_1, t_2 \in J$, we have that

$$|F(h_n)(t_1) - F(h_n)(t_2)| \leq \left| \int_{t_1}^{t_2} \eta(t) dt \right|. \quad (2.15)$$

Hence the sequence $\{F(h_n)\}$ is uniformly bounded and equicontinuous. By Ascoli-Arzelà theorem, there exists a subsequence of $\{F(h_n)\}$ (which we rename the same) that is convergent in PC. Then the sequence

$$\varphi(t, K_b(h_n)'(t)) = \sum_{r_i < t} b_i + F(h_n) \quad (2.16)$$

is convergent according to the norm in PC. Since

$$K_b(h_n)(t) = F \left\{ \varphi^{-1} \left[t, \left(\sum_{r_i < t} b_i + F(h_n) \right) \right] \right\} (t), \quad \forall t \in J, \quad (2.17)$$

according to the continuity of φ^{-1} , we can see that $K_b(h_n)$ is convergent in PC. Thus we conclude that $\{u_n\}$ is convergent in PC^1 . This completes the proof. \square

We denote that $N_f(u) : PC^1 \rightarrow L^1$ is the Nemytski operator associated to f defined by

$$N_f(u)(r) = f(r, u(r), u'(r)), \quad \text{a.e. on } J. \quad (2.18)$$

Define $K : PC^1 \rightarrow PC^1$ as

$$K(u)(r) = F \left\{ \varphi^{-1} \left[r, \left(\sum_{r_i < r} B_i + F(N_f(u)) \right) \right] \right\} (r), \quad \forall r \in J, \quad (2.19)$$

where $B_i = B_i(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r))$.

Lemma 2.3. *u is a solution of (1.1)–(1.4) if and only if u is a solution of the following abstract equation:*

$$u = \sum_{r_i < r} A_i + K(u), \quad (2.20)$$

where $A_i = A_i(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r))$, $B_i = B_i(\lim_{r \rightarrow r_i^-} u(r))$, and $\lim_{r \rightarrow r_i^-} u'(r)$.

Proof. (i) If u is a solution of (1.1)–(1.4), since $u'(0) = 0$ implies that $\varphi(0, u'(0)) = 0$, by integrating (1.1) from 0 to r , then we find that

$$\varphi(r, u'(r)) = \sum_{r_i < r} B_i + \int_0^r f(t, u, u') dt, \quad \forall r \in J'. \quad (2.21)$$

Thus

$$u = \sum_{r_i < r} A_i + K(u). \quad (2.22)$$

Hence u is a solution of (2.20).

(ii) If u is a solution of (2.20), then it is easy to see that (1.2) is satisfied. Let $r = 0$, then we have

$$u(0) = 0. \quad (2.23)$$

From (2.20) we also have

$$\varphi(r, u') = \sum_{r_i < r} B_i + F(N_f(u))(r), \quad r \in (0, T), \quad r \neq r_i, \quad (2.24)$$

$$(\varphi(r, u'))' = f(r, u, u'), \quad r \in (0, T), \quad r \neq r_i. \quad (2.25)$$

From (2.24), we can see that (1.3) is satisfied. Let $r = 0$, from (2.24), then we have

$$\varphi(0, u'(0)) = 0. \tag{2.26}$$

Since $\varphi(r, x) = |x|^{p(r)-1}x / (1 + |x|^{q(r)p(r)})^{1/q(r)} = 0$, we must have $x = 0$; thus,

$$u'(0) = 0. \tag{2.27}$$

Hence u is a solution of (1.1)–(1.4). This completes the proof. □

3. Main Results and Proofs

In this section, we will apply Leray-Schauder’s degree to deal with the existence of solutions for (1.1)–(1.4).

We assume that

$$(H_0) \sum_{i=1}^k |B_i(u, v)| < 1/2, \text{ for all } (u, v) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Theorem 3.1. *If (H_0) is satisfied, then $0 \in \Omega$ is an open bounded set in PC^1 such that the following conditions hold.*

(1⁰) *For any $u \in \overline{\Omega}$, the mapping $r \rightarrow f(r, u, u')$ belongs to $\{v \in L^1 \mid \|v\|_{L^1} < 1/3\}$.*

(2⁰) *For each $\lambda \in (0, 1)$, the problem*

$$\begin{aligned} (\varphi(r, u'))' &= \lambda f(r, u, u'), \quad r \in (0, T), \quad r \neq r_i, \\ \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= \lambda A_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right), \quad i = 1, \dots, k, \\ \lim_{r \rightarrow r_i^+} \varphi(r, u'(r)) - \lim_{r \rightarrow r_i^-} \varphi(r, u'(r)) &= \lambda B_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right), \quad i = 1, \dots, k, \\ u'(0) &= u(0) = 0 \end{aligned} \tag{3.1}$$

has no solution on $\partial\Omega$.

