Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 564836, 14 pages
doi:10.1155/2011/564836

Research Article

Some Vector Inequalities for Continuous Functions of Self-Adjoint Operators in Hilbert Spaces

S. S. Dragomir[1, 2]

[1] *Mathematics, School of Engineering & Science, Victoria University, P.O. Box 14428, Melbourne City, VIC 8001, Australia*
[2] *School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag-3, Johannesburg 2050, South Africa*

Correspondence should be addressed to S. S. Dragomir, sever.dragomir@vu.edu.au

Received 24 November 2010; Accepted 21 February 2011

Academic Editor: Paolo E. Ricci

On utilizing the spectral representation of self-adjoint operators in Hilbert spaces, some inequalities for the composite operator $[f(M)1_H - f(A)][f(A) - f(m)1_H]$, where $\mathrm{Sp}(A) \subseteq [m, M]$ and for various classes of continuous functions $f : [m, M] \to \mathbb{C}$ are given. Applications for the power function and the logarithmic function are also provided.

1. Introduction

Let U be a self-adjoint operator on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the spectrum $\mathrm{Sp}(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then, for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.1}$$

which in terms of vectors can be written as

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle, \tag{1.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0, \qquad g_{x,y}(M) = \langle x, y\rangle, \tag{1.3}$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

Utilising the spectral representation from (1.2), we have established the following Ostrowski-type vector inequality [1].

Theorem 1.1. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $\mathrm{Sp}(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then one has the inequality*

$$\begin{aligned}
&\left|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle\right| \\
&\quad\le \langle E_s x, x\rangle^{1/2}\langle E_s y, y\rangle^{1/2}\bigvee_m^s(f) \\
&\qquad + \langle (1_H - E_s)x, x\rangle^{1/2}\langle (1_H - E_s)y, y\rangle^{1/2}\bigvee_s^M(f) \\
&\quad\le \|x\|\|y\|\left(\frac{1}{2}\bigvee_m^M(f) + \frac{1}{2}\left|\bigvee_m^s(f) - \bigvee_s^M(f)\right|\right) \le \|x\|\|y\|\bigvee_m^M(f),
\end{aligned} \tag{1.4}$$

for any $x, y \in H$ and for any $s \in [m, M]$.

Another result that compares the function of a self-adjoint operator with the integral mean is embodied in the following theorem [2].

Theorem 1.2. *With the assumptions in Theorem 1.1 one has the inequalities*

$$\begin{aligned}
&\left|\langle x, y\rangle \cdot \frac{1}{M-m}\int_m^M f(s)ds - \langle f(A)x, y\rangle\right| \\
&\le \bigvee_m^M(f)\max_{t\in[m,M]}\left[\frac{M-t}{M-m}\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2} + \frac{t-m}{M-m}\langle (1_H - E_t)x, x\rangle^{1/2}\langle (1_H - E_t)y, y\rangle^{1/2}\right] \\
&\le \|x\|\|y\|\bigvee_m^M(f),
\end{aligned} \tag{1.5}$$

for any $x, y \in H$.

The trapezoid version of the above result has been obtained in [3] and is as follows.

Theorem 1.3. *With the assumptions in Theorem 1.1 one has the inequalities*

$$\left|\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right|$$
$$\leq\frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]\bigvee_m^M(f)$$
$$\leq\frac{1}{2}\|x\|\|y\|\bigvee_m^M(f), \tag{1.6}$$

for any $x,y\in H$.

For various inequalities for functions of self-adjoint operators, see [4–8]. For recent results see [1, 9–12].

In this paper, we investigate the quantity

$$\left|\langle[f(M)1_H-f(A)][f(A)-f(m)1_H]x,y\rangle\right|, \tag{1.7}$$

where x,y are vectors in the Hilbert space H and A is a self-adjoint operator with $\mathrm{Sp}(A)\subseteq[m,M]$, and provide different bounds for some classes of continuous functions $f:[m,M]\to\mathbb{C}$. Applications for some particular cases including the power and logarithmic functions are provided as well.

