Five Integral Inequalities; an Inheritance from Hardy and Littlewood

This paper is dedicated to Dan Pedoe, Professor Emeritus, University of Minnesota

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This paper is concerned with five integral inequalities considered as generalisations of an inequality first discovered by G.H. Hardy and J.E. Littlewood in 1932. Subsequently the inequality was considered in greater detail in the now classic text *Inequalities* of 1934, written by Hardy and Littlewood together with G. Pólya.

All these inequalities involve Lebesgue square-integrable functions, together with their first two derivatives, integrated over the positive half-line of the real field.

The method to discuss the analytical properties of these inequalities is based on the Sturm–Liouville theory of the underlying second-order differential equation, and the associated Titchmarsh–Weyl *m*-coefficient.

The five examples are specially chosen so that the corresponding Sturm–Liouville differential equations have solutions in the domain of special functions; in the case of these examples the functions involved are those named as the Airy, Bessel, Gamma and the Weber parabolic cylinder functions. The extensive range of known properties of these functions enables explicit analysis of some of the analytical problems to give definite results in the examples of this paper.

The analytical problems are "hard" in the technical sense and some of them remain unsolved; this position leads to the statement in the paper of a number of conjectures.

In recent years the difficulties involved in the analysis of these problems led to a numerical approach and this method has been remarkably successful. Although such methods, involving standard error analysis and the inevitable introduction of round-off error, cannot by their nature provide analytical proofs; nevertheless the now established record of success of these numerical methods in predicting correct analytical results lends authority to the correctness of the conjectures made in this paper.

1 INTRODUCTION

The inheritance discussed in this paper 1991 stems from a remarkable result of G.H. Hardy and J.E. Littlewood given originally in the seminal paper [17] of 1932, and subsequently entered in the book *Inequalities* of Hardy, Littlewood and G. Pólya [18, Chapter VII, Section 7.8].

THEOREM 1 (Hardy and Littlewood)   *If* $f:[0,\infty) \to \mathbb{R}$, *if* f *and* f' *are locally absolutely continuous on* $[0,\infty)$, *and if both* f *and* f'' *are in* $L^2[0,\infty)$ *then*

$$\left(\int_0^\infty f'(x)^2\, dx \right)^2 \le 4 \int_0^\infty f(x)^2\, dx \int_0^\infty f''(x)^2\, dx \qquad (1.1)$$

with equality if and only if, for some $\alpha \in \mathbb{R}$ *and for some* $\rho > 0$,

$$f(x) = \alpha Y(\rho x), \quad (x \in [0,\infty)) \qquad (1.2)$$

where

$$Y(x) = \exp\left(-\tfrac{1}{2}x\right) \sin\left(\tfrac{1}{2}x\sqrt{3} - \tfrac{1}{3}\pi\right) \quad (x \in [0,\infty)). \qquad (1.3)$$

In this statement $\mathbb{R}$ represents the real field; the prime $'$ denotes differentiation on $\mathbb{R}$; $L^2[0,\infty)$ represents the classical Lebesgue integration space; absolute continuity (AC) is with respect to Lebesgue measure; local on $[0,\infty)$ limits a property to all compact intervals of $[0,\infty)$.

True to the spirit of the book *Inequalities* three distinct proofs of this result are given, see [18, Section 7.8].

Of these three proofs the second proof depends upon an elementary, elegant and ingenious device which leads to showing that the number 4 in (1.1) is best possible, and that all cases of equality are given by (1.2) with the extremal function Y determined by (1.3). However this method seems not to lend itself to generalisation or extension to consider other integral inequalities.

The third proof in [18, Section 7.8] of (1.1) depends on an intricate application of the methods of the calculus of variations to a singular variational problem. The analytical difficulties in the proof suggest that extensions and generalisations of this method will prove to be both limited in extent and difficult to implement.

For a discussion on these two methods of proof of (1.1), in the light of developments in the calculus of variations, see the extensive and detailed survey on variational inequalities by Ahlbrandt [2].

