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*Research Article*

# Riesz Potential on the Heisenberg Group

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The relation between Riesz potential and heat kernel on the Heisenberg group is studied. Moreover, the Hardy-Littlewood-Sobolev inequality is established.

## 1. Introduction

The classical Riesz potential $I_\alpha$ is defined on $R^n$ by

$$I_\alpha(f) = (-\Delta)^{-\alpha/2}(f), \quad 0 < \alpha < n, \tag{1.1}$$

where $\Delta$ is the Laplacian operator. By virtue of the equations

$$\begin{aligned} \left((-\Delta)^{-\alpha/2} f\right)^{\widehat{}}(x) &= (2\pi|x|)^{-\alpha}\widehat{f}(x), \\ \left(|\cdot|^{-n+\alpha}\right)^{\widehat{}}(y) &= \gamma(\alpha)(2\pi)^{-\alpha}|y|^{-\alpha}, \end{aligned} \tag{1.2}$$

where $\gamma(\alpha) = \pi^{n/2}2^\alpha\Gamma(\alpha/2)/\Gamma(n/2-\alpha/2)$, one can get the explicit expression of Riesz potential

$$I_\alpha(f)(x) = \frac{1}{\gamma(\alpha)}\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy. \tag{1.3}$$

In addition, one has the following Hardy-Littlewood-Sobolev theorem (see [1]).

**Theorem 1.1.** *Let $1 < \alpha < n$, $1 \le p < q < \infty$, $1/q = 1/p - \alpha/n$. One has the following.*

(a) *If $p > 1$, then $\|I_\alpha(f)\|_q \le A_{p,q}\|f\|_p$;*

(b) *if $f \in L^1(R^n)$, then for all $\lambda > 0$,*

$$m\{x \in R^n : |I_\alpha(f)| > \lambda\} \le \left(\frac{A\|f\|_1}{\lambda}\right)^q. \tag{1.4}$$

In recent years many interesting works about the Riesz potential have been done by many authors. Thangavelu and Xu [2] discussed the Riesz potential for the Dunkl transform. Garofalo and Tyson [3] proved superposition principle Riesz potentials of nonnegative continuous function on Lie groups of Heisenberg type. Huang and Liu [4] studied the Hardy-Littlewood-Sobolev inequality of this operator on the Laguerre hypergroup. For more results about the Riesz potential, we refer the readers to see [5–9].

It is a remarkable fact that the Heisenberg group, denoted by $H^n$, arises in two aspects. On the one hand, it can be realized as the boundary of the unit ball in several complex variables. On the other hand, an important aspect of the study of the Heisenberg group is the background of physics, namely, the mathematical ideas connected with the fundamental notions of quantum mechanics. In other words, there is its genesis in the context of quantum mechanics which emphasizes its symplectic role in the theory of theta functions and related parts of analysis. Due to this reason, many interesting works were devoted to the theory of harmonic analysis on $H^n$ in [10–15] and the references therein.

In present paper, we consider the Riesz potential associated with the Heisenberg group. We will show a connection between the Riesz potential and the heat kernel, and then get the Hardy-Littlewood-Sobolev inequality.

## 2. Preliminaries

The Heisenberg group $H^n$ is a Lie group with the underlying manifold $C^n \times R$, the multiplication law is

$$(z,t)(z',t') = \left(z + z', t + t' + \frac{1}{2}\operatorname{Im} z\overline{z'}\right), \tag{2.1}$$

where $z\overline{z'} = \sum_{j=1}^n z_j\overline{z'_j}$. The dilation of $H^n$ is defined by $\delta_a(z,t) = (az, a^2t)$ with $a > 0$. For $(z,t) \in H^n$, the homogeneous norm of $(z,t)$ is given by

$$(|z,t|) = \left|(z,t)^{-1}\right| = \left(|z|^4 + |t|^2\right)^{1/4}. \tag{2.2}$$

Note that $|\delta_a(z,t)| = (|az|^4 + |a^2t|^2)^{1/4} = a|(z,t)|$. In addition, $|\cdot|$ satisfies the quasi-triangle inequality

$$\left|(z,t)(z',t')\right| \le |(z,t)| + \left|(z',t')\right|. \tag{2.3}$$

The ball of radius $r$ centered at $(z,t)$ is given by

$$B_r(z,t) = \left\{ (z',t') \in H^n : \left| (z,t)^{-1}(z',t') \right| < r \right\}. \tag{2.4}$$