2. Some Vector Inequalities

The following representation in terms of the spectral family is of interest in itself.

Lemma 2.1. *Let* A *be a self-adjoint operator in the Hilbert space* H *with the spectrum* $\mathrm{Sp}(A)\subseteq[m,M]$ *for some real numbers* $m<M$ *and let* $\{E_\lambda\}_\lambda$ *be its spectral family. If* $f:[m,M]\to\mathbb{C}$ *is a continuous function on* $[m,M]$ *with* $f(M)\neq f(m)$*, then one has the representation*

$$\frac{1}{[f(M)-f(m)]^2}[f(M)1_H-f(A)][f(A)-f(m)1_H]$$
$$=\frac{1}{f(M)-f(m)}\int_{m-0}^{M}\left(E_t-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s\,df(s)\right)\left(E_t-\frac{1}{2}1_H\right)df(t). \tag{2.1}$$

Proof. We observe

$$\begin{aligned}
&\frac{1}{f(M)-f(m)}\int_{m-0}^{M}\left(E_t-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s df(s)\right)\left(E_t-\frac{1}{2}1_H\right)df(t)\\
&\quad=\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t^2 df(t)\\
&\qquad-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s df(s)\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)\\
&\qquad-\frac{1}{2}\int_{m-0}^{M}E_t df(t)+\frac{1}{2}\int_{m-0}^{M}E_s df(s)\\
&\quad=\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t^2 df(t)-\left[\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)\right]^2,
\end{aligned}\tag{2.2}$$

which is an equality of interest in itself.

Since E_t are projections, we have $E_t^2 = E_t$ for any $t \in [m, M]$ and then we can write

$$\begin{aligned}
&\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t^2 df(t)-\left[\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)\right]^2\\
&\quad=\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)-\left[\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)\right]^2\\
&\quad=\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)\left[1_H-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)\right].
\end{aligned}\tag{2.3}$$

Integrating by parts in the Riemann-Stieltjes integral and utilizing the spectral representation (1.1), we have

$$\begin{aligned}
\int_{m-0}^{M}E_t df(t)&=f(M)1_H-f(A),\\
1_H-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_t df(t)&=\frac{f(A)-f(m)1_H}{f(M)-f(m)},
\end{aligned}\tag{2.4}$$

which together with (2.3) and (2.2) produce the desired result (2.1). □

The following vector version may be stated as well.

Corollary 2.2. *With the assumptions of Lemma 2.1 one has the equality*

$$\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\rangle$$
$$= [f(M) - f(m)] \int_{m-0}^{M} \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle df(t), \tag{2.5}$$

for any $x, y \in [m, M]$.

The following result that provides some bounds for continuous functions of bounded variation may be stated as well.

Theorem 2.3. *Let* A *be a self-adjoint operator in the Hilbert space* H *with the spectrum* $\mathrm{Sp}(A) \subseteq [m, M]$ *for some real numbers* $m < M$*, and let* $\{E_\lambda\}_\lambda$ *be its spectral family. If* $f : [m, M] \to \mathbb{C}$ *is a continuous function of bounded variation on* $[m, M]$ *with* $f(M) \neq f(m)$*, then we have the inequality*

$$\begin{aligned} &|\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\rangle| \\ &\quad \leq \frac{1}{2}\|y\| |f(M) - f(m)| \bigvee_m^M (f) \\ &\quad\quad \times \sup_{t \in [m,M]} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s) \right\| \\ &\quad \leq \frac{1}{2}\|x\|\|y\| \left[\bigvee_m^M (f) \right]^2, \end{aligned} \tag{2.6}$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a bounded function, $v : [a, b] \to \mathbb{C}$ is of bounded variation, and the Riemann-Stieltjes integral $\int_a^b p(t) dv(t)$ exists, then the following inequality holds:

$$\left| \int_a^b p(t) dv(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \tag{2.7}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising this property and the representation (2.5), we have by the Schwarz inequality in Hilbert space H that

