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Research Article

Existence of Solutions for η-Generalized Vector Variational-Like Inequalities

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We introduce and study a class of η-generalized vector variational-like inequalities and a class of η-generalized strong vector variational-like inequalities in the setting of Hausdorff topological vector spaces. An equivalence result concerned with two classes of η-generalized vector variational-like inequalities is proved under suitable conditions. By using FKKM theorem, some new existence results of solutions for the η-generalized vector variational-like inequalities and η-generalized strong vector variational-like inequalities are obtained under some suitable conditions.

1. Introduction

Vector variational inequality was first introduced and studied by Giannessi [1] in the setting of finite-dimensional Euclidean spaces. Since then, the theory with applications for vector variational inequalities, vector complementarity problems, vector equilibrium problems, and vector optimization problems have been studied and generalized by many authors (see, e.g., [2–15] and the references therein).

Recently, Yu et al. [16] considered a more general form of weak vector variational inequalities and proved some new results on the existence of solutions of the new class of weak vector variational inequalities in the setting of Hausdorff topological vector spaces.

Very recently, Ahmad and Khan [17] introduced and considered weak vector variational-like inequalities with η-generally convex mapping and gave some existence results.

On the other hand, Fang and Huang [18] studied some existence results of solutions for a class of strong vector variational inequalities in Banach spaces, which give a positive answer to an open problem proposed by Chen and Hou [19].

In 2008, Lee et al. [20] introduced a new class of strong vector variational-type inequalities in Banach spaces. They obtained the existence theorems of solutions for the inequalities without monotonicity in Banach spaces by using Brouwer fixed point theorem and Browder fixed point theorem.

Motivated and inspired by the work mentioned above, in this paper we introduce and study a class of η-generalized vector variational-like inequalities and a class of η-generalized strong vector variational-like inequalities in the setting of Hausdorff topological vector spaces. We first show an equivalence theorem concerned with two classes of η-generalized vector variational-like inequalities under suitable conditions. By using FKKM theorem, we prove some new existence results of solutions for the η-generalized vector variational-like inequalities and η-generalized strong vector variational-like inequalities under some suitable conditions. The results presented in this paper improve and generalize some known results due to Ahmad and Khan [17], Lee et al. [20], and Yu et al. [16].

2. Preliminaries

Let X and Y be two real Hausdorff topological vector spaces, $K \subset X$ a nonempty, closed, and convex subset, and $C \subset Y$ a closed, convex, and pointed cone with apex at the origin. Recall that the Hausdorff topological vector space Y is said to an ordered Hausdorff topological vector space denoted by (Y, C) if ordering relations are defined in Y as follows:

$$\begin{aligned} &\forall x, y \in Y, \quad x \leq y \Longleftrightarrow y - x \in C, \\ &\forall x, y \in Y, \quad x \nleq y \Longleftrightarrow y - x \notin C. \end{aligned} \tag{2.1}$$

If the interior $\operatorname{int} C$ is nonempty, then the weak ordering relations in Y are defined as follows:

$$\begin{aligned} &\forall x, y \in Y, \quad x < y \Longleftrightarrow y - x \in \operatorname{int} C, \\ &\forall x, y \in Y, \quad x \nless y \Longleftrightarrow y - x \notin \operatorname{int} C. \end{aligned} \tag{2.2}$$

Let $L(X, Y)$ be the space of all continuous linear maps from X to Y and $T : X \to L(X, Y)$. We denote the value of $l \in L(X, Y)$ on $x \in X$ by (l, x). Throughout this paper, we assume that $C(x) : x \in K$ is a family of closed, convex, and pointed cones of Y such that $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$, η is a mapping from $K \times K$ into X, and f is a mapping from $K \times K$ into Y.

In this paper, we consider the following two kinds of vector variational inequalities:

η-Generalized Vector Variational-Like Inequality (for short, η-GVVLI): for each $z \in K$ and $\lambda \in (0, 1]$, find $x \in K$ such that

$$\langle T(\lambda x + (1 - \lambda) z), \eta(y, x) \rangle + f(y, x) \notin -\operatorname{int} C(x), \quad \forall y \in K, \tag{2.3}$$

η-Generalized Strong Vector Variational-Like Inequality (for short, η-GSVVLI): for each $z \in K$ and $\lambda \in (0, 1]$, find $x \in K$ such that

$$\langle T(\lambda x + (1 - \lambda) z), \eta(y, x) \rangle + f(y, x) \notin -C(x) \setminus \{0\}, \quad \forall y \in K. \tag{2.4}$$

η-GVVLI and η-GSVVLI encompass many models of variational inequalities. For example, the following problems are the special cases of η-GVVLI and η-GSVVLI.

