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The Mapping $v_{x,y}$ in Normed Linear Spaces with Applications to Inequalities in Analysis

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In this paper we introduce the mapping $v_{x,y}$ connected with the lower and upper semi-inner products $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$, and study its monotonicity, boundedness, convexity and other properties. Applications to theory of inequalities in analysis are given including refinements of the Schwarz inequality.

Keywords: Lower and upper semi-inner products; Normed spaces; Inner product spaces; Inequalities for sums and integrals in analysis

AMS Subject Classification: 46B20, 46B99, 46C20, 46C99, 26D15, 26D99

1 INTRODUCTION

Let $(X, \|\cdot\|)$ be a real normed linear space. Consider the lower and upper semi-inner products

$$(y,x)_i = \lim_{t\to 0-} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

and

$$(y,x)_s = \lim_{t\to 0+} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

which are well defined for every pair x, $y \in X$ (see for example [1]).

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For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see [1,2,3]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(I) $(x, x)_p = \|x\|^2$ for all $x \in X$;
(II) $(\alpha x, \beta y)_p = \alpha\beta(x, y)_p$ if $\alpha\beta \geq 0$ and x, $y \in X$;
(III) $|(x, y)_p| \leq \|x\| \, \|y\|$ for all x, $y \in X$;
(IV) $(\alpha x + y, x)_p = \alpha(x, x)_p + (y, x)_p$ if x, y belong to X and α is a real number;
(V) $(-x, y)_p = -(x, y)_q$ for all x, $y \in X$;
(VI) $(x + y, z)_p \leq \|x\| \, \|z\| + (y, z)_p$ for all x, y, $z \in X$;
(VII) The mapping $(\cdot, \cdot)_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(VIII) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if $(y, x_0)_s = (y, x_0)_i$ for all $y \in X$; in general, $(y, x)_i \leq (y, x)_s$ for all x, $y \in X$;
(IX) If the norm $\|\cdot\|$ is induced by an inner product $(\cdot, \cdot)$, then

$$(y, x)_i = (y, x) = (y, x)_s \quad \text{for all } x,\ y \in X.$$

For other properties of $(\cdot, \cdot)_p$ see the recent papers [1,2,3], where further references are given.

The terminology throughout the paper is standard. We mention that for functions we use the terms ‘increasing’ (and ‘strictly increasing’), ‘decreasing’ (and ‘strictly decreasing’), thus avoiding ‘nondecreasing’ and ‘nonincreasing’.

2 MAPPINGS ASSOCIATED WITH THE LOWER AND UPPER SEMI-INNER PRODUCTS

Let $(X, \|\cdot\|)$ be a real normed linear space and x, y two fixed elements of X. We define the following mappings:

$$n_{x,y}: \mathbb{R} \to \mathbb{R}_+, \quad n_{x,y}(t) = \|x + ty\|,$$
$$v_{x,y}: \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad v_{x,y}(t) = \frac{\|x + ty\| - \|x\|}{t}.$$

We shall list here some pertinent properties of the mappings n and v.

PROPOSITION 2.1 *Let x, y be fixed in X. Then*

(i) $n_{x,y}$ *is convex on* $\mathbb{R}$;
(ii) $n_{x,y}$ *is continuous and has one sided derivatives at each point of* $\mathbb{R}$;
(iii) *If x, y are linearly independent, then*

$$\frac{d^+ n_{x,y}(t)}{dt} = \frac{(y, x+ty)_s}{\|x+ty\|}, \quad t \in \mathbb{R} \tag{2.1}$$

and

$$\frac{d^- n_{x,y}(t)}{dt} = \frac{(y, x+ty)_i}{\|x+ty\|}, \quad t \in \mathbb{R}. \tag{2.2}$$

Proof (i) is well known; (ii) follows from the convexity of $n_{x,y}$ [6, Proposition 5.5.17].

(iii) Let $t \in \mathbb{R}$. Then

$$\begin{aligned}\frac{d^+ n_{x,y}(t)}{dt} &= \lim_{\substack{\alpha \to t \\ \alpha > t}} \left(\frac{\|x+\alpha y\| - \|x+ty\|}{\alpha - t} \right) \\ &= \lim_{\substack{\beta \to 0 \\ \beta > 0}} \left(\frac{\|x+ty+\beta y\| - \|x+ty\|}{\beta} \right) \\ &= \lim_{\beta \to 0+} \frac{\|x+ty+\beta y\|^2 - \|x+ty\|^2}{2\beta} \\ &\quad \times \lim_{\beta \to 0+} \frac{2}{\|x+ty+\beta y\| + \|x+ty\|} \\ &= \frac{(y, x+ty)_s}{\|x+ty\|},\end{aligned}$$

and relation (2.1) is proved.

Equality (2.2) is proved in a similar fashion.

Remark 2.2 In the case of a normed linear space, the graph of the mapping $n_{x,y}$ for fixed linearly independent vectors x, y is depicted in Figure 1. The mapping is convex, but may not be strictly convex; this is suggested by drawing the graph in a dashed line.