For $1 \le p \le \infty$, let $L^p(H^n)$ be the space of measurable functions $f$ on $H^n$, such that

$$\begin{aligned} \|f\|_p &= \left( \int_{H^n} |f(z,t)|^p dz\, dt \right)^{1/p} < \infty, \quad \text{if } p \in [1,\infty), \\ \|f\|_\infty &= \operatorname*{ess\,sup}_{(z,t)\in H^n} |f(z,t)| < \infty, \quad \text{if } p = \infty. \end{aligned} \tag{2.5}$$

Let $\pi_\lambda(z,t)$ $(z = x + iy,\ \lambda \in R^* = R/\{0\})$ be the Schrödinger representations which acts on $\varphi \in L^2(R^n)$ by

$$\pi_\lambda(z,t)\varphi(\zeta) = e^{i\lambda t} e^{i\lambda(x\cdot\zeta + (1/2)x\cdot y)} \varphi(\zeta + y), \tag{2.6}$$

where $x \cdot y = \sum_{j=1}^n x_j y_j$. Suppose that $f$ is a Schwartz function on $H^n$, that is, $f \in S(H^n)$. The Fourier transform of $f$ is defined by

$$\widehat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)dz\, dt. \tag{2.7}$$

This means that, for each $\varphi, \psi \in L^2(R^n)$,

$$\left( \widehat{f}(\lambda)\varphi, \psi \right) = \int_{H^n} f(z,t)\left(\pi_\lambda(z,t)\varphi, \psi\right) dz\, dt, \tag{2.8}$$

where $(,)$ denotes the inner product.

Let us write $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z)$ with $\pi_\lambda(z) = \pi_\lambda(z,0)$ and define

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z,t)e^{i\lambda t} dt. \tag{2.9}$$

Then (2.7) can be written as

$$\widehat{f}(\lambda) = \int_{C^n} f^\lambda(z)\pi_\lambda(z)dz. \tag{2.10}$$

If we set

$$W_\lambda(g) = \int_{C^n} g(z)\pi_\lambda(z)dz, \tag{2.11}$$

then $\widehat{f}(\lambda) = W_\lambda(f^\lambda)$. Let $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$; one has the inversion of Fourier transform

$$f(z,t) = \int_{-\infty}^{\infty} \operatorname{tr}\Big(\pi_\lambda^*(z,t)\widehat{f}(\lambda)\Big) d\mu(\lambda), \tag{2.12}$$

where $\pi_\lambda^*(z,t)$ denotes the adjoint of $\pi_\lambda(z,t)$.

The convolution of $f$ and $g$ is defined by

$$f * g(z,t) = \int_{H^n} f((z,t)(-w,-s))g(w,s)dw\, ds. \tag{2.13}$$

It is clear that $\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda)$. In addition, we have the generalized Yong inequality

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \tag{2.14}$$

where $(1/r) = (1/p) + (1/q) - 1$. More details about the harmonic analysis on Heisenberg group can be found in [14–16].

Let $T$ be a mapping from $L^p(H^n)$ to $L^q(H^n)$, $1 \leq p, q \leq \infty$. Then $T$ is of type $(p,q)$ if

$$\|T(f)\|_{L^q(H^n)} \leq A\|f\|_{L^p(H^n)}, \quad f \in L^p(H^n), \tag{2.15}$$

where $A$ does not depend on $f$. Similarly, $T$ is of weak type $(p,q)$ if

$$m\{(z,t) \in H^n : |T(f)(z,t)| > \lambda\} \leq \left(\frac{A\|f\|_p}{\lambda}\right)^q, \quad \text{when } q < \infty, \tag{2.16}$$

where $A$ does not depend on $f$ or $\lambda$ $(\lambda > 0)$.