(1) If $f(y,x) = 0$ and $C(x) = C$ for all $x, y \in K$, then η-GVVLI reduces to finding $x \in K$, such that for each $z \in K$, $\lambda \in (0,1]$,

$$\langle T(\lambda x + (1-\lambda)z), \eta(y,x)\rangle \notin -\operatorname{int} C, \quad \forall y \in K, \tag{2.5}$$

which is introduced and studied by Ahmad and Khan [17]. In addition, if $\eta(y,x) = y - x$ for each $x, y \in K$, then η-GVVLI reduces to the following model studied by Yu et al. [16].

Find $x \in K$ such that for each $z \in K$, $\lambda \in (0,1]$,

$$\langle T(\lambda x + (1-\lambda)z), y - x\rangle \notin -\operatorname{int} C, \quad \forall y \in K. \tag{2.6}$$

(2) If $\lambda = 1$ and $C(x) = C$ for all $x \in K$, then η-GSVVLI is equivalent to the following vector variational inequality problem introduced and studied by Lee et al. [20].

Find $x \in K$ satisfying

$$\langle T(x), \eta(y,x)\rangle + f(y,x) \notin -C \setminus \{0\}, \quad \forall y \in K. \tag{2.7}$$

For our main results, we need the following definitions and lemmas.

Definition 2.1. Let $T : K \to L(X,Y)$ and $\eta : K \times K \to K$ be two mappings and $C = \bigcap_{x\in K} C(x) \neq \emptyset$. T is said to be η-monotone in C if and only if

$$\langle T(x) - T(y), \eta(x,y)\rangle \in C, \quad \forall x, y \in K. \tag{2.8}$$

Definition 2.2. Let $T : K \to L(X,Y)$ and $\eta : K \times K \to K$ be two mappings. We say that T is η-hemicontinuous if, for any given $x, y, z \in K$ and $\lambda \in (0,1]$, the mapping $t \mapsto \langle T(\lambda(x + (1-t)(y-x)) + (1-\lambda)z), \eta(y,x)\rangle$ is continuous at 0^+.

Definition 2.3. A multivalued mapping $A : X \to 2^Y$ is said to be upper semicontinuous on X if, for all $x \in X$ and for each open set G in Y with $A(x) \subset G$, there exists an open neighbourhood $O(x)$ of $x \in X$ such that $A(x') \subset G$ for all $x' \in O(x)$.

Lemma 2.4 (see [21]). *Let (Y,C) be an ordered topological vector space with a closed, pointed, and convex cone C with $\operatorname{int} C \neq \emptyset$. Then for any $y, z \in Y$, we have*

(1) $y - z \in \operatorname{int} C$ *and* $y \notin \operatorname{int} C$ *imply* $z \notin \operatorname{int} C$;

(2) $y - z \in C$ *and* $y \notin \operatorname{int} C$ *imply* $z \notin \operatorname{int} C$;

(3) $y - z \in -\operatorname{int} C$ *and* $y \notin -\operatorname{int} C$ *imply* $z \notin -\operatorname{int} C$;

(4) $y - z \in -C$ *and* $y \notin -\operatorname{int} C$ *imply* $z \notin -\operatorname{int} C$.