The first proof of the inequality (1.1) in [18, Section 7.8] has subsequently commanded considerable attention. This proof is inspired by the methods of the calculus of variations but then takes on an independence from variational techniques; this has led to extensions of (1.1) which, by their form, the methods of variational analysis are not designed to handle. It is this method of proof that has led, to quote from [2, page 6], "to a fertile area of research".

An account of recent developments of the original Hardy–Littlewood inequality (1.1) can be found in the survey article of Brown, Evans and Everitt [7]. See also the book [20] of Mitrinović, Pečarić and Fink in which the Hardy–Littlewood type integral inequalities are considered in a number of places in the text. The Hardy–Littlewood integral inequality has been extended to infinite series; for a survey of the properties and generalisations of the resulting series inequality see [6].

A catalogue of all known special cases (up to 1996) of the extensions of the Hardy–Littlewood inequality, for both integrals and series, is given in [3].

The inequality (1.1) is connected with the spectral theory of the linear, ordinary differential equation

$$-y''(x) = \lambda y(x) \quad (x \in [0, \infty)) \tag{1.4}$$

where $\lambda \in \mathbb{C}$ (the complex field) is the spectral parameter. The solutions of this equation that lie in the space $L^2[0, \infty)$ provide one method, based on the first proof in [18, Section 7.8], to establish all the results quoted in Theorem 1 within a context that allows extensive generalisation.

In this paper we consider five special examples of an extension of the inequality (1.1) based on properties of the differential equation

$$-y''(x) + (q(x) - \tau)y(x) = \lambda y(x) \quad (x \in [0, \infty)) \tag{1.5}$$

of which (1.4) is a special case. In this equation

(i) the coefficient $q : [0, \infty) \to \mathbb{R}$ and satisfies the integrability condition

$$q \in L_{\mathrm{loc}}[0, \infty), \tag{1.6}$$

(ii) the parameter $\tau \in \mathbb{R}$ and is the translation parameter that allows any point on $\mathbb{R} \subset \mathbb{C}$ to be considered as the origin of the spectral λ-plane.

The extension of the Hardy–Littlewood integral inequality then takes the form, for all $f \in D$,

$$\left(\int_0^{\to\infty} \{f'(x)^2 + (q(x) - \tau)f(x)^2\} \, dx \right)^2$$
$$\leq K(\tau) \int_0^\infty f(x)^2 \, dx \int_0^\infty \{f''(x) - (q(x) - \tau)f(x)\}^2 \, dx \tag{1.7}$$

where:

(i) the domain D is defined by

$$D := \{ f : [0, \infty) \to \mathbb{R} : f, f' \in AC_{\mathrm{loc}}[0, \infty)$$

and

$$f, f'' - qf \in L^2[0, \infty) \}, \tag{1.8}$$

(ii) the notation $\to \infty$ on the left-hand side of (1.7) indicates that the integral may only be conditionally convergent,

(iii) the number $K(\tau)$ depends upon the shift parameter τ.

Properties of the inequality (1.7):

(i) The inequality can be considered if the integral on the left-hand side is well-defined in $\mathbb{R}$ for all $f \in D$; a sufficient condition for this requirement to hold is that the differential expression

$$-f'' + (q - \tau)f \tag{1.9}$$

is in the strong limit-point condition at ∞ in the space $L^2[0, \infty)$; for the technical details of this condition see [9, Section 3]; there is a large literature devoted to sufficiency conditions for strong limit-point to hold; the strong limit-point condition is independent of the shift parameter τ, all the five examples of (1.7) to be considered in this paper satisfy the strong limit-point condition.

(ii) With the strong limit-point condition assumed to hold the inequality is said to be *valid*, for $\tau \in \mathbb{R}$, if (1.7) holds, for all $f \in D$, for some positive number $K(\tau)$; the notation is then adopted that $K(\tau)$ is the best possible number, *i.e.* the smallest number, for which the inequality is valid.

(iii) If the inequality is not valid then the notation $K(\tau) = \infty$ is used.

(iv) If the inequality is valid then any element of D that gives equality in (1.7) is termed a case of equality; the null function is always a case of equality but there may or may not be any non-null cases of equality.