In the case of an inner product space, the mapping $v_{x,y}$ is strictly convex and attains a unique minimum at the point $t_0 = -(y, x)/\|y\|^2$

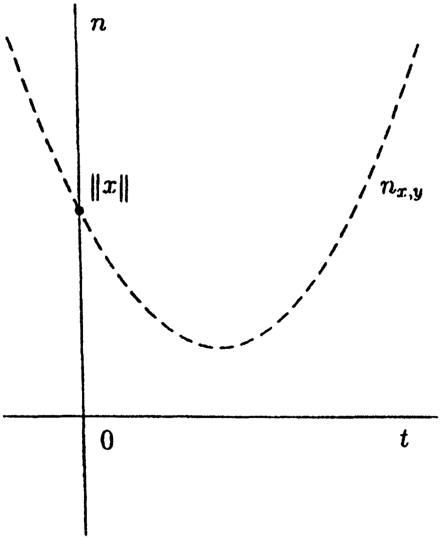

FIGURE 1

equal to

$$n_0 := n_{x,y}(t_0) = \frac{(\|x\|^2\|y\|^2 - (y,x)^2)^{1/2}}{\|y\|}.$$

Indeed, $n'_{x,y}(t) = 0$ if and only if $t = t_0$ and

$$n_{x,y}(t_0) = \left\| x - \frac{(y,x)y}{\|y\|^2} \right\| = \frac{(\|x\|^2\|y\|^2 - (y,x)^2)^{1/2}}{\|y\|}.$$

The graph of $n_{x,y}$ for the case of an inner product is depicted in Figure 2.

The mapping $v_{x,y}$ has the following properties.

THEOREM 2.3 *Let x, y be fixed in X. Then*:

(i) $v_{x,y}$ *is increasing on* $\mathbb{R}\backslash\{0\}$;
(ii) $v_{x,y}$ *is bounded and*

$$|v_{x,y}(t)| \le \|y\| \quad \textit{for all } t \in \mathbb{R}\backslash\{0\}; \tag{2.3}$$

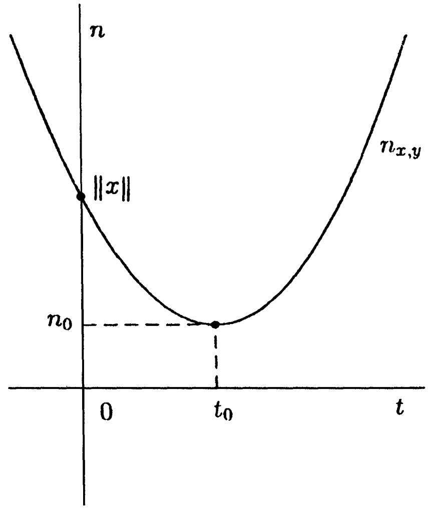

FIGURE 2

(iii) *We have the inequalities*

$$\frac{(y, x+uy)_s}{\|x+uy\|} \leq v_{x,y}(u) \leq \frac{(y,x)_i}{\|x\|} \quad \textit{for all } u<0, \tag{2.4}$$

$$\frac{(y, x+ty)_i}{\|x+ty\|} \geq v_{x,y}(t) \geq \frac{(y,x)_s}{\|x\|} \quad \textit{for all } t>0, \tag{2.5}$$

assuming that x, y *are linearly independent.*

(iv) *We have the limits*

$$\lim_{u \to -\infty} v_{x,y}(u) = -\|y\|, \quad \lim_{t \to \infty} v_{x,y}(t) = \|y\| \tag{2.6}$$

and

$$\lim_{u \to 0-} v_{x,y}(u) = \frac{(y,x)_i}{\|x\|}, \quad \lim_{t \to 0+} v_{x,y}(t) = \frac{(y,x)_s}{\|x\|} \tag{2.7}$$

assuming that $x \neq 0$;

(v) $v_{x,y}$ *has one sided derivatives at each point of* $\mathbb{R}\backslash\{0\}$ *and, if* x, y *are linearly independent, then*

$$\frac{d^+ v_{x,y}(t)}{dt} = \frac{1}{t^2}\left[t\frac{(y,x+ty)_s}{\|x+ty\|} - n_{x,y}(t) + \|x\|\right] \tag{2.8}$$

and

$$\frac{d^- v_{x,y}(t)}{dt} = \frac{1}{t^2}\left[t\frac{(y,\, x+ty)_i}{\|x+ty\|} - n_{x,y}(t) + \|x\|\right] \tag{2.9}$$

for all $t \in \mathbb{R}\backslash\{0\}$.

Proof (i) Since $n_{x,y}$ is convex, the mapping

$$v_{x,y}(t) = \frac{n_{x,y}(t) - n_{x,y}(0)}{t-0}$$

is known to be increasing on $\mathbb{R}\backslash\{0\}$ [6, Lemma 5.5.16].