Lemma 2.5 (see [22]). *Let M be a nonempty, closed, and convex subset of a Hausdorff topological space, and $G : M \to 2^M$ a multivalued map. Suppose that for any finite set $\{x_1, \ldots, x_n\} \subset M$, one has $\operatorname{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$ (i.e., F is a KKM mapping) and $G(x)$ is closed for each $x \in M$ and compact for some $x \in M$, where conv denotes the convex hull operator. Then $\bigcap_{x\in M} G(x) \neq \emptyset$.*

Lemma 2.6 (see [23]). *Let X be a Hausdorff topological space, $A_1, A_2, \ldots, A_n$ be nonempty compact convex subsets of X. Then $\operatorname{conv}(\bigcup_{i=1}^{n} A_i)$ is compact.*

Lemma 2.7 (see [24]). *Let X and Y be two topological spaces. If $A : X \to 2^Y$ is upper semicontinuous with closed values, then A is closed.*

3. Main Results

Theorem 3.1. *Let X be a Hausdorff topological linear space, $K \subset X$ a nonempty, closed, and convex subset, and $(Y, C(x))$ an ordered topological vector space with $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$. Let $\eta : K \times K \to X$ and $f : K \times K \to X$ be affine mappings such that $\eta(x,x) = f(x,x) = 0$ for each $x \in K$. Let $T : K \to L(X,Y)$ be an η-hemicontinuous mapping. If $C = \bigcap_{x \in K} C(x) \neq \emptyset$ and T is η-monotone in C, then for each $z \in K$, $\lambda \in (0,1]$, the following statements are equivalent*

(i) *find $x_0 \in K$, such that $\langle T_z(x_0), \eta(y,x_0)\rangle + f(y,x_0) \notin -\operatorname{int} C(x_0)$, for all $y \in K$;*

(ii) *find $x_0 \in K$, such that $\langle T_z(y), \eta(y,x_0)\rangle + f(y,x_0) \notin -\operatorname{int} C(x_0)$, for all $y \in K$,*

where T_z is defined by $T_z(x) = T(\lambda x + (1-\lambda)z)$ for all $x \in K$.

Proof. Suppose that (i) holds. We can find $x_0 \in K$, such that

$$\langle T_z(x_0), \eta(y,x_0)\rangle + f(y,x_0) \notin -\operatorname{int} C(x_0), \quad \forall y \in K. \tag{3.1}$$

Since T is η-monotone, for each $x, y \in K$, we have

$$\langle T(\lambda y + (1-\lambda)z) - T(\lambda x + (1-\lambda)z), \eta(\lambda y + (1-\lambda)z, \lambda x + (1-\lambda)z)\rangle \in C. \tag{3.2}$$

On the other hand, we know η is affine and $\eta(x,x) = 0$. It follows that

$$\begin{aligned} &\langle T_z(y) - T_z(x), \eta(y,x)\rangle \\ &\quad = \frac{1}{\lambda}\langle T(\lambda y + (1-\lambda)z) - T(\lambda x + (1-\lambda)z), \eta(\lambda y + (1-\lambda)z, \lambda x + (1-\lambda)z)\rangle \in C. \end{aligned} \tag{3.3}$$

Hence T_z is also η-monotone. That is

$$\langle T_z(x_0), \eta(y,x_0)\rangle - \langle T_z(y), \eta(y,x_0)\rangle \in -C, \quad \forall y \in K. \tag{3.4}$$

Since $C = \bigcap_{x \in K} C(x)$, for all $y \in K$, we obtain

$$\langle T_z(x_0), \eta(y,x_0)\rangle + f(y,x_0) - \langle T_z(y), \eta(y,x_0)\rangle - f(y,x_0) \in -C \subset -C(x_0). \tag{3.5}$$

By Lemma 2.4,

$$\langle T_z(y), \eta(y,x_0)\rangle + f(y,x_0) \notin -\operatorname{int} C(x_0), \quad \forall y \in K, \tag{3.6}$$

and so x_0 is a solution of (ii).

Conversely, suppose that (ii) holds. Then there exists $x_0 \in K$ such that

$$\langle T_z(y), \eta(y, x_0)\rangle + f(y, x_0) \notin -\operatorname{int} C(x_0), \quad \forall y \in K. \tag{3.7}$$

For each $y \in K$, $t \in (0,1)$, we let $y_t = ty + (1-t)x_0$. Obviously, $y_t \in K$. It follows that

$$\langle T_z(y_t), \eta(y_t, x_0)\rangle + f(y_t, x_0) \notin -\operatorname{int} C(x_0). \tag{3.8}$$

Since f and η are affine and $\eta(x_0, x_0) = f(x_0, x_0) = 0$, we have

$$\langle T(\lambda(ty + (1-t)x_0) + (1-\lambda)z), t\eta(y, x_0)\rangle + tf(y, x_0) \notin -\operatorname{int} C(x_0). \tag{3.9}$$