$$\begin{aligned}
&\left|\left\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\right\rangle\right| \\
&\le |f(M) - f(m)| \bigvee_m^M (f) \\
&\quad \times \sup_{t\in[m,M]} \left|\left\langle \left(E_t - \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_s df(s)\right)x, \left(E_t - \frac{1}{2}1_H\right)y\right\rangle\right| \\
&\le |f(M) - f(m)| \bigvee_m^M (f) \\
&\quad \times \sup_{t\in[m,M]} \left[\left\| E_t x - \frac{1}{f(M)-f(m)}\int_{m-0}^{M} E_s x df(s)\right\| \left\| E_t y - \frac{1}{2}y\right\|\right],
\end{aligned} \tag{2.8}$$

for any $x, y \in [m, M]$.

Since E_t are projections, in this case we have

$$\begin{aligned}
\left\| E_t y - \frac{1}{2}y\right\|^2 &= \|E_t y\|^2 - \langle E_t y, y\rangle + \frac{1}{4}\|y\|^2 \\
&= \left\langle E_t^2 y, y\right\rangle - \langle E_t y, y\rangle + \frac{1}{4}\|y\|^2 = \frac{1}{4}\|y\|^2,
\end{aligned} \tag{2.9}$$

then from (2.8), we deduce the first part of (2.6).

Now, by the same property (2.7) for vector-valued functions p with values in Hilbert spaces, we also have

$$\begin{aligned}
&\left\| [f(M) - f(m)]E_t x - \int_{m-0}^{M} E_s x df(s)\right\| \\
&\quad = \left\| \int_{m-0}^{M} (E_t x - E_s x) df(s)\right\| \le \bigvee_m^M (f) \sup_{s\in[m,M]} \|E_t x - E_s x\|,
\end{aligned} \tag{2.10}$$

for any $t \in [m, M]$ and $x \in H$.

Since $0 \le E_t \le 1_H$ in the operator order, then $-1_H \le E_t - E_s \le 1$ which gives that $-\|x\|^2 \le \langle (E_t - E_s)x, x\rangle \le \|x\|^2$, that is, $|\langle (E_t - E_s)x, x\rangle| \le \|x\|^2$ for any $x \in H$, which implies that $\|E_t - E_s\| \le 1$ for any $t, s \in [m, M]$. Therefore, $\sup_{s\in[m,M]} \|E_t x - E_s x\| \le \|x\|$ which together with (2.10) prove the last part of (2.6). □

The case of Lipschitzian functions is as follows.

Theorem 2.4. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $\mathrm{Sp}(A) \subseteq [m, M]$ for some real numbers $m < M$, and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$ and with $f(M) \neq f(m)$, then one has the inequality*

$$\begin{aligned}
&|\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\rangle| \\
&\quad \leq \frac{1}{2}L\|y\||f(M) - f(m)|\int_{m-0}^{M}\left\|E_t x - \frac{1}{f(M) - f(m)}\int_{m-0}^{M} E_s x df(s)\right\|dt \\
&\quad \leq \frac{1}{2}L^2\|y\|\iint_{m-0}^{M}\|E_t x - E_s x\|ds\,dt \\
&\quad \leq \frac{\sqrt{2}}{2}L^2\|y\|(M - m)\langle Ax - mx, Mx - Ax\rangle^{1/2} \\
&\quad \leq \frac{\sqrt{2}}{4}L^2\|y\|\|x\|(M - m)^2,
\end{aligned} \tag{2.11}$$

for any $x, y \in H$.