(ii) By the triangle inequality for the norm we have

$$|\,\|x+ty\| - \|x\|\,| \le \|x+ty-x\| = |t|\,\|y\|, \quad t \in \mathbb{R},$$

from which (2.3) follows.

(iii) Let $u<0$. Then, by the Schwarz inequality,

$$(x, x+uy)_s \le \|x\|\,\|x+uy\|.$$

From the properties of semi-inner product $(\cdot,\,\cdot)_s$ we obtain

$$\begin{aligned}(x, x+uy)_s &= (x+uy-uy, x+uy)_s\\ &= \|x+uy\|^2 + (-uy, x+uy)_s\\ &= \|x+uy\|^2 - u(y, x+uy)_s,\end{aligned}$$

and so, by the previous inequality,

$$\|x+uy\|^2 - u(y, x+uy)_s \le \|x\|\,\|x+uy\|$$

for all $u<0$, from which we get

$$v_{x,y}(u) = \frac{\|x+uy\| - \|x\|}{u} \ge \frac{(y, x+uy)_s}{\|x+uy\|},$$

and the first inequality in (2.4) is proved.

By the Schwarz inequality,

$$\|x\|\,\|x+uy\| \geq (x+uy, x)_s$$

for all $u<0$. A simple calculation shows that

$$(x+uy, x)_s = \|x\|^2 + (uy, x)_s = \|x\|^2 - u(-y, x)_s = \|x\|^2 + u(y, x)_i$$

for all $u<0$, and thus the above inequality gives

$$\|x\|\,\|x+uy\| - \|x\|^2 \geq u(y, x)_i, \quad u<0,$$

from where we get

$$v_{x,y}(u) = \frac{\|x+uy\| - \|x\|}{u} \leq \frac{(y, x)_s}{\|x\|}$$

for all $u<0$, and the second inequality in (2.4) is also proved.
Inequality (2.5) is proved analogously.
(iv) We have

$$\begin{aligned}\lim_{t\to+\infty} v_{x,y}(t) &= \lim_{\alpha\to 0+} v_{x,y}(1/\alpha) = \lim_{\alpha\to 0+} \frac{\|x+y/\alpha\| - \|x\|}{1/\alpha}\\ &= \lim_{\alpha\to 0+}(\|y+\alpha x\| - \alpha\|x\|) = \|y\|.\end{aligned}$$

The second limit in (2.6) is proved similarly.
Now let us observe that

$$\begin{aligned}\lim_{t\to 0+} v_{x,y}(t) &= \lim_{t\to 0+} \frac{\|x+ty\|^2 - \|x\|^2}{2t} \cdot \lim_{t\to 0+} \frac{2}{\|x+ty\| + \|x\|}\\ &= \frac{(y, x)_s}{\|x\|}\end{aligned}$$

for all $x \in X\backslash\{0\}$.
The second limit in (2.7) is proved similarly.

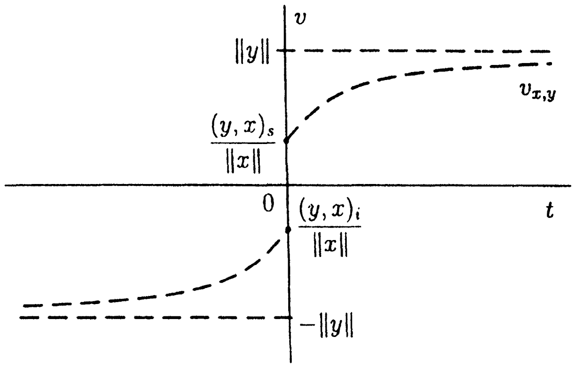

FIGURE 3

(v) The fact that $v_{x,y}$ has one sided derivatives at each point $t \in \mathbb{R}\backslash\{0\}$ is obvious. Let us compute the one sided derivatives:

$$\begin{aligned}\frac{d^+ v_{x,y}(t)}{dt} &= \frac{d^+}{dt}\left(\frac{n_{x,y}(t) - \|x\|}{t}\right)\\ &= \frac{1}{t^2}\left[\frac{d^+ n_{x,y}(t)}{dt}t - (n_{x,y}(t) - \|x\|)\right]\\ &= \frac{1}{t^2}\left[t\frac{(y, x+ty)_s}{\|x+ty\|} - n_{x,y}(t) + \|x\|\right],\end{aligned}$$

and relation (2.8) is obtained.

Identity (2.9) is established similarly.

Remark 2.4 In the case of a normed linear space, the graph of the mapping $v_{x,y}$ is depicted in Figure 3. The dashed line suggests that the convexity of the mapping is not known.
Note that if the space $(X, \|\cdot\|)$ is smooth at x, then $(y,x)_s = (y,x)_i$.

The lines $v = \|y\|$ and $v = -\|y\|$ are asymptotes to the graph as $t \to \infty$ and $t \to -\infty$, respectively.