Let $S^n$ be the unit sphere in $H^n$ and $\Sigma^{n-1}$ the unit Euclidean sphere in $R^n$. Suppose that $f$ is a measurable function on $H^n$, and we have (see [10])

$$\int_{H^n} f(z,t)dz\, dt = \int_{-\infty}^{\infty}\int_{\Sigma^{2n-1}}\int_0^{\infty} f(\rho\zeta,t)\rho^{2n-1} d\rho\, d\zeta\, dt. \tag{2.17}$$

We set $\rho^2 = r^2\cos\theta$, $t = r^2\sin\theta$, then

$$\int_{H^n} f(z,t)dz\, dt = \int_{\Sigma^{2n-1}}\int_{-\pi/2}^{\pi/2}\int_0^{\infty} f\Big(r(\cos\theta)^{1/2}\zeta, r^2\sin\theta\Big)(\cos\theta)^{n-1} r^{2n+1} dr\, d\theta\, d\zeta. \tag{2.18}$$

Hence

$$\int_{S^n} f\, d\sigma = \int_{\Sigma^{2n-1}}\int_{-\pi/2}^{(\pi/2)} f\Big((\cos\theta)^{1/2}\zeta, \sin\theta\Big)(\cos\theta)^{n-1} d\theta\, d\zeta. \tag{2.19}$$

A direct calculation shows that the area of $S^n$ is

$$\sigma(S^n) = \frac{4\pi^{n+1/2}\Gamma(1/2)}{\Gamma(n)\Gamma((n+1)/2)}. \tag{2.20}$$

In addition, we have the volume of unit ball in $H^n$

$$m(B^n) = \frac{2\pi^{n+1/2}\Gamma(1/2)}{(n+1)\Gamma(n)\Gamma((n+1)/2)}, \tag{2.21}$$

and thus the volume of $B_r(z,t)$

$$m(B_r(z,t)) = r^{2n+2}m(B^n). \tag{2.22}$$

For a radial function $f$, we have

$$\int_{H^n} f(z,t)dz\,dt = \sigma(S^n)\int_0^\infty f(r)r^{2n+1}dr. \tag{2.23}$$

The Hardy-Littlewood maximal operator $M$ is defined on $H^n$ by

$$Mf(z,t) = \sup_{r>0}\frac{1}{m(B_r)}\int_{B_r}\left|f((z,t)(-w,-s))\right|dw\,ds, \tag{2.24}$$

which is of type $(p,p)$ for $1 < p \le \infty$ and is of weak type (1,1) (see [17, 18]).

## 3. The Sublaplacian and the Heat Kernel on the Heisenberg Group

As it is known, the following vector fields

$$\begin{gathered} X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}, \quad j = 1,2,\ldots,n, \\ Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}, \quad j = 1,2,\ldots,n, \\ T = \frac{\partial}{\partial t} \end{gathered} \tag{3.1}$$

form a basis for the Lie algebra of left-invariant vector fields on $H^n$. The sublaplacian is defined by

$$\mathcal{L} = -\sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right), \tag{3.2}$$

which also has another explicit form

$$\mathcal{L} = -\Delta_z - \frac{1}{4}|z|^2\partial_t^2 + N\partial_t, \tag{3.3}$$

where $\Delta_z$ is the standard Laplacian on $C^n$ and

$$N = \sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right). \tag{3.4}$$

For the Schrödinger representations $\pi_\lambda$ one easily calculates that

$$\pi_\lambda^*(X_j)\varphi(\zeta) = i\lambda\zeta_j\varphi(\zeta), \qquad \pi_\lambda^*(Y_j)\varphi(\zeta) = \frac{\partial}{\partial\zeta_j}\varphi(\zeta). \tag{3.5}$$

So that $\pi_\lambda^*(\mathcal{L}) = -\Delta + \lambda^2|\zeta|^2 = H(\lambda)$.

Let $h_k(t)$ $(k \in Z_+ = \{0,1,2,\dots\})$ be the normalized Hermite functions given by

$$h_k(t) = \left(2^k\sqrt{\pi}k!\right)^{-(1/2)}H_k(t)e^{-(1/2)t^2}, \tag{3.6}$$

where $H_k(t) = (-1)^k(d^k/dt^k)\{e^{-t^2}\}e^{t^2}$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in Z_+^n$ and $\zeta \in R^n$, we define

$$\Phi_\alpha(\zeta) = \prod_{j=1}^{n}h_{\alpha_j}(\zeta_j). \tag{3.7}$$

Then $\{\Phi_\alpha\}$ forms an orthonormal basis for $L^2(R^n)$.