Proof. Recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, that is,

$$|f(s) - f(t)| \leq L|s - t| \quad \text{for any } t, s \in [a, b], \tag{2.12}$$

then the Riemann-Stieltjes integral $\int_a^b p(t)dv(t)$ exists and the following inequality holds:

$$\left|\int_a^b p(t)dv(t)\right| \leq L\int_a^b |p(t)|dt. \tag{2.13}$$

Now, on applying this property of the Riemann-Stieltjes integral, then we have from the representation (2.5) that

$$\begin{aligned}
&|\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\rangle| \\
&\quad \leq |f(M) - f(m)|\int_{m-0}^{M}\left|\left\langle\left(E_t - \frac{1}{f(M) - f(m)}\int_{m-0}^{M} E_s df(s)\right)x, \left(E_t - \frac{1}{2}1_H\right)y\right\rangle\right|df(t), \\
&\quad \leq L|f(M) - f(m)|\int_{m-0}^{M}\left\|E_t x - \frac{1}{f(M) - f(m)}\int_{m-0}^{M} E_s x df(s)\right\|\left\|E_t y - \frac{1}{2}y\right\|dt \\
&\quad = \frac{1}{2}L\|y\||f(M) - f(m)|\int_{m-0}^{M}\left\|E_t x - \frac{1}{f(M) - f(m)}\int_{m-0}^{M} E_s x df(s)\right\|dt,
\end{aligned} \tag{2.14}$$

for any $x, y \in H$ and the first inequality in (2.11) is proved.

Further, observe that

$$\begin{aligned}
&|f(M)-f(m)|\int_{m-0}^{M}\left\|E_t x-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x\,df(s)\right\|dt\\
&\quad=\int_{m-0}^{M}\left\|[f(M)-f(m)]E_t x-\int_{m-0}^{M}E_s x\,df(s)\right\|dt\\
&\quad=\int_{m-0}^{M}\left\|\int_{m-0}^{M}(E_t x-E_s x)\,df(s)\right\|dt,
\end{aligned}\tag{2.15}$$

for any $x\in H$.

If we use the vector-valued version of the property (2.13), then we have

$$\int_{m-0}^{M}\left\|\int_{m-0}^{M}(E_t x-E_s x)\,df(s)\right\|dt\le L\iint_{m-0}^{M}\|E_t x-E_s x\|\,ds\,dt,\tag{2.16}$$

for any $x\in H$ and the second part of (2.11) is proved.

Further on, by applying the double-integral version of the Cauchy-Buniakowski-Schwarz inequality, we have

$$\iint_{m-0}^{M}\|E_t x-E_s x\|\,ds\,dt\le(M-m)\left(\iint_{m-0}^{M}\|E_t x-E_s x\|^2\,ds\,dt\right)^{1/2},\tag{2.17}$$

for any $x\in H$.

Now, by utilizing the fact that E_s are projections for each $s\in[m,M]$, then we have

$$\begin{aligned}
&\iint_{m-0}^{M}\|E_t x-E_s x\|^2\,ds\,dt\\
&\quad=2\left[(M-m)\int_{m-0}^{M}\|E_t x\|^2dt-\left\|\int_{m-0}^{M}E_t x\,dt\right\|^2\right]\\
&\quad=2\left[(M-m)\int_{m-0}^{M}\langle E_t x,x\rangle dt-\left\|\int_{m-0}^{M}E_t x\,dt\right\|^2\right],
\end{aligned}\tag{2.18}$$

for any $x\in H$.

If we integrate by parts and use the spectral representation (1.2), then we get

$$\int_{m-0}^{M} \langle E_t x, x\rangle dt = \langle Mx - Ax, x\rangle, \qquad \int_{m-0}^{M} E_t x dt = Mx - Ax \tag{2.19}$$

and by (2.18), we then obtain the following equality of interest:

$$\iint_{m-0}^{M} \|E_t x - E_s x\|^2 ds\, dt = 2\langle Ax - mx, Mx - Ax\rangle, \tag{2.20}$$

for any $x \in H$.

On making use of (2.20) and (2.17), we then deduce the third part of (2.11).

Finally, by utilizing the elementary inequality in inner product spaces

$$\operatorname{Re}\langle a, b\rangle \le \frac{1}{4}\|a + b\|^2, \quad a, b \in H, \tag{2.21}$$

we also have that

$$\langle Ax - mx, Mx - Ax\rangle \le \frac{1}{4}(M - m)^2\|x\|^2, \tag{2.22}$$

for any $x \in H$, which proves the last inequality in (2.11). □

The case of nondecreasing monotonic functions is as follows.