3 THE CASE OF INNER PRODUCT SPACES

We address ourselves to an important question whether $v_{x,y}$ is convex (concave) by considering the case when $(X, \|\cdot\|)$ is an inner product space. In this case the following proposition holds.

PROPOSITION 3.1 *If* $(X;(\cdot,\cdot))$ *is a real inner product space and* x, y *linearly independent vectors in* X, *then the mapping* $v_{x,y}$ *is twice differentiable on* $\mathbb{R}\backslash\{0\}$ *with*

$$\frac{d^2 v_{x,y}(t)}{dt^2} = t\,\frac{n_{x,y}^2(t)(n_{x,y}(t) - \|x\|)^2 - t^2(y, x+ty)^2}{t^4 n_{x,y}^3(t)}, \quad t \in \mathbb{R}\backslash\{0\}. \tag{3.1}$$

Proof If $(X;(\cdot,\cdot))$ is an inner product space, then $v_{x,y}$ is differentiable on $\mathbb{R}\backslash\{0\}$, and

$$\frac{dv_{x,y}(t)}{dt} = \frac{1}{t^2}\left[t\,\frac{(y,x) + t\|y\|^2}{n_{x,y}(t)} - n_{x,y}(t) + \|x\|\right].$$

The second derivative of $v_{x,y}$ also exists, and

$$\frac{d^2 v_{x,y}(t)}{dt^2} = \frac{I}{t^4 n_{x,y}^2(t)},$$

where $I = I_{x,y}(t)$ is calculated as follows:

$$\begin{aligned}
I &= \frac{d}{dt}\Big(t(y,x) + t^2\|y\|^2 - n_{x,y}^2(t) + \|x\| n_{x,y}(t)\Big) t^2 n_{x,y}(t) \\
&\quad - (t(y,x) + t^2\|y\|^2 - n_{x,y}^2(t) + \|x\| n_{x,y}(t))\,\frac{d}{dt}(t^2 n_{x,y}(t)) \\
&= ((y,x) + 2t\|y\|^2 - 2n_{x,y}(t)n'_{x,y}(t) + \|x\| n'_{x,y}(t))t^2 n_{x,y}(t) \\
&\quad - (t(y,x) + t^2\|y\|^2 - n_{x,y}^2(t) + \|x\| n_{x,y}(t))(2tn_{x,y}(t) + t^2 n'_{x,y}(t)) \\
&= t^2(y,x)n_{x,y}(t) + 2t^3\|y\|^2 n_{x,y}(t) - 2t^2 n_{x,y}^2(t)n'_{x,y}(t) \\
&\quad + t^2\|x\| n_{x,y}(t)n'_{x,y}(t) - 2t^2(y,x)n_{x,y}(t) - 2t^3\|y\|^2 n_{x,y}(t) \\
&\quad + 2tn_{x,y}^3(t) - 2t\|x\| n_{x,y}^2(t) - t^3(y,x)n'_{x,y}(t) - t^4\|y\|^2 n'_{x,y}(t) \\
&\quad + t^2 n_{x,y}^2(t)n'_{x,y}(t) - t^2\|x\| n_{x,y}(t)n'_{x,y}(t) \\
&= -t^2(y,x)n_{x,y}(t) - t^2 n_{x,y}^2(t)n'_{x,y}(t) + 2tn_{x,y}^3(t) - 2t\|x\| n_{x,y}^2(t) \\
&\quad - t^3(y,x)n'_{x,y}(t) - t^4\|y\|^2 n'_{x,y}(t)
\end{aligned}$$

$$\begin{aligned}
&= -t^2(y,x)n_{x,y}(t) + 2tn^3_{x,y}(t) - 2t\|x\|n^2_{x,y}(t) \\
&\quad - t^2 n^2_{x,y}(t)\frac{(y,x)+t\|y\|^2}{n_{x,y}(t)} - t^3(y,x)\frac{(y,x)+t\|y\|^2}{n_{x,y}(t)} \\
&\quad - t^4\|y\|^2\frac{(y,x)+t\|y\|^2}{n_{x,y}(t)} = \frac{J}{n_{x,y}(t)},
\end{aligned}$$

where

$$\begin{aligned}
J = &-t^2(y,x)n^2_{x,y}(t) + 2tn^4_{x,y}(t) - 2t\|x\|n^3_{x,y}(t) - t^2n^2_{x,y}(t)(y,x) \\
&- t^3n^2_{x,y}(t)\|y\|^2 - t^3(y,x)^2 - 2t^4\|y\|^2(y,x) - t^5\|y\|^4.
\end{aligned}$$