That is

$$\langle T(\lambda(x_0 + t(y - x_0)) + (1-\lambda)z), \eta(y, x_0)\rangle + f(y, x_0) \notin -\operatorname{int} C(x_0). \tag{3.10}$$

Considering the η-hemicontinuity of T and letting $t \to 0^+$, we have

$$\langle T_z(x_0), \eta(y, x_0)\rangle + f(y, x_0) \notin -\operatorname{int} C(x_0), \quad \forall y \in K. \tag{3.11}$$

This completes the proof. □

Remark 3.2. If $C(x) = C$ and $f(y, x) = 0$ for all $x, y \in K$, then Theorem 3.1 is reduced to Lemma 5 of [17].

Let K be a closed convex subset of a topological linear space X, and $\{C(x) : x \in K\}$ a family of closed, convex, and pointed cones of a topological space Y such that $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$. Throughout this paper, we define a set-valued mapping $\overline{C} : K \to 2^Y$ as follows:

$$\overline{C}(x) = Y \setminus \{-\operatorname{int} C(x)\}, \quad \forall x \in K. \tag{3.12}$$

Theorem 3.3. *Let X be a Hausdorff topological linear space, $K \subset X$ a nonempty, closed, compact, and convex subset, and $(Y, C(x))$ an ordered topological vector space with $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$. Let $\eta : K \times K \to X$ and $f : K \times K \to X$ be affine mappings such that $\eta(x, x) = f(x, x) = 0$ for each $x \in K$. Let $T : K \to L(X, Y)$ be an η-hemicontinuous mapping. Assume that the following conditions are satisfied*

(i) $C = \bigcap_{x \in K} C(x) \neq \emptyset$ *and T is η-monotone in C;*

(ii) $\overline{C} : K \to 2^Y$ *is an upper semicontinuous set-valued mapping.*

Then for each $z \in K$, $\lambda \in (0,1]$, there exist $x_0 \in K$ such that

$$\langle T(\lambda x_0 + (1-\lambda)z), \eta(y, x_0)\rangle + f(y, x_0) \notin -\operatorname{int} C(x_0), \quad \forall y \in K. \tag{3.13}$$

Proof. For each $y \in K$, we denote $T_z(x) = T(\lambda x + (1-\lambda)z)$, and define

$$\begin{aligned} F_1(y) &= \{x \in K : \langle T_z(x), \eta(y,x)\rangle + f(y,x) \notin -\operatorname{int} C(x)\}, \\ F_2(y) &= \{x \in K : \langle T_z(y), \eta(y,x)\rangle + f(y,x) \notin -\operatorname{int} C(x)\}. \end{aligned} \tag{3.14}$$

Then $F_1(y)$ and $F_2(y)$ are nonempty since $y \in F_1(y)$ and $y \in F_2(y)$. The proof is divided into the following three steps.

(I) First, we prove the following conclusion: F_1 is a KKM mapping. Indeed, assume that F_1 is not a KKM mapping; then there exist $u_1, u_2, \dots, u_m \in K$, $t_1 \geq 0, t_2 \geq 0, \dots, t_m \geq 0$ with $\sum_{i=1}^m t_i = 1$ and $w = \sum_{i=1}^m t_i u_i$ such that

$$w \notin \bigcup_{i=1}^{m} F_1(u_i), \quad i = 1,2,\dots,m. \tag{3.15}$$

That is,

$$\forall i = 1,2,\dots,m, \quad \langle T_z(w), \eta(u_i,w)\rangle + f(u_i,w) \in -\operatorname{int} C(w). \tag{3.16}$$

Since η and f are affine, we have

$$\begin{aligned} \langle T_z(w), \eta(w,w)\rangle + f(w,w) &= \left\langle T_z(w), \eta\left(\sum_{i=1}^m t_i u_i, w\right)\right\rangle + f\left(\sum_{i=1}^m t_i u_i, w\right) \\ &= \sum_{i=1}^m t_i\left(\langle T_z(w), \eta(u_i,w)\rangle + f(u_i,w)\right) \in -\operatorname{int} C(w). \end{aligned} \tag{3.17}$$

On the other hand, we know $\eta(w,w) = f(w,w) = 0$. Then we have $0 = \langle T_z(w), \eta(w,w)\rangle + f(w,w) \in -\operatorname{int} C(w)$. It is impossible and so $F_1 : K \to 2^K$ is a KKM mapping.