Theorem 2.5. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $\mathrm{Sp}(A) \subseteq [m, M]$ for some real numbers $m < M$, and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a monotonic nondecreasing function on $[m, M]$, then one has the inequality*

$$\begin{aligned} &\left|\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\rangle\right| \\ &\quad\le \frac{1}{2}\|y\|[f(M) - f(m)] \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s)\right\| df(t) \\ &\quad\le \frac{1}{2}\|y\|[f(M) - f(m)]\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, x\rangle^{1/2} \\ &\quad\le \frac{1}{4}\|y\|\|x\|[f(M) - f(m)]^2, \end{aligned} \tag{2.23}$$

for any $x, y \in H$.

Proof. From the theory of Riemann-Stieltjes integral, it is also well known that if $p : [a,b] \to \mathbb{C}$ is of bounded variation and $v : [a,b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)dv(t)$ and $\int_a^b |p(t)|dv(t)$ exist and

$$\left|\int_a^b p(t)dv(t)\right| \le \int_a^b |p(t)|dv(t). \tag{2.24}$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (2.5) that

$$\begin{aligned}
&\left|\left\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y\right\rangle\right| \\
&\quad\le [f(M) - f(m)] \int_{m-0}^{M} \left|\left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s df(s)\right)x, \left(E_t - \frac{1}{2}1_H\right)y \right\rangle\right| df(t), \\
&\quad\le [f(M) - f(m)] \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s)\right\| \left\| E_t y - \frac{1}{2}y\right\| df(t) \\
&\quad= \frac{1}{2}\|y\| [f(M) - f(m)] \int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s)\right\| df(t),
\end{aligned} \tag{2.25}$$

for any $x, y \in H$, which proves the first inequality in (2.23).

On utilizing the Cauchy-Buniakowski-Schwarz-type inequality for the Riemann-Stieltjes integral of monotonic nondecreasing integrators, we have

$$\begin{aligned}
&\int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s)\right\| df(t) \\
&\quad\le \left[\int_{m-0}^{M} df(t)\right]^{1/2} \left[\int_{m-0}^{M} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^{M} E_s x df(s)\right\|^2 df(t)\right]^{1/2},
\end{aligned} \tag{2.26}$$

for any $x, y \in H$.

Observe that

$$\begin{aligned}
&\int_{m-0}^{M}\left\|E_t x-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x\,df(s)\right\|^2 df(t)\\
&\quad=\int_{m-0}^{M}\left[\|E_t x\|^2-2\operatorname{Re}\left\langle E_t x,\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x\,df(s)\right\rangle\right.\\
&\qquad\left.+\left\|\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x\,df(s)\right\|^2\right]df(t)\\
&\quad=[f(M)-f(m)]\left[\frac{1}{f(M)-f(m)}\int_{m-0}^{M}\|E_t x\|^2 df(t)-\left\|\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x\,df(s)\right\|^2\right]
\end{aligned}\tag{2.27}$$

and, integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
\int_{m-0}^{M}\|E_t x\|^2 df(t)&=\int_{m-0}^{M}\langle E_t x,E_t x\rangle df(t)=\int_{m-0}^{M}\langle E_t x,x\rangle df(t)\\
&=f(M)\|x\|^2-\int_{m-0}^{M}f(t)d\langle E_t x,x\rangle\\
&=f(M)\|x\|^2-\langle f(A)x,x\rangle=\langle[f(M)1_H-f(A)]x,x\rangle,\\
\int_{m-0}^{M}E_s x\,df(s)&=f(M)x-f(A)x,
\end{aligned}\tag{2.28}$$

for any $x\in H$.