But

$$\begin{aligned}
&- t^5\|y\|^4 - 2t^4\|y\|^2(y,x) - t^3(y,x)^2 \\
&\quad = -t^3(t\|y\|^2 + (y,x))^2 = -t^3(y,x+ty)^2,
\end{aligned}$$

and

$$\begin{aligned}
&2tn^4_{x,y}(t) - 2t\|x\|n^3_{x,y}(t) - 2t^2(y,x)n^2_{x,y}(t) - t^3n^2_{x,y}(t)\|y\|^2 \\
&\quad = tn^2_{x,y}(t)(2n^2_{x,y}(t) - 2\|x\|n_{x,y}(t) - 2t(y,x) - t^2\|y\|^2) \\
&\quad = tn^2_{x,y}(t)(n^2_{x,y}(t) + \|x+ty\|^2 - 2\|x\|n_{x,y}(t) - 2t(y,x) - t^2\|y\|^2) \\
&\quad = tn^2_{x,y}(t)(n^2_{x,y}(t) + \|x\|^2 + 2t(y,x) + t^2\|y\|^2 - 2\|x\|n_{x,y}(t) \\
&\qquad - 2t(y,x) - t^2\|y\|^2) \\
&\quad = tn^2_{x,y}(t)(n_{x,y}(t) - \|x\|)^2.
\end{aligned}$$

In conclusion, for each $t \in \mathbb{R}\backslash\{0\}$ we get

$$\frac{d^2v_{x,y}(t)}{dt^2} = t\,\frac{n^2_{x,y}(t)(n_{x,y}(t) - \|x\|)^2 - t^2(y,x+ty)^2}{t^4n^3_{x,y}(t)}. \tag{3.2}$$

In the case that the vectors x, y are orthogonal, the preceding proposition gives information about the convexity and concavity of $v_{x,y}$.

PROPOSITION 3.2 *If x, y are nonzero orthogonal vectors in an inner product space $(X;(\cdot,\cdot))$, then the mapping $v_{x,y}$ is strictly convex on $(-\infty,0)$ and strictly concave on $(0,\infty)$.*

Proof By the preceding proposition we have

$$\frac{d^2 v_{x,y}(t)}{dt^2} = t\frac{c_{x,y}(t)}{t^4 n_{x,y}^2(t)},$$

where

$$\begin{aligned} c_{x,y}(t) &:= n_{x,y}^2(t)(n_{x,y}(t) - \|x\|)^2 - t^2(y, x+ty)^2 \\ &= \left[n_{x,y}(t)(n_{x,y}(t) - \|x\|) - t(y, x+ty)\right] \\ &\quad \times \left[n_{x,y}(t)(n_{x,y}(t) - \|x\|) + t(y, x+ty)\right]. \end{aligned}$$

As x, y are linearly independent (being orthogonal and nonzero), we have by (2.4) and (2.5) that

$$\frac{(y, x+ty)}{\|x+ty\|} < \frac{\|x+ty\| - \|x\|}{t} \quad \text{if } t < 0$$

and

$$\frac{(y, x+ty)}{\|x+ty\|} > \frac{\|x+ty\| - \|x\|}{t} \quad \text{if } t > 0;$$

in both cases we have

$$n_{x,y}(t)(n_{x,y}(t) - \|x\|) - t(y, x+ty) < 0.$$

If $x \perp y$, then $(x, y) = 0$ and $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$. Then

$$\begin{aligned} &n_{x,y}(t)(n_{x,y}(t) - \|x\|) + t(y, x+ty) \\ &\quad = \|x+ty\|(\|x+ty\| - \|x\|) + t^2\|y\|^2 > 0. \end{aligned}$$

Consequently $v''_{x,y}(t) > 0$ if $t \in (-\infty, 0)$ and $v''_{x,y}(t) < 0$ if $t \in (0, \infty)$, and the proposition is proved.

Remark 3.3 If it is assumed that X is an inner product space and x, y are nonzero orthogonal vectors in X, then the mapping $v_{x,y}$ is strictly increasing in $\mathbb{R}\backslash\{0\}$, strictly convex on $(-\infty, 0)$, and strictly concave on $(0, \infty)$.

It can be extended by continuity to all of $\mathbb{R}$ as

$$\lim_{t \to 0} v_{x,y}(t) = (y, x)/\|x\| = 0$$

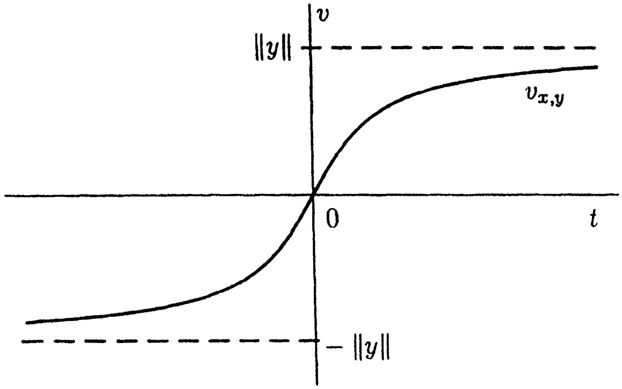

FIGURE 4

in view of equation (2.7). In this case the graph passes through the origin; it is depicted in Figure 4.

We note that the lines $v=\|y\|$ and $v=-\|y\|$ are asymptotes for $t\to\infty$ and $t\to-\infty$, respectively.