(v) A *weak* case of equality is a non-null element $f \in D$ such that equality holds in (1.7) by virtue of f being a solution of the differential equation (1.5) when $\lambda = 0$, *i.e.*

$$\int_0^\infty \{f''(x) - (q(x) - \tau)f(x)\}^2 \, dx = 0$$

and

$$\int_0^\infty \{f'(x)^2 + (q(x) - \tau)f(x)^2\} \, dx = 0.$$

Such examples are known to exist but the term weak is used since these cases will not, by their nature, determine or verify the value of the best possible number $K(\tau)$.

A *normal* case of equality is one in which

$$\int_0^\infty \{f'(x)^2 + (x - \tau)f(x)^2\} \, dx > 0$$

and

$$\int_0^\infty \{f''(x) - (x - \tau)f(x)\}^2 \, dx > 0$$

in which case, since f is not null,

$$K(\tau) = \left(\int_0^\infty \{f'(x)^2 + (x - \tau)f(x)^2\} \, dx \right)^2$$
$$\times \left(\int_0^\infty f(x)^2 \, dx \int_0^\infty \{f''(x) - (x - \tau)f(x)\}^2 \, dx \right)^{-1}.$$

The analysis of any particular inequality (1.7) consists of: (i) trying to determine if the inequality is valid or invalid; (ii) if valid then to determine or characterise the best possible number $K(\tau)$; (iii) and then to determine or characterise all the corresponding cases of equality. The inequality (1.1) of Hardy and Littlewood is an example of the general inequality (1.7) for which it is possible to give complete answers to all these three stages.

The main analytical tool involved in the analysis of the inequality (1.7) is the Titchmarsh–Weyl m-coefficient, see [21, Chapters II and III]. For the introduction of the m-coefficient into the theory of this inequality see [9,13,14]. For a number of special results that have influenced the content of this paper see [4,5,11,15,16].

The analytical problems for establishing the validity, the character-isation of the best possible number $K(\tau)$ and all the cases of equality of (1.7), can be very demanding. For this reason numerical techniques have been established to seek out reliable information on the possible answers to these problems. These numerical techniques are now so well tried and established as to inspire confidence in their findings. Results of this form are given in later sections of this paper. For a survey of these numerical methods see [7, Section 4].

A detailed analysis of general inequalities of the form (1.7), with a number of examples, is given in [9]; an overall view of the analytical and numerical techniques required is given in [7]; a catalogue of all integral and series inequalities of this Hardy–Littlewood type is to be found in the recent survey paper [3].

The five special cases of (1.7) that form the content of this paper are given by the following determinations of the coefficient q:

$$\left.\begin{array}{l} 1.\ q(x) = x^2 \\ 2.\ q(x) = -x \\ 3.\ q(x) = x \\ 4.\ q(x) = -x^2 \\ 5.\ q(x) = \tfrac{1}{2}(x+1)^{-2} \end{array}\right\} \quad (x \in [0,\infty)). \qquad (1.10)$$

The reason for making this choice of coefficients is that the resulting differential equations (1.5) all have solutions that can be represented in terms of known special functions of mathematical analysis. This

explicit information allows detailed mathematical and numerical analysis of all the five cases of (1.7) given by the specific q cases of (1.10). The properties of the special functions required are all given in the handbook [1].

The parameter τ plays a very significant role in the properties of the general inequality (1.7); the validity of the inequality for one value of the shift parameter τ gives no information about the inequality for other values of this parameter. To illustrate this phenomenon the results for the Hardy–Littlewood inequality (1.1) are given; with the addition of the shift parameter τ this inequality takes the form

$$\left(\int_0^\infty \{f'(x)^2 - \tau f(x)^2\} \, dx \right)^2$$
$$\leq K(\tau) \int_0^\infty f(x)^2 dx \int_0^\infty \{f''(x) + \tau f(x)\}^2 \, dx \qquad (1.11)$$

where f continues to satisfy the conditions given in Theorem 1. Then it may be shown, see [9, Section 9, Example 2], that

$$\left. \begin{array}{l} K(\tau) = \infty \; (\tau \in (-\infty, 0)) \\ K(0) = 4 \text{ with equality as given by (1.2) and (1.3)} \\ K(\tau) = 4 \; (\tau \in (0, \infty)) \text{ with equality given only by the null function.} \end{array} \right\}$$
$$(1.12)$$