We set $\Phi_\alpha^\lambda(\zeta) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}\zeta)$ and denote

$$H(\lambda)\Phi_\alpha^\lambda = (2|\alpha| + n)|\lambda|\Phi_\alpha^\lambda. \tag{3.8}$$

Therefore,

$$\mathcal{L}\left(\pi_\lambda(z,t)\Phi_\alpha^\lambda, \Phi_\alpha^\beta\right) = (2|\alpha| + n)|\lambda|\left(\pi_\lambda(z,t)\Phi_\alpha^\lambda, \Phi_\alpha^\beta\right). \tag{3.9}$$

Moreover, one has

$$\widehat{\mathcal{L}f}(\lambda) = \hat{f}(\lambda)H(\lambda). \tag{3.10}$$

Now let $\mathcal{L}(e^{i\lambda t}f(z)) = e^{i\lambda t}L_\lambda f(z)$, then $L_\lambda$ has the form

$$L_\lambda = -\Delta_z + \frac{1}{4}\lambda^2|z|^2 + i\lambda N. \tag{3.11}$$

From (3.9) we know that the functions

$$\Phi^{\lambda}_{\alpha,\beta}(z) = (2\pi)^{-(n/2)}|\lambda|^{n/2}\left(\pi_\lambda(z)\Phi^{\lambda}_{\alpha}, \Phi^{\lambda}_{\beta}\right) \tag{3.12}$$

are eigenfunctions of the operator $L_\lambda$:

$$L_\lambda \Phi^{\lambda}_{\alpha,\beta}(z) = (2|\alpha| + n)|\lambda|\Phi^{\lambda}_{\alpha,\beta}(z). \tag{3.13}$$

Let $\varphi^{n-1}_{k}(z)$ be the Laguerre functions defined on $C^n$ by

$$\varphi^{n-1}_{k}(z) = L^{n-1}_{k}\left(\frac{1}{2}|z|^2\right)e^{-|z|^2/4} \tag{3.14}$$

and set $\varphi^{n-1}_{k,\lambda}(z) = \varphi^{n-1}_{k}(\sqrt{\lambda}z)$ for $\lambda \in R^*$. Then from [19, (2.3.26)] we have

$$|\lambda|^{n/2}\sum_{|\alpha|=k}\Phi^{\lambda}_{\alpha,\alpha}(z) = (2\pi)^{-n/2}\varphi^{n-1}_{k,\lambda}(z). \tag{3.15}$$

In view of this equation we have the following.

**Proposition 3.1.** *One has*

$$W_\lambda\left(\varphi^{n-1}_{k,\lambda}\right) = (2\pi)^{n}|\lambda|^{-n}P_k(\lambda), \tag{3.16}$$

*where $P_k(\lambda)$ stands for the projection of $L^2(R^n)$ onto the $k$th eigenspace of $H(\lambda)$, that is,*

$$P_k(\lambda)\varphi = \sum_{|\alpha|=k}\left(\varphi, \Phi^{\lambda}_{\alpha}\right)\Phi^{\lambda}_{\alpha}. \tag{3.17}$$

Now we consider the heat equation associated to the sublaplacian

$$\partial_s F(z,t;s) = -\mathcal{L}F(z,t;s) \tag{3.18}$$

with the initial condition $F(z,t;0) = f(z,t)$. In fact, the function $q_s$ given by

$$q_s(z,t) = c_n\int_{-\infty}^{\infty} e^{-i\lambda t}\left(\frac{\lambda}{\sinh \lambda s}\right)^{n} e^{-(1/4)(\lambda \coth \lambda s)|z|^2}d\lambda \tag{3.19}$$

is just the solution of the heat equation and satisfies

$$F(z,t;s) = \left(e^{-s\mathcal{L}}f\right)(z,t) = q_s * f(z,t). \tag{3.20}$$

Moreover, we have the Fourier transform of $q_s(z,t)$ (see [18, page 86])

$$\widehat{q}_s(\lambda) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty} e^{-(2k+n)|\lambda|s} W_\lambda\left(\varphi_{k,\lambda}^{n-1}\right). \tag{3.21}$$

## 4. Riesz Potential on the Heisenberg Group

In Section 1 we have recalled some properties about the Riesz potential on $R^n$; now we are going to discuss the Riesz potential on the Heisenberg group.

