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Research Article

On Shafer and Carlson Inequalities

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We present a generalized and sharp version of Shafer's inequality for the inverse tangent function and a new lower bound of Carlson's inequality by means of a third order estimate of the inverse cosine function.

1. Introduction

For $x > 0$, it is known in the literature that

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan x. \tag{1.1}$$

This inequality was first presented without proof by Shafer [1]. Three proofs of it were later given in [2]. Shafer's inequality (1.1) was recently sharpened and generalized by Qi et al. in [3].

In view of inequality (1.1), we now ask: for each $a > 0$, what is the largest number b and what is the smallest number c such that the inequalities

$$\frac{bx}{1+a\sqrt{1+x^2}} \le \arctan x \le \frac{cx}{1+a\sqrt{1+x^2}} \tag{1.2}$$

are valid for all $x \ge 0$? Theorem 2.1 below answers this question.

For $0 \le x < 1$, it is known in the literature that

$$\frac{6\sqrt{1-x}}{2\sqrt{2}+\sqrt{1+x}} < \arccos x < \frac{\sqrt[3]{4}\cdot\sqrt{1-x}}{(1+x)^{1/6}}. \tag{1.3}$$

The inequalities (1.3) were established by Carlson [4] (see also [5, page 246]). Carlson's inequalities (1.3) were recently sharpened and generalized by and Guo and Qi in [6, 7]. In view of the first inequality in (1.3), the following question has been asked: for each $\nu > 0$, what is the largest number λ and what is the smallest number μ such that the inequalities

$$\frac{\lambda\sqrt{1-x}}{\nu+\sqrt{1+x}} \le \arccos x \le \frac{\mu\sqrt{1-x}}{\nu+\sqrt{1+x}} \tag{1.4}$$

are valid for all $0 \le x \le 1$? In [8], Chen and Mortici answered this question. Also in [8], the authors proved that for all $0 \le x \le 1$, the inequalities

$$\frac{\sqrt[3]{4}\cdot\sqrt{1-x}}{\alpha+(1+x)^{1/6}} \le \arccos x \le \frac{\sqrt[3]{4}\cdot\sqrt{1-x}}{\beta+(1+x)^{1/6}} \tag{1.5}$$

hold with best possible constants

$$\alpha = \frac{2\sqrt[3]{4}-\pi}{\pi} = 0.0105708962\ldots, \quad \beta = 0. \tag{1.6}$$

In view of the second inequality in (1.3), we now define the function $P(x)$ by

$$P(x) = \frac{r(1-x)^p}{(1+x)^q}, \quad 0 \le x \le 1. \tag{1.7}$$

We are interested in finding the values of the parameters p, q and r such that $P(x)$ is the best 3rd order approximation of $\arccos x$ in a neighborhood of the origin. This is addressed in Theorem 3.1. Motivated by the result of Theorem 3.1, we establish a new lower bound for the inverse cosine function in Theorem 3.2.

The following lemma is needed in our present investigation.

Lemma 1.1 (see [9–11]). *Let $-\infty < a < b < \infty$, and $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b). Suppose $g' \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{[f(x)-f(a)]}{[g(x)-g(a)]} \quad \text{and} \quad \frac{[f(x)-f(b)]}{[g(x)-g(b)]}. \tag{1.8}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

2. Generalized and Sharp Shafer's Inequality

Theorem 2.1. *The largest number b and the smallest number c required by inequality* (1.2) *are:*

$$\begin{aligned}
&\text{when } 0 < a \le \pi/2, \quad b = (\pi/2)a, \quad c = 1 + a,\\
&\text{when } \pi/2 < a \le 2/(\pi-2), \quad b = \left(4\left(a^2-1\right)\right)/a^2, \quad c = 1 + a,\\
&\text{when } 2/(\pi-2) < a < 2, \quad b = \left(4\left(a^2-1\right)\right)/a^2, \quad c = (\pi/2)a,\\
&\text{when } 2 \le a < \infty, \quad b = 1 + a, \quad c = (\pi/2)a.
\end{aligned} \tag{2.1}$$

Proof. For $x = 0$, inequality (1.2) holds for all values of b and c. For $x > 0$ and for $a > 0$, inequality (1.2) is equivalent to

$$b \le \frac{\left(1 + a\sqrt{1+x^2}\right)\arctan x}{x} \le c. \tag{2.2}$$

Consider the function $f(x)$ defined by

$$\begin{gathered}
f(x) := \frac{\left(1 + a\sqrt{1+x^2}\right)\arctan x}{x}, \quad x > 0,\\
f(0) := 1 + a.
\end{gathered} \tag{2.3}$$