Then (1.1)–(1.4) has at least one solution on $\overline{\Omega}$.

Proof. Denote that

$$A_i = A_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right), \quad B_i = B_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right). \tag{3.2}$$

For any $\lambda \in [0, 1]$, define $K_\lambda : PC^1 \rightarrow PC^1$ as

$$K_\lambda(u)(r) = F \left\{ \varphi^{-1} \left[r, \left(\sum_{r_i < r} \lambda B_i + F(\lambda N_f(u)) \right) \right] \right\} (r), \quad \forall r \in J. \tag{3.3}$$

Denote that $\Psi(u, \lambda) := \lambda \sum_{r_i < r} A_i + K_\lambda(u)$.

We know that (1.1)–(1.4) has the same solutions of

$$u = \Psi(u, 1) = \sum_{r_i < r} A_i + K_1(u). \quad (3.4)$$

Since f is Caratheodory, it is easy to see that $N_f(\cdot)$ is continuous and sends bounded sets into equiintegrable sets. According to Lemma 2.2, we can conclude that Ψ is compact continuous on $\overline{\Omega}$ for any $\lambda \in [0, 1]$. We assume that for $\lambda = 1$ (3.4) does not have a solution on $\partial\Omega$; otherwise, we complete the proof. Now from hypothesis (2^0) , it follows that (3.4) has no solution for $(u, \lambda) \in \partial\Omega \times (0, 1]$.

When $\lambda = 0$, (3.1) is equivalent to the following usual problem:

$$-(\varphi(r, u'))' = 0, \quad u'(0) = u(0) = 0, \quad (3.5)$$

where obviously 0 is the unique solution to this problem.

Since $0 \in \Omega$, we have proved that (3.4) has no solution (u, λ) on $\partial\Omega \times [0, 1]$, then we get that, for each $\lambda \in [0, 1]$, Leray-Schauder's degree $d_{\text{LS}}[I - \Psi(\cdot, \lambda), \Omega, 0]$ is well defined. From the homotopy invariant property of that degree, we have

$$d_{\text{LS}}[I - \Psi(u, 1), \Omega, 0] = d_{\text{LS}}[I - \Psi(u, 0), \Omega, 0] = 1. \quad (3.6)$$

This completes the proof. \square

In the following, we will give an application of Theorem 3.1.

Denote that $\sigma = 4T/(4T + 1)$ and

$$\Omega_\varepsilon = \left\{ u \in \text{PC}^1 \mid \max_{1 \leq j \leq N} \left(|u^j|_0 + \left| (u^j)' \right|_0 \right) < \varepsilon \right\}. \quad (3.7)$$

Obviously, Ω_ε is an open subset of PC^1 .

Assume that

(H₁) $f(r, u, v) = \delta g(r, u, v)$, where δ is a positive parameter, and g is Caratheodory.

(H₂) $\sum_{i=1}^k |A_i(u, v)| \leq (\sigma/2)\varepsilon$, $\sum_{i=1}^k |B_i(u, v)| \leq \{\min_{r \in I} |\varepsilon / (4T + 1)|^{p(r)} / 2(1 + |N\varepsilon|^{q(r)p(r)})^{1/q(r)}, 1/2\}$, for all $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$.

Theorem 3.2. *If (H₁)–(H₂) are satisfied, then problem (1.1)–(1.4) has at least one solution on $\overline{\Omega_\varepsilon}$, when positive parameter δ is small enough.*

Proof. Let one consider the problem

$$u = \Psi(u, \lambda) = \lambda \sum_{r_i < r} A_i + K_\lambda(u), \quad (3.8)$$

where $K_\lambda(\cdot)$ is defined in (3.3).