*Definition 4.1.* For $0 < \gamma < 2n+2$, the Riesz potential $I_\gamma$ is defined on $S(H^n)$ by

$$I_\gamma f(z,t) = \mathcal{L}^{-\gamma/2} f(z,t). \tag{4.1}$$

From above definition and (3.10) it is easy to see that

$$\widehat{I_\gamma f}(\lambda) = \widehat{f}(\lambda) H(\lambda)^{-(\gamma/2)}. \tag{4.2}$$

If $\gamma, \tau > 0$, $\gamma + \tau < 2n+2$, then we have

$$\begin{aligned} (I_\gamma(I_\tau f))^\wedge(\lambda) &= (I_\tau f)^\wedge(\lambda) H(\lambda)^{-(\gamma/2)} \\ &= \widehat{f}(\lambda) H(\lambda)^{-(\gamma+\tau)/2} \\ &= (I_{\gamma+\tau} f)^\wedge(\lambda), \end{aligned} \tag{4.3}$$

which suggests that $I_\gamma(I_\tau f) = I_{\gamma+\tau} f$. Especially, for $2 \le \gamma < 2n+2$, one has

$$\mathcal{L}(I_\gamma f) = I_\gamma(\mathcal{L} f) = I_{\gamma-2} f. \tag{4.4}$$

At present we do not prepare to gain the expression of $I_\gamma$ analogues to (1.3) because it is hard to calculate the Fourier transform of $|(z,t)|$. But the following theorem will give us another expression of $I_\gamma$, which provides a bridge to discuss the boundedness of the Riesz potential.

**Theorem 4.2.** *Let $q_s(z,t)$ be the heat kernel on $H^n$. For $0 < \gamma < 2n+2$, one has for $f \in S(H^n)$*

$$I_\gamma f(z,t) = \int_0^\infty \Gamma\left(\frac{\gamma}{2}\right)^{-1} s^{\gamma/2-1} q_s(\cdot) ds * f(z,t). \tag{4.5}$$

*Proof.* By (3.17), (3.21), and Proposition 3.1 we have

$$\begin{aligned}
&\left(\widehat{f}(\lambda)\int_0^\infty \Gamma\left(\frac{\gamma}{2}\right)^{-1} s^{\gamma/2-1}\widehat{q}_s(\lambda)ds\,\Phi_\alpha^\lambda,\Phi_\beta^\lambda\right)\\
&\quad=\left(\widehat{f}(\lambda)\int_0^\infty \Gamma\left(\frac{\gamma}{2}\right)^{-1} s^{\gamma/2-1}(2\pi)^{-n}|\lambda|^n\sum_{k=0}^{\infty}e^{-(2k+n)|\lambda|s}W_\lambda\left(\varphi_{k,\lambda}^{n-1}\right)ds\,\Phi_\alpha^\lambda,\Phi_\beta^\lambda\right)\\
&\quad=\left(\widehat{f}(\lambda)\sum_{k=0}^{\infty}\int_0^\infty \Gamma\left(\frac{\gamma}{2}\right)^{-1} s^{\gamma/2-1}e^{-(2k+n)|\lambda|s}ds\,P_k(\lambda)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\right)\\
&\quad=\left(\widehat{f}(\lambda)\sum_{k=0}^{\infty}((2k+n)|\lambda|s)^{-\gamma/2}P_k(\lambda)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\right)\\
&\quad=\left(\widehat{f}(\lambda)H(\lambda)^{-\gamma/2}\Phi_\alpha^\lambda,\Phi_\beta^\lambda\right)\\
&\quad=\left(\widehat{I_\gamma f}(\lambda)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\right).
\end{aligned}\tag{4.6}$$

Then we get the desired result. □

**Lemma 4.3.** *The heat kernel $q_s(z,t)$ satisfies the estimate*

$$q_s(z,t)\le Cs^{-n-1}e^{(A/s)|(z,t)|^{1/2}}\tag{4.7}$$

*with some positive constants $C$ and $A$.*

*Proof.* Since $(|z|^4+|t|^2)^{1/2}\le|z|^2+|t|$, then by [19, Proposition 2.8.2] we obtain this lemma. □

The following theorem is an immediate consequence of Theorem 4.2 and Lemma 4.3.

**Theorem 4.4.** *The Riesz potential $I_\gamma$ satisfies the estimate*

$$\left|I_\gamma f(z,t)\right|\le C\left|f\right|*|(w,s)|^{\gamma-(2n+2)}(z,t),\tag{4.8}$$

*where $C$ is a positive constant.*

Using Theorems 4.2 and 4.4, we get the Hardy-Littlewood-Sobolev theorem on the Heisenberg group.