By an elementary change of variable

$$x = \tan t, \quad 0 \le t < \frac{\pi}{2}, \tag{2.4}$$

we obtain

$$\begin{gathered}
f(x) = g(t) := \frac{t(1 + a\sec t)}{\tan t}, \quad 0 < t < \frac{\pi}{2},\\
f(0) = g(0) := 1 + a.
\end{gathered} \tag{2.5}$$

Differentiating with respect to t yields

$$\frac{\sin^2 t}{\sin t - t\cos t} g'(t) = a - h(t), \quad 0 < t < \frac{\pi}{2}, \tag{2.6}$$

where

$$h(t) = \frac{2t - \sin(2t)}{2(\sin t - t\cos t)}. \tag{2.7}$$

For $0 \le t \le \pi/2$, let

$$h_1(t) = 2t - \sin(2t), \qquad h_2(t) = 2(\sin t - t\cos t). \tag{2.8}$$

Then,

$$\frac{h_1'(t)}{h_2'(t)} = \frac{2\sin t}{t} \tag{2.9}$$

is strictly decreasing on $(0, \pi/2)$. By Lemma 1.1, the function

$$h(t) = \frac{h_1(t)}{h_2(t)} = \frac{h_1(t) - h_1(0)}{h_2(t) - h_2(0)} \tag{2.10}$$

is strictly decreasing on $(0, \pi/2)$, and we have

$$\frac{\pi}{2} = \lim_{s \to (\pi/2)^-} h(s) < h(t) < \lim_{s \to 0^+} h(s) = 2, \quad \forall t \in \left(0, \frac{\pi}{2}\right). \tag{2.11}$$

We split into several cases.

Case 1. $0 < a \le \pi/2$.

By (2.6) and (2.11), $g'(t) < 0$ on $(0, \pi/2)$. Therefore, the function $g(t)$ is strictly decreasing on $[0, \pi/2)$. As $x = \tan t$ is strictly increasing for $t \in [0, \pi/2)$, we see that the function $f(x)$ is strictly decreasing for $x \in [0, \infty)$, and we have

$$\frac{\pi}{2}a = f(\infty) < f(x) = \frac{\left(1 + a\sqrt{1+x^2}\right)\arctan x}{x} \le f(0) = 1 + a, \quad \forall x \ge 0. \tag{2.12}$$

Hence, inequality (1.2) holds for $x \ge 0$ with best possible constants

$$b = \frac{\pi}{2}a, \qquad c = 1 + a. \tag{2.13}$$

Case 2. $\pi/2 < a < 2$.

By (2.11), the function $h(t)$ is strictly decreasing from $(0, \pi/2)$ onto $(\pi/2, 2)$. Therefore, for each a with $\pi/2 < a < 2$, there exists a unique $\xi = \xi(a) \in (0, \pi/2)$ such that $h(\xi) = a$, that is,

$$\frac{2\xi - \sin(2\xi)}{2(\sin\xi - \xi\cos\xi)} = a, \tag{2.14}$$

or equivalently

$$\frac{\xi}{\sin \xi} = \frac{a + \cos \xi}{1 + a \cos \xi}. \tag{2.15}$$

Moreover, it follows from (2.6) that $g'(t) < 0$ on $(0, \xi)$, and $g'(t) > 0$ on $(\xi, \pi/2)$. Therefore, the function $g(t)$ is strictly decreasing on $(0, \xi)$ and strictly increasing on $(\xi, \pi/2)$, thus it takes its unique minimum $g(\xi)$ at $t = \xi$. Write (2.5) as

$$g(t) = \frac{t(a + \cos t)}{\sin t}, \quad 0 < t < \frac{\pi}{2}. \tag{2.16}$$

Substituting $t = \xi$ into (2.16) and using (2.15), we get

$$g_{\min} := g(\xi) = \frac{(a + \cos \xi)^2}{1 + a \cos \xi}, \quad \xi \in \left(0, \frac{\pi}{2}\right), \tag{2.17}$$

or equivalently,

$$y^2 + a(2 - g)y + a^2 - g = 0, \quad \text{where } y = \cos \xi. \tag{2.18}$$

From discriminant

$$\Delta = (a(2 - g))^2 - 4\left(a^2 - g\right) \geq 0, \tag{2.19}$$

we obtain

$$g \geq \frac{4(a^2 - 1)}{a^2}. \tag{2.20}$$

So to summarize, we have

$$g(0) = 1 + a, \qquad g\left(\frac{\pi}{2}\right) = \lim_{t \to (\pi/2)^-} g(t) = \frac{\pi}{2} a, \tag{2.21}$$

$g(t)$ decreases strictly on $(0, \xi)$ with minimum value $g_{\min} = g(\xi) = 4(a^2 - 1)/a^2$ at $t = \xi = \arccos[(a^2 - 2)/a]$, and increases strictly on $(\xi, \pi/2)$.