(II) Further, we prove that

$$\bigcap_{y \in K} F_1(y) = \bigcap_{y \in K} F_2(y). \tag{3.18}$$

In fact, if $x \in F_1(y)$, then $\langle T_z(x), \eta(y,x)\rangle + f(y,x) \notin \operatorname{int} C(x)$. From the proof of Theorem 3.1, we know that T_z is η-monotone in $C(z)$. It follows that

$$\langle T_z(y) - T_z(x), \eta(y,x)\rangle \in C, \tag{3.19}$$

and so

$$\langle T_z(x), \eta(y,x)\rangle + f(y,x) - \langle T_z(y), \eta(y,x)\rangle - f(y,x) \in -C \subset -C(x). \tag{3.20}$$

By Lemma 2.4, we have

$$\langle T_z(y), \eta(y,x) \rangle + f(y,x) \notin -\operatorname{int} C(x), \tag{3.21}$$

and so $x \in F_2(y)$ for each $y \in K$. That is, $F_1(y) \subset F_2(y)$ and so

$$\bigcap_{y\in K} F_1(y) \subset \bigcap_{y\in K} F_2(y). \tag{3.22}$$

Conversely, suppose that $x \in \bigcap_{y\in K} F_2(y)$. Then

$$\langle T_z(y), \eta(y,x) \rangle + f(y,x) \notin -\operatorname{int} C(x), \quad \forall y \in K. \tag{3.23}$$

It follows from Theorem 3.1 that

$$\langle T_z(x), \eta(y,x) \rangle + f(y,x) \notin -\operatorname{int} C(x), \quad \forall y \in K. \tag{3.24}$$

That is, $x \in \bigcap_{y\in K} F_1(y)$ and so

$$\bigcap_{y\in K} F_2(y) \subset \bigcap_{y\in K} F_1(y), \tag{3.25}$$

which implies that

$$\bigcap_{y\in K} F_1(y) = \bigcap_{y\in K} F_2(y). \tag{3.26}$$

(III) Last, we prove that $\bigcap_{y\in K} F_2(y) \neq \emptyset$. Indeed, since F_1 is a KKM mapping, we know that, for any finite set $\{y_1, y_2, \ldots, y_n\} \subset K$, one has

$$\operatorname{conv}\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^{n} F_1(y_i) \subset \bigcup_{i=1}^{n} F_2(y_i). \tag{3.27}$$

This shows that F_2 is also a KKM mapping.

Now, we prove that $F_2(y)$ is closed for all $y \in K$. Assume that there exists a net $\{x_\alpha\} \subset F_2(y)$ with $x_\alpha \to x \in K$. Then

$$\langle T_z(y), \eta(y,x_\alpha) \rangle + f(y,x_\alpha) \notin -\operatorname{int} C(x_\alpha). \tag{3.28}$$

Using the definition of $\overline{C}$, we have

$$\langle T_z(y), \eta(y,x_\alpha) \rangle + f(y,x_\alpha) \in \overline{C}(x_\alpha). \tag{3.29}$$

Since η and f are continuous, it follows that

$$\langle T_z(y), \eta(y, x_\alpha)\rangle + f(y, x_\alpha) \longrightarrow \langle T_z(y), \eta(y, x)\rangle + f(y, x). \tag{3.30}$$

Since $\overline{C}$ is upper semicontinuous mapping with close values, by Lemma 2.7, we know that $\overline{C}$ is closed, and so

$$\langle T_z(y), \eta(y, x)\rangle + f(y, x) \in \overline{C}(x). \tag{3.31}$$

This implies that

$$\langle T_z(y), \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} C(x), \tag{3.32}$$

and so $F_2(y)$ is closed. Considering the compactness of K and closeness of $F_2(y) \subset K$, we know that $F_2(y)$ is compact. By Lemma 2.5, we have $\bigcap_{y\in K} F_2(y) \neq \emptyset$, and it follows that $\bigcap_{y\in K} F_1(y) \neq \emptyset$, that is, for each $z \in K$ and $\lambda \in (0,1]$, there exists $x_0 \in K$ such that

$$\langle T(\lambda x_0 + (1-\lambda)z), \eta(y, x_0)\rangle + f(y, x_0) \notin -\operatorname{int} C(x_0), \quad \forall y \in K. \tag{3.33}$$

Thus, η-GVVLI is solvable. This completes the proof. □

Remark 3.4. The condition (ii) in Theorem 3.3 can be found in several papers (see, e.g., [25, 26]).