**Theorem 4.5.** *Let $0<\gamma<2n+2$, $1\le p<q<\infty$, and $1/p=1/q+\gamma/(2n+2)$. For $f\in L^p(H^n)$, one has the following.*

(a) *If $p>1$, then $I_\gamma$ is of type $(p,q)$.*

(b) *If $p=1$, then $I_\gamma$ is of weak type $(1,q)$.*

*Proof.* Let $M$ be the maximal operator defined by (2.24). We claim that

$$\left|f\right|*|(\cdot)|^{\gamma-(2n+2)}(z,t)\le c\left(Mf(z,t)\right)^{p/q}\|f\|_p^{1-p/q}.\tag{4.9}$$

Let $B_R$ be the ball of radius $R$ centered at (0,0), and let $\chi_{B_R}$ be its characteristic function. We set

$$|(w,s)|^{\gamma-(2n+2)} = g_1 + g_2 \tag{4.10}$$

with $g_1(w,s) = |(w,s)|^{\gamma-(2n+2)}\chi_{B_R}(w,s)$. Obviously, $g_1 > 0$ is radial and decreasing, then we can write

$$g_1 = \sum_{j=1}^{N} a_j \chi_{B_j}, \tag{4.11}$$

where $a_j > 0$ and $B_j$ is the ball centered at origin. By (2.23) and (2.24) we have

$$\begin{aligned} |f| * g_1(z,t) &= \int_{H^n} |f((z,t)(-w,-s))| \sum_{j=1}^{N} a_j \chi_{B_j}(w,s)\,dw\,ds \\ &= \sum_{j=1}^{N} a_j \frac{m(B_j)}{m(B_j)} \int_{B_j} |f((z,t)(-w,-s))|\,dw\,ds \\ &\leq \sum_{j=1}^{N} a_j m(B_j) Mf(z,t) \\ &= \sigma(S^n) \int_0^R r^{\gamma-(2n+2)} r^{2n+1}\,dr\,Mf(z,t) \\ &= c_1 R^{\gamma} Mf(z,t). \end{aligned} \tag{4.12}$$

Let $p'$ be the conjugate exponent of $p$. Since $1/p - 1/q = \gamma/(2n+2)$, we have

$$\begin{aligned} \|g_2\|_{p'} &= \left(\sigma(S^n) \int_R^{\infty} r^{[\gamma-(2n+2)]p'} r^{2n+1}\,dr\right)^{1/p'} \\ &= c_2 R^{-(2n+2)/q}. \end{aligned} \tag{4.13}$$

Then by Hölder's inequality we get

$$|f| * g_2(z,t) \leq c_2 R^{-(2n+2)/q} \|f\|_p. \tag{4.14}$$

Therefore,

$$\left(|f| * |(w,s)|^{\gamma-(2n+2)}\right)(z,t) \leq c_1 R^{\gamma} Mf(z,t) + c_2 R^{-(2n+2)/q} \|f\|_p. \tag{4.15}$$

We choose $R$ such that

$$c_1 R^{\gamma} Mf(z,t) = c_2 R^{-(2n+2)/q} \|f\|_p. \tag{4.16}$$

That is, $R = c_3 (Mf(z,t)/\|f\|_p)^{-p/(2n+2)}$. Substituting this in the above then gives (4.9).

Now if $p > 1$, we obtain (a) by the virtue of the type $(p,p)$ of the maximal operator $M$. If $p = 1$ and $q = (2n+2)/(2n+2-\gamma)$, we have

$$\begin{aligned}
\int_{\{(z,t)\in H^n : |I_\gamma f(z,t)|>\lambda\}} dz\,dt &\le \int_{\{(z,t): c(Mf(z,t))^{1/q}\|f\|_1^{1-1/q}>\lambda\}} dz\,dt \\
&\le \int_{\{(z,t): Mf(z,t)>(\lambda/c)^q \|f\|_1^{1-q}\}} dz\,dt \\
&\le \left(\frac{c}{\lambda}\|f\|_1\right)^q .
\end{aligned} \tag{4.17}$$

This proves our main theorem. □

**Theorem 4.6.** *Let* $0 < \gamma < 2n+2$ *and* $1 \le p < q < \infty$. *For* $f \in L^p(H^n)$, *one has the following.*

(a) *If* $p = 1$, *then the condition* $1/q + \gamma/(2n+2) = 1$ *is necessary and sufficient for the weak type* $(1,q)$ *of* $I_\gamma$.