4 APPLICATIONS TO THEORY OF INEQUALITIES IN ANALYSIS

The results of the preceding sections can be used to improve on some classical theorems of theory of inequalities in analysis, such as the Cauchy–Buniakovski–Schwarz inequality for sums and integrals, Hölder's inequality for sums and integrals, and other.

The following proposition which follows from Theorem 2.3 gives an improvement on the Schwarz inequality in normed linear spaces.

PROPOSITION 4.1 *Let x, y be two linearly independent vectors in a normed linear space $(X, \|\cdot\|)$. Then the following inequalities hold for all $u<0<t$:*

$$
\begin{aligned}
-\|x\|\|y\| &\le \frac{\|x\|(y, x+uy)_i}{\|x+uy\|} \le \frac{\|x\|(y, x+uy)_s}{\|x+uy\|} \\
&\le \frac{\|x\|(\|x+uy\|-\|x\|)}{u}
\end{aligned}
$$

$$
\begin{aligned}
&\le (y,x)_i \le (y,x)_s \\
&\le \frac{\|x\|(\|x+ty\| - \|x\|)}{t} \\
&\le \frac{\|x\|(y, x+ty)_i}{\|x+ty\|} \le \frac{\|x\|(y, x+ty)_s}{\|x+ty\|} \\
&\le \|x\|\|y\|. \qquad (4.1)
\end{aligned}
$$

We give some concrete examples of the preceding proposition.

Example 4.2 Let $\ell^1(\mathbb{R}) := \{x = (x_i)_{i\in\mathbb{N}} | \sum_{i=1}^{\infty} |x_i| < \infty\}$ and let $x, y \in \ell^1(\mathbb{R})$. Then the following inequalities hold for all $u < 0 < t$:

$$
\begin{aligned}
-\sum_{i=1}^{\infty} |y_i| &\le \sum_{x_i+uy_i\neq 0} \operatorname{sgn}(x_i+uy_i)y_i - \sum_{x_i+uy_i=0} |y_i| \\
&\le \sum_{x_i+uy_i\neq 0} \operatorname{sgn}(x_i+uy_i)y_i + \sum_{x_i+uy_i=0} |y_i| \\
&\le \sum_{i=1}^{\infty} \frac{|x_i+uy_i| - |x_i|}{u} \\
&\le \sum_{x_i\neq 0} \operatorname{sgn}(x_i)y_i - \sum_{x_i=0} |y_i| \\
&\le \sum_{x_i\neq 0} \operatorname{sgn}(x_i)y_i + \sum_{x_i=0} |y_i| \\
&\le \sum_{i=1}^{\infty} \frac{|x_i+ty_i| - |x_i|}{t} \\
&\le \sum_{x_i+ty_i\neq 0} \operatorname{sgn}(x_i+ty_i)y_i - \sum_{x_i+ty_i=0} |y_i| \\
&\le \sum_{x_i+ty_i\neq 0} \operatorname{sgn}(x_i+ty_i)y_i + \sum_{x_i+ty_i=0} |y_i| \\
&\le \sum_{i=1}^{\infty} |y_i|. \qquad (4.2)
\end{aligned}
$$

This follows from (4.1) taking into account that in $\ell^1(\mathbb{R})$ we have [4]

$$
(x,y)_i = \sum_{i=1}^{\infty} |y_i| \left(\sum_{y_i\neq 0} \operatorname{sgn}(y_i)x_i - \sum_{y_i=0} |x_i| \right)
$$

and

$$(x, y)_s = \sum_{i=1}^{\infty} |y_i| \left(\sum_{y_i \neq 0} \operatorname{sgn}(y_i) x_i + \sum_{y_i = 0} |x_i| \right),$$

where $\operatorname{sgn}(t) = |t|/t$ for $t \neq 0$.

Example 4.3 Let x, y be two linearly independent elements of $\ell^2(\mathbb{R}) := \{x = (x_i)_{i \in \mathbb{N}} | \sum_{i=1}^{\infty} x_i^2 < \infty\}$. If $u < 0 < t$, the following inequalities hold:

$$\begin{aligned}
-\left(\sum_{i=1}^{\infty} x_i^2 \sum_{i=1}^{\infty} y_i^2\right)^{1/2} &\leq \left[\frac{\sum_{i=1}^{\infty} x_i^2}{\sum_{i=1}^{\infty} (x_i + u y_i)^2}\right]^{1/2} \sum_{i=1}^{\infty} y_i (x_i + u y_i) \\
&\leq \left[\left(\sum_{i=1}^{\infty} (x_i + u y_i)^2 \sum_{i=1}^{\infty} x_i^2\right)^{1/2} - \sum_{i=1}^{\infty} x_i^2\right] \Big/ u \\
&\leq \sum_{i=1}^{\infty} x_i y_i \\
&\leq \left[\left(\sum_{i=1}^{\infty} (x_i + t y_i)^2 \sum_{i=1}^{\infty} x_i^2\right)^{1/2} - \sum_{i=1}^{\infty} x_i^2\right] \Big/ t \\
&\leq \left[\frac{\sum_{i=1}^{\infty} x_i^2}{\sum_{i=1}^{\infty} (x_i + t y_i)^2}\right]^{1/2} \sum_{i=1}^{\infty} y_i (x_i + t y_i) \\
&\leq \left(\sum_{i=1}^{\infty} x_i^2 \sum_{i=1}^{\infty} y_i^2\right)^{1/2}. \qquad (4.3)
\end{aligned}$$

This follows from (4.1) applied to the inner product $(x, y) = \sum_{i=1}^{\infty} x_i y_i$.