Subcase 2.1. $1 + a = g(0) \geq g(\pi/2) = (\pi/2)a$, that is, $\pi/2 < a \leq 2/(\pi - 2)$:

We have

$$\frac{4(a^2 - 1)}{a^2} = g(\xi) \leq g(t) = \frac{t(1 + a \sec t)}{\tan t} \leq g(0) = 1 + a, \quad 0 \leq t < \frac{\pi}{2}, \tag{2.22}$$

which, by the elementary change of variable (2.4), can be transformed into

$$\frac{4(a^2-1)}{a^2} = f(\tan\xi) \le f(x) = \frac{\left(1+a\sqrt{1+x^2}\right)\arctan x}{x} \le f(0) = 1+a, \quad x \ge 0. \tag{2.23}$$

Hence, inequality (1.2) holds with best possible constants

$$b = \frac{4(a^2-1)}{a^2}, \quad c = 1+a. \tag{2.24}$$

Subcase 2.2. $1+a = g(0) < g(\pi/2) = (\pi/2)a$, that is, $2/(\pi-2) < a < 2$:

We have

$$\frac{4(a^2-1)}{a^2} = g(\xi) \le g(t) = \frac{t(1+a\sec t)}{\tan t} < \lim_{t\to(\pi/2)^-} g(t) = \frac{\pi}{2}a, \quad 0 \le t < \frac{\pi}{2}, \tag{2.25}$$

which, by the elementary change of variable (2.4), can be transformed into

$$\frac{4(a^2-1)}{a^2} = f(\tan\xi) \le f(x) = \frac{\left(1+a\sqrt{1+x^2}\right)\arctan x}{x} < f(\infty) = \frac{\pi}{2}a, \quad x \ge 0. \tag{2.26}$$

Hence, inequality (1.2) holds with best possible constants

$$b = \frac{4(a^2-1)}{a^2}, \qquad c = \frac{\pi}{2}a. \tag{2.27}$$

Case 3. $2 \le a < \infty$.

By (2.6) and (2.11), $g'(t) > 0$ on $(0,\pi/2)$. Therefore, the function $g(t)$ is strictly increasing on $[0,\pi/2)$. As $x = \tan t$ is strictly increasing for $t \in [0,\pi/2)$, we see that the function $f(x)$ is strictly increasing for $x \in [0,\infty)$, and we have

$$1+a = f(0) \le f(x) = \frac{\left(1+a\sqrt{1+x^2}\right)\arctan x}{x} < f(\infty) = \frac{\pi}{2}a, \quad \forall x \ge 0. \tag{2.28}$$

Hence inequality (1.2) holds for $x \ge 0$ with best possible constants

$$b = 1+a, \qquad c = \frac{\pi}{2}a \tag{2.29}$$

The proof of Theorem 2.1 is complete. □

Remark 2.2. We would like to remark on three special cases of Theorem 2.1.

(i) Let $a = \pi/2$. Then $b = \pi^2/4$ and $c = 1 + (\pi/2)$. Thus inequality (1.2) becomes

$$\frac{(\pi^2/2)x}{2+\pi\sqrt{1+x^2}} \le \arctan x \le \frac{(2+\pi)x}{2+\pi\sqrt{1+x^2}}, \quad x \ge 0. \tag{2.30}$$

(ii) Let $a = 2/(\pi - 2)$. Then $b = \pi(4-\pi)$ and $c = \pi/(\pi - 2)$. Thus inequality (1.2) becomes

$$\frac{\pi(4-\pi)(\pi-2)x}{(\pi-2)+2\sqrt{1+x^2}} \le \arctan x \le \frac{\pi x}{(\pi-2)+2\sqrt{1+x^2}}, \quad x \ge 0. \tag{2.31}$$

(iii) Let $a = 2$. Then $b = 3$ and $c = \pi$. Thus inequality (1.2) becomes

$$\frac{3x}{1+2\sqrt{1+x^2}} \le \arctan x \le \frac{\pi x}{1+2\sqrt{1+x^2}}, \quad x \ge 0. \tag{2.32}$$

Among inequalities (2.30)–(2.32), the upper bound

$$\frac{\pi x}{(\pi-2)+2\sqrt{1+x^2}} \tag{2.33}$$

is the best, in the sense that it is the smallest one among the three upper bounds in (2.30)–(2.32). There is no strict comparison among the three lower bounds in (2.30)–(2.32).