On making use of the equalities (2.28), we have

$$\begin{aligned}
&\frac{1}{f(M)-f(m)}\int_{m-0}^{M}\|E_t x\|^2 df(t)-\left\|\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x\,df(s)\right\|^2\\
&\quad=\frac{1}{[f(M)-f(m)]^2}\left[[f(M)-f(m)]\langle[f(M)1_H-f(A)]x,x\rangle-\|f(M)x-f(A)x\|^2\right]\\
&\quad=\frac{[f(M)-f(m)]\langle[f(M)1_H-f(A)]x,x\rangle-\langle f(M)x-f(A)x,f(M)x-f(A)x\rangle}{[f(M)-f(m)]^2}\\
&\quad=\frac{[f(M)-f(m)]\langle[f(M)1_H-f(A)]x,x\rangle-\langle f(M)x-f(A)x,f(M)x-f(A)x\rangle}{[f(M)-f(m)]^2}\\
&\quad=\frac{\langle f(M)x-f(A)x,f(A)x-f(m)x\rangle}{[f(M)-f(m)]^2},
\end{aligned}\tag{2.29}$$

for any $x\in H$.

Therefore, we obtain the following equality of interest in itself as well:

$$\frac{1}{f(M)-f(m)}\int_{m-0}^{M}\left\|E_t x-\frac{1}{f(M)-f(m)}\int_{m-0}^{M}E_s x\,df(s)\right\|^2 df(t)$$
$$=\frac{\langle f(M)x-f(A)x, f(A)x-f(m)x\rangle}{[f(M)-f(m)]^2} \tag{2.30}$$
$$=\frac{\langle [f(M)1_H-f(A)][f(A)-f(m)1_H]x,x\rangle}{[f(M)-f(m)]^2},$$

for any $x \in H$

On making use of the inequality (2.26), we deduce the second inequality in (2.23).

The last part follows by (2.21), and the details are omitted. □

3. Applications

We consider the power function $f(t) := t^p$, where $p \in \mathbb{R} \setminus \{0\}$ and $t > 0$. The following power inequalities hold.

Proposition 3.1. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum* $\mathrm{Sp}(A) \subseteq [m, M]$ *for some real numbers with $0 \le m < M$.*

If $p > 0$, then for any $x, y \in H$,

$$\left|\langle (M^p 1_H - A^p)(A^p - m^p 1_H)x, y\rangle\right|$$
$$\le \frac{\sqrt{2}}{2}B_p^2\|y\|(M-m)\langle Ax-mx, Mx-Ax\rangle^{1/2} \tag{3.1}$$
$$\le \frac{\sqrt{2}}{4}B_p^2\|y\|\|x\|(M-m)^2,$$

where

$$B_p = p\times\begin{cases} M^{p-1}, & \text{if } p\ge 1,\\ m^{p-1}, & \text{if } 0<p<1,\ m>0,\end{cases}$$

$$\left|\langle (A^{-p} - M^{-p}1_H)(m^{-p}1_H - A^{-p})x, y\rangle\right| \tag{3.2}$$
$$\le \frac{\sqrt{2}}{2}C_p^2\|y\|(M-m)\langle Ax-mx, Mx-Ax\rangle^{1/2}$$
$$\le \frac{\sqrt{2}}{4}C_p^2\|y\|\|x\|(M-m)^2,$$

where

$$C_p = pm^{-p-1}, \quad m > 0. \tag{3.3}$$

The proof follows from Theorem 2.4 applied for the power function.

Proposition 3.2. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $\mathrm{Sp}(A) \subseteq [m, M]$ for some real numbers with $0 \leq m < M$.*

If $p > 0$, then for any $x, y \in H$

$$\begin{aligned}
&\left|\langle (M^p 1_H - A^p)(A^p - m^p 1_H)x, y\rangle\right| \\
&\quad \leq \frac{1}{2}\|y\|(M^p - m^p)\langle (M^p 1_H - A^p)(A^p - m^p 1_H)x, x\rangle^{1/2} \\
&\quad \leq \frac{1}{4}\|y\|\|x\|(M^p - m^p)^2, \\
&\left|\langle (A^{-p} - M^{-p}1_H)(m^{-p}1_H - A^{-p})x, y\rangle\right| \\
&\quad \leq \frac{1}{2}\|y\|(m^{-p} - M^{-p})\langle (A^{-p} - M^{-p}1_H)(m^{-p}1_H - A^{-p})x, x\rangle^{1/2} \\
&\quad \leq \frac{1}{4}\|y\|\|x\|(m^{-p} - M^{-p})^2.
\end{aligned} \tag{3.4}$$

The proof follows from Theorem 2.5.