Example 4.4 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a positive measure μ, and let $p > 1$. If $L^p(\Omega)$ is the Banach space of all real valued functions on Ω, p-integrable with respect to μ, then it is known [7] that

$$\lim_{t \to 0} (\|x + ty\|_p - \|x\|_p)/t = \|x\|_p^{1-p} \int_{\Omega} |x(s)|^{p-1} \operatorname{sgn}(x(s)) y(s) d\mu(s).$$

Then

$$[y, x]_p := \lim_{t\to 0} \frac{\|x+ty\|_p^2 - \|x\|_p^2}{2t} = \|x\|_p^{2-p} \int_\Omega |x(s)|^{p-1} \operatorname{sgn}(x(s)) y(s) d\mu(s),$$

where $x,\ y \in L^p(\Omega)$.

Let x, y be linearly independent elements of $L^p(\Omega)$ and $u<0<t$. Applying inequality (4.1) to the semi-inner product $[\cdot,\cdot]_p = (\cdot,\cdot)_i = (\cdot,\cdot)_s$, we obtain the following inequalities:

$$\begin{aligned}
&-\left(\int_\Omega |y(s)|^p d\mu(s)\right)^{1/p} \\
&\le \frac{\int_\Omega |x(s)+uy(s)|^{p-1} \operatorname{sgn}(x(s)+uy(s)) y(s) d\mu(s)}{\left(\int_\Omega |x(s)+uy(s)|^p d\mu(s)\right)^{(p-1)/p}} \\
&\le \frac{1}{u}\left[\left(\int_\Omega |x(s)+uy(s)|^p d\mu(s)\right)^{1/p} - \left(\int_\Omega |x(s)|^p d\mu(s)\right)^{1/p}\right] \\
&\le \frac{\int_\Omega |x(s)|^{p-1} \operatorname{sgn}(x(s)) y(s) d\mu(s)}{\left(\int_\Omega |x(s)|^p d\mu(s)\right)^{(p-1)/p}} \\
&\le \frac{1}{t}\left[\left(\int_\Omega |x(s)+ty(s)|^p d\mu(s)\right)^{1/p} - \left(\int_\Omega |x(s)|^p d\mu(s)\right)^{1/p}\right] \\
&\le \frac{\int_\Omega |x(s)+ty(s)|^{p-1} \operatorname{sgn}(x(s)+ty(s)) y(s) d\mu(s)}{\left(\int_\Omega |x(s)+ty(s)|^p d\mu(s)\right)^{(p-1)/p}} \\
&\le \left(\int_\Omega |y(s)|^p d\mu(s)\right)^{1/p}. \qquad (4.4)
\end{aligned}$$

We give further applications of the previously obtained results. First we have the following proposition.

PROPOSITION 4.5 *Let $a, b \in \mathbb{R}$ with $a<b$. Then the following inequalities hold*:

$$\begin{aligned}
\left\|x+\tfrac{1}{2}(a+b)y\right\| &\le (b-a)^{-1}\int_a^b \|x+ty\| dt \\
&\le \tfrac{1}{2}\left[\left\|x+\tfrac{1}{2}(a+b)y\right\| + \tfrac{1}{2}(\|x+ay\| + \|x+by\|)\right]
\end{aligned}$$

$$\leq \tfrac{1}{2}(\|x+ay\| + \|x+by\|)$$
$$\leq \|x\| + \tfrac{1}{2}(|a|+|b|)\|y\|. \tag{4.5}$$

Proof The mapping $f:\mathbb{R}\to\mathbb{R}$, $f(t):=\|x+ty\|$, is convex on $\mathbb{R}$. We deduce (4.5) by using properties of norm and the following Hermite–Hadamard's inequality for convex functions [5, p. 10]:

$$\begin{aligned} f\left(\tfrac{1}{2}(a+b)\right) &\leq (b-a)^{-1}\int_a^b f(t)dt \\ &\leq \tfrac{1}{2}\left[f\left(\tfrac{1}{2}(a+b)\right)+\tfrac{1}{2}(f(a)+f(b))\right] \\ &\leq \tfrac{1}{2}(f(a)+f(b)), \quad a<b, \quad a,b\in\mathbb{R}. \end{aligned}$$

PROPOSITION 4.6 *Let a, $b\in\mathbb{R}$, $0<a<b$. The following refinement of the Schwarz inequality holds*:

$$\begin{aligned} (y,x)_s &\leq \frac{\|x\|}{b-a}\int_a^b v_{x,y}(t)dt \\ &\leq \frac{2}{a+b}\left[\frac{1}{2}\left(\left\|x+\frac{a+b}{2}y\right\|+\frac{\|x+ay\|+\|x+by\|}{2}\right)-\|x\|\right]\|x\| \\ &\leq \frac{2}{a+b}\left[\frac{\|x+ay\|+\|x+by\|}{2}-\|x\|\right]\|x\| \\ &\leq \|x\|\|y\| \end{aligned} \tag{4.6}$$

for all x, $y\in X$.