Obviously, u is a solution of (1.1)–(1.4) if and only if u is a solution of the abstract equation (3.8) when $\lambda = 1$. We only need to prove that the conditions of Theorem 3.1 are satisfied.

(1⁰) When positive parameter δ is small enough, for any $u \in \overline{\Omega_\varepsilon}$, we can see that the mapping $r \mapsto \delta g(r, u, u')$ belongs to $\{u \in L^1 \mid \|u\|_{L^1} < 1/3\}$.

(2⁰) We shall prove that for each $\lambda \in (0, 1)$ the problem

$$\begin{aligned}
 (\varphi(r, u'))' &= \lambda f(r, u, u'), \quad r \in (0, T), \quad r \neq r_i, \\
 \lim_{r \rightarrow r_i^+} u(r) - \lim_{r \rightarrow r_i^-} u(r) &= \lambda A_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right), \quad i = 1, \dots, k, \\
 \lim_{r \rightarrow r_i^+} \varphi(r, u'(r)) - \lim_{r \rightarrow r_i^-} \varphi(r, u'(r)) &= \lambda B_i \left(\lim_{r \rightarrow r_i^-} u(r), \lim_{r \rightarrow r_i^-} u'(r) \right), \quad i = 1, \dots, k,
 \end{aligned}
 \tag{3.9}$$

$$u'(0) = u(0) = 0$$

has no solution on $\partial\Omega_\varepsilon$.

If it is false, then there exists a $\lambda \in (0, 1)$ and $u \in \partial\Omega_\varepsilon$ is a solution of (3.8). Then there exists an $j \in \{1, \dots, N\}$ such that $|u^j|_0 + |(u^j)'|_0 = \varepsilon$.

(i) Suppose that $|u^j|_0 \geq \sigma\varepsilon$, then $|(u^j)'|_0 \leq (1 - \sigma)\varepsilon$. For any $r \in J$, since $u(0) = 0$, according to (H₂) and (1.2), we have

$$\begin{aligned}
 |u^j(r)| &= |u^j(r) - u^j(0)| = \left| \int_0^r (u^j)'(t) dt + \sum_{0 < r_i < r} A_i \right| \\
 &\leq \int_0^r |(u^j)'(t)| dt + \sum_{0 < r_i < r} |A_i| \\
 &\leq \int_0^r (1 - \sigma)\varepsilon dr + \sum_{i=1}^k |A_i| \\
 &\leq \frac{\sigma}{4}\varepsilon + \frac{\sigma}{2}\varepsilon = \frac{3\sigma}{4}\varepsilon.
 \end{aligned}
 \tag{3.10}$$

It is a contradiction.

(ii) Suppose that $|u^j|_0 < \sigma\varepsilon$, $(1 - \sigma)\varepsilon < |(u^j)'|_0 \leq \varepsilon$. This implies that $|(u^j)'(r_*)| > (1 - \sigma)\varepsilon = (1/(4T + 1))\varepsilon$ for some $r_* \in J$.

Denote that

$$\varphi^j(r, u')(r) = \frac{|u'|^{p(r)-1} (u^j)'(r)}{(1 + |u'|^{q(r)p(r)})^{1/q(r)}}.
 \tag{3.11}$$

Since $u'(0) = 0$, we have

$$\varphi^j(r_*, u')(r_*) = \lambda \int_0^{r_*} f^j(r, u, u') dr + \lambda \sum_{0 < r_i < r_*} B_i^j. \quad (3.12)$$

According to (H₁)-(H₂), when positive parameter δ is small enough, we have

$$\begin{aligned} \frac{|\varepsilon/(4T+1)|^{p(r_*)}}{(1+|N\varepsilon|^{q(r_*)p(r_*)})^{1/q(r_*)}} &\leq \left| \varphi^j(r_*, u')(r_*) \right| \leq \lambda \int_0^{r_*} |f^j(r, u, u')| dr + \lambda \left| \sum_{0 < r_i < r_*} B_i^j \right| \\ &\leq \delta \int_0^T \beta_{N\varepsilon}(r) dr + \sum_{i=1}^k |B_i| < \frac{|\varepsilon/(4T+1)|^{p(r_*)}}{(1+|N\varepsilon|^{q(r_*)p(r_*)})^{1/q(r_*)}}. \end{aligned} \quad (3.13)$$

It is a contradiction.