3. A New Lower Bound of Carlson's Inequality

Theorem 3.1 below determines the values of the parameters p, q, and r which provides the best function $P(x)$ approximating $\arccos x$.

Theorem 3.1. *Let $P(x)$ be defined by* (1.7). *Then for*

$$p = \frac{\pi+2}{\pi^2}, \qquad q = \frac{\pi-2}{\pi^2}, \qquad r = \frac{\pi}{2}, \tag{3.1}$$

one has

$$\lim_{x\to 0}\frac{\arccos x - P(x)}{x^3} = \frac{\pi^2-8}{6\pi^2}. \tag{3.2}$$

In particular, the speed of the function $P(x)$ approximating $\arccos x$ *is given by the order estimate $O(x^3)$ as $x \to 0$.*

Proof. The power series expansion of $\arccos x - P(x)$ near 0 is

$$\begin{aligned}\arccos x - P(x) = {} & \frac{\pi}{2} - r + (pr + qr - 1)x \\ & + \left(-\frac{1}{2}p^2r + \frac{1}{2}pr - \frac{1}{2}q^2r - \frac{1}{2}qr - pqr\right)x^2 \\ & + \left(\frac{1}{2}pq^2r + \frac{1}{6}q^3r + \frac{1}{2}q^2r + \frac{1}{3}qr + \frac{1}{6}p^3r + \frac{1}{2}p^2qr\right. \\ & \left. -\frac{1}{2}p^2r + \frac{1}{3}pr - \frac{1}{6}\right)x^3 + O\left(x^4\right).\end{aligned} \tag{3.3}$$

It is easy to check that for p, q, r as defined in (3.1), we have

$$\begin{gathered}\frac{\pi}{2} - r = 0, \\ pr + qr - 1 = 0 \\ -\frac{1}{2}p^2r + \frac{1}{2}pr - \frac{1}{2}q^2r - \frac{1}{2}qr - pqr = 0,\end{gathered} \tag{3.4}$$

and so

$$\arccos x - P(x) = \arccos x - \frac{(\pi/2)(1-x)^{(\pi+2)/\pi^2}}{(1+x)^{(\pi-2)/\pi^2}} = \frac{\pi^2 - 8}{6\pi^2}x^3 + O\left(x^4\right) \quad (x \longrightarrow 0). \tag{3.5}$$

□

The next theorem provides a new lower bound for the inverse cosine function.

Theorem 3.2. *For* $0 \le x \le 1$,

$$\frac{(\pi/2)(1-x)^{(\pi+2)/\pi^2}}{(1+x)^{(\pi-2)/\pi^2}} \le \arccos x. \tag{3.6}$$

Proof. For $x = 1$, inequality (3.6) clearly holds. We now consider the function

$$F(x) := \frac{(1+x)^{(\pi-2)/\pi^2}\arccos x}{(1-x)^{(\pi+2)/\pi^2}}, \quad 0 \le x < 1. \tag{3.7}$$

By an elementary change of variable

$$x = \cos(2t), \quad 0 < t \le \frac{\pi}{4}, \tag{3.8}$$

we have

$$\sqrt{1+x} = \sqrt{2}\cos t, \qquad \sqrt{1-x} = \sqrt{2}\sin t, \tag{3.9}$$

and $F(x)$ can be rewritten as

$$F(x) = f(t) := \frac{2t\left(\sqrt{2}\cos t\right)^{2(\pi-2)/\pi^2}}{\left(\sqrt{2}\sin t\right)^{2(\pi+2)/\pi^2}}, \quad 0 < t \le \frac{\pi}{4}. \tag{3.10}$$

Differentiating with respect to t yields, for $0 < t \le \pi/4$,

$$-\frac{\pi^2 (\sin t)^{(\pi^2+2\pi+4)/\pi^2} (\cos t)^{(\pi^2-2\pi+4)/\pi^2}}{2^{(\pi^2-4)/\pi^2}} f'(t) = 4t\cos(2t) - \frac{\pi^2}{2}\sin(2t) + 2\pi t. \tag{3.11}$$

Write

$$g(t) := 4t\cos(2t) - \frac{\pi^2}{2}\sin(2t) + 2\pi t, \quad 0 < t \le \frac{\pi}{4}. \tag{3.12}$$