Remark 3.5. If $C(x) = C$ and $f(y, x) = 0$ for all $x, y \in K$ in Theorem 3.3, then condition (ii) holds and condition (i) is equivalent to the η-monotonicity of T. Thus, it is easy to see that Theorem 3.3 is a generalization of [17, Theorem 6].

In the above theorem, K is compact. In the following theorem, under some suitable conditions, we prove a new existence result of solutions for η-GVVLI without the compactness of K.

Theorem 3.6. *Let X be a Hausdorff topological linear space, $K \subset X$ a nonempty, closed, and convex subset, and $(Y, C(x))$ be an ordered topological vector space with $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$. Let $\eta : K \times K \to X$ and $f : K \times K \to X$ be affine mappings such that $\eta(x, x) = f(x, x) = 0$ for each $x \in K$. Let $T : K \to L(X, Y)$ be an η-hemicontinuous mapping. Assume that the following conditions are satisfied:*

(i) *$C =\subset \bigcap_{x\in K} C(x) \neq \emptyset$ and T is η-monotone in C;*

(ii) *$\overline{C} : K \to 2^Y$ is an upper semicontinuous set-valued mapping;*

(iii) *there exists a nonempty compact and convex subset D of K and for each $z \in K$, $\lambda \in (0,1]$, $x \in K \setminus D$, there exist $y_0 \in D$ such that*

$$\langle T(\lambda y_0 + (1-\lambda)z), \eta(y_0, x)\rangle + f(y_0, x) \in -\operatorname{int} C(y_0). \tag{3.34}$$

Then for each $z \in K$, $\lambda \in (0,1]$, there exist $x_0 \in D$ such that

$$\langle T(\lambda x_0 + (1-\lambda)z), \eta(y, x_0)\rangle + f(y, x_0) \notin -\operatorname{int} C(x_0), \quad \forall y \in K. \tag{3.35}$$

Proof. By Theorem 3.1, we know that the solution set of the problem (ii) in Theorem 3.1 is equivalent to the solution set of following variational inequality: find $x \in K$, such that

$$\langle T(\lambda y + (1-\lambda)z), \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} C(x), \quad \forall y \in K. \tag{3.36}$$

For each $z \in K$ and $\lambda \in (0,1]$, we denote $T_z(x) = T(\lambda x + (1-\lambda)z)$. Let $G : K \to 2^D$ be defined as follows:

$$G(y) = \{x \in D : \langle T_z(y), \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} C(x)\}, \quad \forall y \in K. \tag{3.37}$$

Obviously, for each $y \in K$,

$$G(y) = \{x \in K : \langle T_z(y), \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} C(x)\} \cap D. \tag{3.38}$$

Using the proof of Theorem 3.3, we obtain that $G(y)$ is a closed subset of D. Considering the compactness of D and closedness of $G(y)$, we know that $G(y)$ is compact.

Now we prove that for any finite set $\{y_1, y_2, \ldots, y_n\} \subset K$, one has $\bigcap_{i=1}^n G(y_i) \neq \emptyset$. Let $Y_n = \bigcup_{i=1}^n \{y_i\}$. Since Y is a real Hausdorff topological vector space, for each $y_i \in \{y_1, y_2, \ldots, y_n\}$, $\{y_i\}$ is compact and convex. Let $N = \operatorname{conv}(D \cup Y_n)$. By Lemma 2.6, we know that N is a compact and convex subset of K.