Now, consider the logarithmic function $f(t) = \ln t,\ t > 0$. We have the following

Proposition 3.3. *Let A be a self-adjoint operator in the Hilbert space H with the spectrum $\mathrm{Sp}(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$. Then one has the inequalities*

$$\begin{aligned}
&\left|\langle [(\ln M)1_H - \ln A][\ln A - (\ln m)1_H]x, y\rangle\right| \\
&\quad \leq \frac{\sqrt{2}}{2m^2}\|y\|(M - m)\langle Ax - mx, Mx - Ax\rangle^{1/2} \\
&\quad \leq \frac{\sqrt{2}}{4}\|y\|\|x\|\left(\frac{M}{m} - 1\right)^2, \\
&\left|\langle [(\ln M)1_H - \ln A][\ln A - (\ln m)1_H]x, y\rangle\right| \\
&\quad \leq \frac{1}{2}\|y\|\langle [(\ln M)1_H - \ln A][\ln A - (\ln m)1_H]x, x\rangle^{1/2}\ln\left(\frac{M}{m}\right) \\
&\quad \leq \frac{1}{4}\|y\|\|x\|\left[\ln\left(\frac{M}{m}\right)\right]^2.
\end{aligned} \tag{3.5}$$

The proof follows from Theorems 2.4 and 2.5 applied for the logarithmic function.

References

[1] S. S. Dragomir, "Some Ostrowski's type vector inequalities for functions of selfadjoint operators in Hilbert spaces," *RGMIA Research Report Collection*, vol. 13, no. 2, article 7, 2010.

[2] S. S. Dragomir, "Comparison between functions of selfadjoint operators in Hilbert spaces and integral means," *RGMIA Research Report Collection*, vol. 13, no. 2, article 10, 2010.

[3] S. S. Dragomir, "Approximating n-time differentiable functions of selfadjoint operators in Hilbert spaces by two point Taylor's type expansion," *RGMIA Research Report Collection*, vol. 13, no. 2, article 13, 2010.

[4] T. Furuta, J. Mićić Hot, J. Pečarić, and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[5] A. Matković, J. Pečarić, and I. Perić, "A variant of Jensen's inequality of Mercer's type for operators with applications," *Linear Algebra and Its Applications*, vol. 418, no. 2-3, pp. 551–564, 2006.

[6] B. Mond and J. E. Pečarić, "Convex inequalities in Hilbert space," *Houston Journal of Mathematics*, vol. 19, no. 3, pp. 405–420, 1993.

[7] B. Mond and J. E. Pečarić, "Classical inequalities for matrix functions," *Utilitas Mathematica*, vol. 46, pp. 155–166, 1994.

[8] J. Pečarić, J. Mićić, and Y. Seo, "Inequalities between operator means based on the Mond-Pečarić method," *Houston Journal of Mathematics*, vol. 30, no. 1, pp. 191–207, 2004.

[9] S. S. Dragomir, "Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces," *Linear and Multilinear Algebra*, vol. 58, no. 7-8, pp. 805–814, 2010.

[10] S. S. Dragomir, "Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces," *RGMIA Research Report Collection*, vol. 11, article 11, 2008.

[11] S. S. Dragomir, "Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces," *RGMIA Research Report Collection*, vol. 11, article 17, 2008.

[12] S. S. Dragomir, "Some trapezoidal vector inequalities for continuous functions of delfadjoint operators in Hilbert spaces," *RGMIA Research Report Collection*, vol. 13, no. 2, article 14, 2010.