Motivated by the investigations in [12], we are in a position to prove $g(t) > 0$ for $t \in (0, \pi/4)$. Let

$$G(t) = \begin{cases} \lambda, & t = 0, \\ \dfrac{g(t)}{t((\pi/4) - t)^2}, & 0 < t < \dfrac{\pi}{4}, \\ \mu, & t = \dfrac{\pi}{4}, \end{cases} \tag{3.13}$$

where λ and μ are constants determined with limits:

$$\begin{aligned} \lambda &= \lim_{t \to 0+} \frac{g(t)}{t((\pi/4) - t)^2} = \frac{64 - 16\pi^2 + 32\pi}{\pi^2} = 0.6704721009\ldots, \\ \mu &= \lim_{t \to \pi/4-} \frac{g(t)}{t((\pi/4) - t)^2} = \frac{4\pi^2 - 32}{\pi} = 7.47841762\ldots. \end{aligned} \tag{3.14}$$

Using Maple we determine Taylor approximation for the function $G(t)$ by the polynomial of the first order:

$$P_1(t) = \frac{128\left(4 - \pi^2 + 2\pi\right)}{\pi^3} t + \frac{16\left(4 - \pi^2 + 2\pi\right)}{\pi^2}, \tag{3.15}$$

which has a bound of absolute error

$$\varepsilon_1 = \frac{4\pi^3 + 48\pi^2 - 128\pi - 192}{\pi^2} = 0.3690379422\ldots \tag{3.16}$$

for values $t \in [0, \pi/4]$. It is true that

$$G(t) - (P_1(t) - \varepsilon_1) \geq 0, \quad P_1(t) - \varepsilon_1 > 0, \tag{3.17}$$

for $t \in [0, \pi/4]$. Hence, for $t \in [0, \pi/4]$ it is true that $G(t) > 0$ and therefore $g(t) > 0$ and $f'(t) < 0$ for $t \in (0, \pi/4]$. Therefore, the function $f(t)$ is strictly decreasing on $(0, \pi/4]$. As $x = \cos(2t)$ is strictly decreasing on $(0, \pi/4]$, we see that $F(x)$ is strictly increasing for $x \in [0, 1)$, and hence

$$\frac{\pi}{2} = F(0) \leq F(x) = \frac{(1+x)^{(\pi-2)/\pi^2} \arccos x}{(1-x)^{(\pi+2)/\pi^2}} \quad \forall x \in [0, 1). \tag{3.18}$$

By rearranging terms in the last expression, Theorem 3.2 follows. □

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References

[1] R. E. Shafer, "Problem E1867," *The American Mathematical Monthly*, vol. 73, no. 3, pp. 309–310, 1966.
[2] R. E. Shafer, L. S. Grinstein, D. C. B. Marsh, and J. D. E. Konhauser, "Problems and solutions: Solutions of elementary problems: E1867," *The American Mathematical Monthly*, vol. 74, no. 6, pp. 726–727, 1967.
[3] F. Qi, S.-Q. Zhang, and B.-N. Guo, "Sharpening and generalizations of Shafer's inequality for the arc tangent function," *Journal of Inequalities and Applications*, vol. 2009, Article ID 930294, 9 pages, 2009.
[4] B. C. Carlson, "Inequalities for a symmetric elliptic integral," *Proceedings of the American Mathematical Society*, vol. 25, pp. 698–703, 1970.
[5] D. S. Mitrinović, *Analytic inequalities*, Springer, New York, NY, USA, 1970.
[6] B.-N. Guo and F. Qi, "Sharpening and generalizations of Carlson's double inequality for the arc cosine function," http://arxiv.org/abs/0902.3039.
[7] B.-N. Guo and F. Qi, "Sharpening and generalizations of Carlson's inequality for the arc cosine function," *Hacettepe Journal of Mathematics and Statistics*, vol. 39, no. 3, pp. 403–409, 2010.
[8] C.-P. Chen and C. Mortici, "Generalization and sharpness of Carlson's inequality for the inverse cosine function," submitted.
[9] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen, "Generalized elliptic integrals and modular equations," *Pacific Journal of Mathematics*, vol. 192, no. 1, pp. 1–37, 2000.
[10] G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1997.
[11] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Monotonicity of some functions in calculus," preprint.
[12] B. J. Malešević, "One method for proving inequalities by computer," *Journal of Inequalities and Applications*, vol. 2007, Article ID 78691, 8 pages, 2007.