(b) *If* $1 < p < q < \infty$, *then the condition* $1/p = 1/q + \gamma/(2n+2)$ *is necessary and sufficient for the type* $(p,q)$ *of* $I_\gamma$.

*Proof.* The sufficiency follows from Theorem 4.5. We now begin to prove the necessity. Suppose that $f \in S(H^n)$, and let $f_a(z,t) = f(\delta_a(z,t))$. Note that

$$q_{a^2 s}(z,t) = a^{-(2n+2)} q_s(\delta_{a^{-1}}(z,t)). \tag{4.18}$$

Then by (4.9) we get

$$\begin{aligned}
I_\gamma f_a(z,t) &= \int_{H^n}\int_0^\infty \Gamma\left(\frac{\gamma}{2}\right)^{-1} s^{\gamma/2-1} q_s(z',t')\,ds\, f\left((-z',-t')\left(az, a^2 t\right)\right) dz'\,dt' \\
&= \int_{H^n}\int_0^\infty \Gamma\left(\frac{\gamma}{2}\right)^{-1} \left(a^{-2}s\right)^{\gamma/2-1} q_{a^{-2}s}(z',t')\,da^2 s\, f\left(\delta_a\left(\left(-a^{-1}z', -a^{-2}t'\right)(z,t)\right)\right) dz'\,dt' \\
&= \int_{H^n}\int_0^\infty \Gamma\left(\frac{\gamma}{2}\right)^{-1} a^{-\gamma} s^{\gamma/2-1} q_s(\delta_{a^{-1}}(z',t'))\,ds\, f(\delta_a((\delta_{a^{-1}}(z',t'))(z,t)))\,da^{-1}z'\,da^{-2}t' \\
&= a^{-\gamma}(I_\gamma f)_a(z,t).
\end{aligned} \tag{4.19}$$

Thus for $1 \le q < \infty$, we have

$$\begin{aligned} \left\| I_\gamma f_a \right\|_q &= a^{-\gamma-(2n+2)/q} \left\| I_\gamma f \right\|_q \\ m\{(z,t) \in H^n : \left| I_\gamma f_a(z,t) \right| > \lambda\}^{1/q} &= a^{-\gamma-(2n+2)/q} m\{(z,t) \in H^n : \left| I_\gamma f(z,t) \right| > \lambda\}^{1/q}. \end{aligned} \tag{4.20}$$

□

*Case 1* ($p = 1$). It follows from the hypothesis that

$$\begin{aligned} m\{(z,t) \in H^n : \left| I_\gamma f(z,t) \right| > \lambda\} &= a^{q\gamma+2n+2} m\{(z,t) \in H^n : \left| I_\gamma f_a(z,t) \right| > \lambda\} \\ &\le a^{q\gamma+2n+2} \left( \frac{A\left\| f_a \right\|_1}{\lambda} \right)^q \\ &= a^{q\gamma+(2n+2)-q(2n+2)} \left( \frac{A\left\| f \right\|_1}{\lambda} \right)^q. \end{aligned} \tag{4.21}$$

If $1/q+\gamma/(2n+2) > 1$, then $m\{(z,t) \in H^n : |I_\gamma f(z,t)| > \lambda\} = 0$ as $a \to 0$. If $1/q+\gamma/(2n+2) < 1$, then $m\{(z,t) \in H^n : |I_\gamma f(z,t)| > \lambda\} = 0$ as $a \to \infty$. Thus we have $1/q + \gamma/(2n+2) = 1$.

*Case 2* ($1 < p < q < \infty$). Similarly, we have

$$\begin{aligned} \left\| I_\gamma f \right\|_q &= a^{\gamma+(2n+2)/q} \left\| I_\gamma f_a \right\|_q \\ &\le c a^{\gamma+(2n+2)/q-(2n+2)/p} \left\| f \right\|_p, \end{aligned} \tag{4.22}$$

which implies that $1/p = 1/q + \gamma/(2n+2)$.

Then we complete the proof of this theorem.

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