Let $F_1, F_2 : N \to 2^N$ be defined as follows:

$$\begin{aligned} F_1(y) &= \{x \in N : \langle T_z(x), \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} C(x)\}, \quad \forall y \in N; \\ F_2(y) &= \{x \in N : \langle T_z(y), \eta(y, x)\rangle + f(y, x) \notin -\operatorname{int} C(x)\}, \quad \forall y \in N. \end{aligned} \tag{3.39}$$

Using the proof of Theorem 3.3, we obtain

$$\bigcap_{y \in N} F_1(y) = \bigcap_{y \in N} F_2(y) \neq \emptyset, \tag{3.40}$$

and so there exists $y_0 \in \bigcap_{y \in N} F_2(y)$.

Next we prove that $y_0 \in D$. In fact, if $y_0 \in K \setminus D$, then the assumption implies that there exists $u \in D$ such that

$$\langle T(\lambda u + (1-\lambda)z), \eta(u, y_0)\rangle + f(u, y_0) \in -\operatorname{int} C(u), \tag{3.41}$$

which contradicts $y_0 \in F_2(u)$ and so $y_0 \in D$.

Since $\{y_1, y_2, \ldots, y_n\} \subset N$ and $G(y_i) = F_2(y_i) \cap D$ for each $y_i \in \{y_1, y_2, \ldots, y_n\}$, it follows that $y_0 \in \bigcap_{i=1}^n G(y_i)$. Thus, for any finite set $\{y_1, y_2, \ldots, y_n\} \subset K$, we have $\bigcap_{i=1}^n G(y_i) \neq \emptyset$.

Considering the compactness of $G(y)$ for each $y \in K$, we know that there exists $x_0 \in D$ such that $x_0 \in \bigcap_{y\in K} G(y) \neq \emptyset$. Therefore, the solution set of η-GVVLI is nonempty. This completes the proof. □

In the following, we prove the solvability of η-GSVVLI under some suitable conditions by using FKKM theorem.

Theorem 3.7. *Let X be a Hausdorff topological linear space, $K \subset X$ a nonempty, closed, and convex set, and $(Y, C(x))$ an ordered Hausdorff topological vector space with $\operatorname{int} C(x) \neq \emptyset$ for all $x \in K$. Assume that for each $y \in K$, $x \to \eta(x,y)$ and $x \to f(x,y)$ are affine, $\eta(x,y) + \eta(y,x) = 0$, and $f(x,y) + f(y,x) = 0$ for all $x \in K$. Let $T : K \to L(X,Y)$ be a mapping such that*

(i) *for each $z, y \in K$, $\lambda \in (0,1]$, the set $\{x \in K : \langle T(\lambda x + (1-\lambda)z), \eta(y,x)\rangle + f(y,x) \in -C(x) \setminus \{0\}\}$ is open in K;*

(ii) *there exists a nonempty compact and convex subset D of K and for each $z \in K$, $\lambda \in (0,1]$, $x \in K \setminus D$, there exists $u \in D$ such that*

$$\left\langle T(\lambda x + (1-\lambda)z), \eta(u,x)\right\rangle + f(y,x) \in -C(x) \setminus \{0\}. \tag{3.42}$$

Then for each $z \in K$, $\lambda \in (0,1]$, there exists $x_0 \in K$ such that

$$\left\langle T(\lambda x_0 + (1-\lambda)z), \eta(y,x_0)\right\rangle + f(y,x_0) \notin -C(x_0) \setminus \{0\}, \quad \forall y \in K. \tag{3.43}$$

Proof. For each $z \in K$ and $\lambda \in (0,1]$, we denote $T_z(x) = T(\lambda x + (1-\lambda)z)$. Let $G : K \to 2^D$ be defined as follows:

$$G(y) = \left\{x \in D : \left\langle T_z(x), \eta(y,x)\right\rangle + f(y,x) \notin -C(x) \setminus \{0\}\right\}, \quad \forall y \in K. \tag{3.44}$$

Obviously, for each $y \in K$,

$$G(y) = \left\{x \in K : \left\langle T_z(x), \eta(y,x)\right\rangle + f(y,x) \notin -C(x) \setminus \{0\}\right\} \cap D. \tag{3.45}$$

Since $G(y)$ is a closed subset of D, considering the compactness of D and closedness of $G(y)$, we know that $G(y)$ is compact.