Proof We can apply Čebyšev's inequality [5, p. 239] to the mapping $v_{x,y}$, which is increasing on $[a,b]$, to obtain

$$(b-a)\int_a^b v_{x,y}(t)tdt \geq \int_a^b v_{x,y}(t)dt\int_a^b tdt,$$

which is equivalent to

$$(b-a)\int_a^b (\|x+ty\|-\|x\|)dt \geq \frac{b^2-a^2}{2}\int_a^b v_{x,y}(t)dt,$$

that is,

$$\frac{1}{b-a}\int_a^b v_{x,y}(t)dt \le \frac{2}{a+b}\left[\frac{1}{b-a}\int_a^b \|x+ty\|dt - \|x\|\right].$$

By the preceding proposition,

$$\frac{1}{b-a}\int_a^b \|x+ty\|dt \le \frac{1}{2}\left[\left\|x+\frac{a+b}{2}y\right\| + \frac{\|x+ay\|+\|x+by\|}{2}\right]$$
$$\le \frac{\|x+ay\|+\|x+by\|}{2} \le \|x\| + \frac{a+b}{2}\|y\|,$$

and then

$$\frac{1}{b-a}\int_a^b v_{x,y}(t)dt$$
$$\le \frac{2}{a+b}\left[\frac{1}{2}\left(\left\|x+\frac{a+b}{2}y\right\| + \frac{\|x+ay\|+\|x+by\|}{2}\right) - \|x\|\right]$$
$$\le \frac{2}{a+b}\left[\frac{\|x+ay\|+\|x+by\|}{2} - \|x\|\right] \le \|y\|,$$

and (4.6) is proved except for the first part which follows from Proposition 4.1.

Remark 4.7 The inequality

$$\frac{1}{b-a}\int_a^b v_{x,y}(t)dt \le \frac{2}{a+b}\left[\tfrac{1}{2}(\|x+ay\|+\|x+by\|) - \|x\|\right]$$

is equivalent to the following interesting inequality for the mapping $v_{x,y}$:

$$\frac{1}{b-a}\int_a^b v_{x,y}(t)dt \le \frac{av_{x,y}(a)+bv_{x,y}(b)}{a+b}, \quad 0<a<b.$$

Remark 4.8 If we assume that $a<b<0$, then we prove in a similar way that

$$\begin{aligned}
(y,x)_i &\geq \frac{\|x\|}{b-a}\int_a^b v_{x,y}(t)dt \\
&\geq \frac{2}{a+b}\left[\frac{1}{2}\left(\left\|x+\frac{a+b}{2}y\right\| + \frac{\|x+ay\|+\|x+by\|}{2}\right) - \|x\|\right]\|x\| \\
&\geq \frac{2}{a+b}\left[\frac{\|x+ay\|+\|x+by\|}{2} - \|x\|\right]\|x\| \\
&\geq -\|x\|\|y\| \qquad (4.7)
\end{aligned}$$

for all $x, y \in X$.

If $(X;(\cdot,\cdot))$ is an inner product space, the mapping $v_{x,y}$ is strictly convex in $(-\infty,0)$ and strictly concave in $(0,\infty)$ provided that x, y are orthogonal and nonzero. The following proposition holds.

PROPOSITION 4.9 *Let $(X;(\cdot,\cdot))$ be an inner product space and let $x \perp y$, $x \neq 0 \neq y$.*

(i) *If $0<a<b$, then*

$$\begin{aligned}
\frac{2}{a+b}\left(\left\|x+\tfrac{1}{2}(a+b)y\right\| - \|x\|\right) &\geq \frac{1}{b-a}\int_a^b \frac{\|x+ty\|-\|x\|}{t}dt \\
&\geq \frac{1}{2}\left[\frac{2}{a+b}\left(\left\|x+\tfrac{1}{2}(a+b)y\right\| - \|x\|\right)\right. \\
&\quad \left. + \frac{\|x+ay\|-\|x\|}{2a} + \frac{\|x+by\|-\|x\|}{2b}\right] \\
&\geq \frac{1}{2}\left[\frac{\|x+ay\|-\|x\|}{a} + \frac{\|x+by\|-\|x\|}{b}\right] \geq 0;
\end{aligned}$$

(ii) *If $a<b<0$, then the reverse inequalities hold.*

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