Summarizing this argument, for each $\lambda \in (0, 1)$, the problem (3.8) with (1.4) has no solution on $\partial\Omega_\varepsilon$.

Since $0 \in \Omega_\varepsilon$ and 0 is the unique solution of $u = \Psi(u, 0)$, then the Leray-Schauder's degree

$$d_{\text{LS}}[I - \Psi(\cdot, 0), \Omega, 0] = 1 \neq 0. \quad (3.14)$$

This completes the proof. \square

In the following, we will discuss the existence of nonnegative solutions of (1.1)–(1.4). For any $x = \{x^1, \dots, x^N\} \in \mathbb{R}^N$, the notation $x \geq 0$ ($x > 0$) means that $x^l \geq 0$ ($x^l > 0$) for every $l \in \{1, \dots, N\}$.

Assume the following

(H₃) $f(r, u, v) = \delta g(r, u, v)$, where δ is a positive parameter, and

$$g(r, u, v) = \tau(r) \left[|u|^{q_1(r)-1} u + \mu(r) |v|^{q_2(r)-1} v \right] + \gamma(r), \quad (3.15)$$

where $q_1, q_2 \in C(J, \mathbb{R})$, $0 \leq q_1(r)$, and $0 \leq q_2(r)$, for all $r \in J$.

(H₄) $A_i^j(x, y) y^j \geq 0$, and $B_i^j(x, y) y^j \geq 0$, for all $x, y \in \mathbb{R}^N$, $i = 1, \dots, k$, $j = 1, \dots, N$.

(H₅) $\mu, \tau \in C(J, \mathbb{R}^+)$.

(H₆) $\gamma = (\gamma^1, \dots, \gamma^N) \in C(J, \mathbb{R}^N)$ satisfies $\gamma(0) > 0$, $\int_0^t \gamma(s) ds \geq 0$, for all $t \in [0, T]$.

Theorem 3.3. *If (H₂)–(H₆) are satisfied, then the problem (1.1)–(1.4) has a nonnegative solution, when positive parameter δ is small enough.*

Proof. From Theorem 3.2, we can get the existence of solutions of (1.1)–(1.4). If u is a solution of (1.1)–(1.4), according to (1.4) and (H_6) , then we have

$$f(0, u(0), u'(0)) = \delta\gamma(0) > 0. \quad (3.16)$$

Obviously

$$\varphi(r, u')(r) = \int_0^r \delta \left\{ \tau(r) \left[|u|^{q_1(r)-1} u + \mu(r) |u'|^{q_2(r)-1} u' \right] + \gamma(r) \right\} ds, \quad \forall r \in [0, r_1]. \quad (3.17)$$

When $r \rightarrow 0^+$, we have

$$\tau(r) \left[|u|^{q_1(r)-1} u + \mu(r) |u'|^{q_2(r)-1} u' \right] + \gamma(r) > 0, \quad (3.18)$$

then we can see that there exists a $\xi \in (0, r_1)$ such that $\varphi(r, u')(r) > 0$ when $r \in (0, \xi)$. Thus $u'(r) > 0$ for any $r \in (0, \xi)$. Thus $u(r)$ is increasing in $(0, \xi)$, that is, $u(\eta_2) \geq u(\eta_1)$ for any $\eta_1, \eta_2 \in (0, \xi)$ with $\eta_1 < \eta_2$. Since $u(0) = 0$, it is easy to see that $u(r) > 0$ for any $r \in (0, \xi)$. From (3.17) and (H_5) , we can easily see that

$$\begin{aligned} u(r) > 0, \quad u'(r) > 0, \quad \forall r \in (0, r_1), \\ \lim_{r \rightarrow r_1^-} u(r) > 0, \quad \lim_{r \rightarrow r_1^-} u'(r) > 0. \end{aligned} \quad (3.19)$$

From (H_4) , we can see that

$$\lim_{r \rightarrow r_1^+} u(r) > 0, \quad \lim_{r \rightarrow r_1^+} u'(r) > 0. \quad (3.20)$$

Similarly, we can see that

$$u(r) > 0, \quad u'(r) > 0, \quad \forall r \in (r_1, r_2). \quad (3.21)$$

Repeating the step, we can see that

$$u(r) > 0, \quad u'(r) > 0, \quad \forall r \in J'. \quad (3.22)$$

Hence u is nonnegative. This completes the proof. \square

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