Now we prove that for any finite set $\{y_1, y_2, \ldots, y_n\} \subset K$, one has $\bigcap_{i=1}^n G(y_i) \neq \emptyset$. Let $Y_n = \bigcup_{i=1}^n \{y_i\}$. Since Y is a real Hausdorff topological vector space, for each $y_i \in \{y_1, y_2, \ldots, y_n\}$, $\{y_i\}$ is compact and convex. Let $N = \operatorname{conv}(D \cup Y_n)$. By Lemma 2.6, we know that N is a compact and convex subset of K.

Let $F : N \to 2^N$ be defined as follows:

$$F(y) = \left\{x \in N : \left\langle T_z(x), \eta(y,x)\right\rangle + f(y,x) \notin -C(x) \setminus \{0\}\right\}, \quad \forall y \in N. \tag{3.46}$$

We claim that F is a KKM mapping. Indeed, assume that F is not a KKM mapping. Then there exist $u_1, u_2, \ldots, u_m \in K$, $t_1 \geq 0$, $t_2 \geq 0, \ldots, t_m \geq 0$ with $\sum_{i=1}^m t_i = 1$ and $w = \sum_{i=1}^m t_i u_i$ such that

$$w \notin \bigcup_{i=1}^m F(u_i), \quad i = 1, 2, \ldots, m. \tag{3.47}$$

That is,

$$\forall i = 1, 2, \ldots, m \quad \langle T_z(w), \eta(u_i, w) \rangle + f(u_i, w) \in -C(w) \setminus \{0\}. \tag{3.48}$$

Since η and f are affine, we have

$$\begin{aligned} \langle T_z(w), \eta(w, w) \rangle + f(w, w) &= \left\langle T_z(w), \eta\left(\sum_{i=1}^m t_i u_i, w\right)\right\rangle + f\left(\sum_{i=1}^m t_i u_i, w\right) \\ &= \sum_{i=1}^m t_i \left(\langle T_z(w), \eta(u_i, w) \rangle + f(u_i, w) \right) \in -C(w) \setminus \{0\}. \end{aligned} \tag{3.49}$$

On the other hand, we know $\eta(w, w) = f(w, w) = 0$, and so

$$0 = \langle T_z(w), \eta(w, w) \rangle + f(w, w) \in -C(w) \setminus \{0\}, \tag{3.50}$$

which is impossible. Therefore, $F : N \to 2^N$ is a KKM mapping.

Since $F(y)$ is a closed subset of N, it follows that $F(y)$ is compact. By Lemma 2.5, we have

$$\bigcap_{y \in N} F(y) \neq \emptyset. \tag{3.51}$$

Thus, there exists $y_0 \in \bigcap_{y \in N} F(y)$.

Next we prove that $y_0 \in D$. In fact, if $y_0 \in N \setminus D$, then the condition (ii) implies that there exists $u \in D$ such that

$$\langle T(\lambda y_0 + (1 - \lambda) z), \eta(u, y_0) \rangle + f(u, y_0) \in -C(y_0) \setminus \{0\}, \tag{3.52}$$

which contradicts $y_0 \in F(u)$ and so $y_0 \in D$.

Since $\{y_1, y_2, \ldots, y_n\} \subset N$ and $G(y_i) = F(y_i) \cap D$ for each $y_i \in \{y_1, y_2, \ldots, y_n\}$, it follows that $y_0 \in \bigcap_{i=1}^n G(y_i)$. Thus, for any finite set $\{y_1, y_2, \ldots, y_n\} \subset K$, we have $\bigcap_{i=1}^n G(y_i) \neq \emptyset$. Considering the compactness of $G(y)$ for each $y \in K$, it is easy to know that there exists $x_0 \in D$ such that $x_0 \in \bigcap_{y \in K} G(y) \neq \emptyset$. Therefore, for each $z \in K$, $\lambda \in (0, 1]$, there exists $x_0 \in K$ such that

$$\langle T(\lambda x_0 + (1 - \lambda) z), \eta(y, x_0) \rangle + f(y, x_0) \notin -C(x_0) \setminus \{0\}, \quad \forall y \in K. \tag{3.53}$$

Thus, η-GSVVI is solvable. This completes the proof. □

Remark 3.8. If K is compact, $C(x) = C$, and $\lambda = 1$, then Theorem 3.7 is reduced to Theorem 2.1 in [20].

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