For the purposes of this Introduction one result for each of the five cases of (1.10) is given below:

1. If $q(x) = x^2$ and $\tau = 1$ then $K(1) = 4$ and there is equality if and only if

$$f(x) = \alpha \exp\left(-\tfrac{1}{2}x^2\right),$$

in which case both sides of the inequality are zero even if $\alpha \neq 0$.
2. If $q(x) = -x$ and $\tau = 0$ then $K(0) = 4$ and there is a continuum of cases of equality similar in structure to the functions for the Hardy–Littlewood case given by (1.2) and (1.3).
3. If $q(x) = x$ and $\tau = \mu_0$, where μ_0 is the first negative zero of the derivative of the Airy function $x \mapsto Ai'(x)$ $(x \in \mathbb{R})$, then $K(\mu_0) = 4$

and there is equality if and only if $f(x) = \alpha \mathrm{Ai}(x - \mu_0)$ in which case both sides of the inequality are zero.

4. If $q(x) = -x^2$ and $\tau = 0$ then $K(0) = 4 + 2\sqrt{2}$ with only the null function as a case of equality.

5. If $q(x) = \frac{1}{2}(x+1)^{-2}$ and $\tau = 0$ then $K(0) = +\infty$ and the inequality is not valid.

In Section 2 some details of the theory of the m-coefficient and the role the coefficient plays in the analysis of the general inequality, are discussed. The results of this analysis for each of the five cases of the coefficient q are given in Sections 3 to 7 respectively.

2   THE TITCHMARSH–WEYL m-COEFFICIENT

We first consider the case of the differential equation (1.5) when the shift parameter $\tau = 0$.

Consider then the equation

$$-y''(x) + q(x)y(x) = \lambda y(x) \quad (x \in [0, \infty) \text{ and } \lambda \in \mathbb{C}), \qquad (2.1)$$

where the coefficient q satisfies the basic conditions given in (1.6). This is the Titchmarsh equation considered in detail in the book [21]; see in particular Chapters I to III.

Let $\theta, \varphi: [0, \infty) \times \mathbb{C} \to \mathbb{C}$ be solutions of (2.1) determined by the initial conditions

$$\left.\begin{array}{ll} \theta(0, \lambda) = 0 & \theta'(0, \lambda) = 1 \\[2mm] \varphi(0, \lambda) = -1 & \varphi'(0, \lambda) = 0 \end{array}\right\} \quad (\lambda \in \mathbb{C}); \qquad (2.2)$$

then θ and φ have the properties, see [21, Chapter II],

1. $\theta(\cdot, \lambda)$ and $\varphi(\cdot, \lambda)$ are solutions of (2.1) on $[0, \infty)$ for all $\lambda \in \mathbb{C}$
2. $\theta(x, \cdot)$ and $\varphi(x, \cdot)$ are holomorphic functions on $\mathbb{C}$ for all $x \in [0, \infty)$.

Define $\mathbb{C}_{\pm} := \{\lambda \in \mathbb{C}: \mathrm{Im}(\lambda) \lessgtr 0\}$.

Now let the coefficient q be chosen so that the differential equation is in the strong limit-point condition at ∞ in $L^2[0, \infty)$; then θ and φ have

the additional property, see [21, Chapter II],

$$\theta(\cdot, \lambda) \neq L^2[0, \infty) \text{ and } \varphi(\cdot, \lambda) \neq L^2[0, \infty) \text{ for all } \lambda \in \mathbb{C}\backslash\mathbb{R}. \qquad (2.3)$$

However the general theory shows, see [21, Chapter II], that there exists a unique pair of analytic functions $\{m_+, m_-\}$ with the properties:

1. $m_\pm : \mathbb{C}_\pm \to \mathbb{C}_\pm$
2. $m_\pm$ is holomorphic in $\mathbb{C}_\pm$
3. $\overline{m_\pm}(\lambda) = m_\mp(\overline{\lambda}) \quad (\lambda \in \mathbb{C}_\pm)$

such that, in contrast to the result (2.3),

$$\theta(\cdot, \lambda) + m_\pm(\lambda)\varphi(\cdot, \lambda) \in L^2[0, \infty) \quad (\lambda \in \mathbb{C}_\pm). \qquad (2.4)$$

Properties 1, 2, 3 above imply that the pair $\{m_+, m_-\}$ is a Nevanlinna (Herglotz, Pick, Riesz) function on $\mathbb{C}$. Notwithstanding property 3, m_+ may or may not have an analytical continuation across $\mathbb{R} \subset \mathbb{C}$; even if this continuation exists m_- on $\mathbb{C}_-$ may or may not be the continuation; some of these possibilities arise in the cases of the m-coefficient in this paper, and reference is made at appropriate places in the following sections.

For the Nevanlinna function $\{m_+, m_-\}$ define $m : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ by

$$m(\lambda) := m_\pm(\lambda) \quad (\lambda \in \mathbb{C}_\pm).$$

The m-coefficient was introduced by Titchmarsh in 1941 on the basis of work by Weyl in 1910.

As an example of these results and properties we have for the differential equation (1.4)

$$\theta(x, \lambda) = \sin(x\sqrt{\lambda})/\sqrt{\lambda} \qquad \varphi(x, \lambda) = -\cos(x\sqrt{\lambda})$$
$$m(\lambda) \equiv m_\pm(\lambda) = i/\sqrt{\lambda} \quad (\lambda \in \mathbb{C}\backslash[0, \infty))$$

and

$$\theta(x, \lambda) + m(\lambda)\varphi(x, \lambda) = -i\exp(ix\sqrt{\lambda})/\sqrt{\lambda},$$

all for $\lambda \in \mathbb{C} \setminus [0, \infty)$. In these formulae the function $\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ is defined by, cutting the λ-plane on the real axis from the origin 0 to $+\infty$,

$$\lambda = \rho \exp(i\psi) \quad \text{with} \quad \rho \geq 0 \text{ and } 0 \leq \psi < 2\pi \tag{2.5}$$

and then

$$\sqrt{\lambda} := \rho^{1/2} \exp\left(\tfrac{1}{2} i\psi\right), \tag{2.6}$$

so that

$$\text{Im}(\sqrt{\lambda}) > 0 \quad \text{for all } \lambda \in \mathbb{C} \setminus [0, \infty). \tag{2.7}$$

To indicate how the m-coefficient is used to assess the properties of the inequality (1.7) the following given properties and definitions are required:

1. Let the line segments $L_{\pm}(\psi)$ for $\psi \in (0, \tfrac{1}{2}\pi]$ be defined by

$$L_+(\psi) := \{\rho \exp(i\psi) : \rho \in (0, \infty)\}$$
$$L_-(\psi) := \{\rho \exp(i\psi + i\pi) : \rho \in (0, \infty)\}.$$

2. Let the arguments $\theta_{\pm} \in [0, \tfrac{1}{2}\pi]$ be defined by

$$\theta_{\pm} := \inf\left\{\theta \in \left(0, \tfrac{1}{2}\pi\right] : \text{for all } \psi \in \left[\theta, \tfrac{1}{2}\pi\right] \right.$$
$$\left. \mp \text{Im}\left(\lambda^2 m_{\pm}(\lambda) \geq 0 \ (\lambda \in L_{\pm}(\psi))\right)\right\}.$$

3. Let the argument $\theta_0 \in [0, \tfrac{1}{2}\pi]$ be defined by

$$\theta_0 := \max(\theta_+, \theta_-).$$

4. Let the real number sets $E_{\pm} \subseteq (0, \infty)$ be defined by

$$E_{\pm} := \left\{\rho \in (0, \infty) : \lambda \in L_{\pm}(\theta_0) \quad \text{and} \quad \text{Im}\left(\lambda^2 m_{\pm}(\lambda)\right) = 0\right\}.$$

5. Let $Y_{\pm} : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be defined by

$$Y_{\pm}(x; \rho) := \text{Im}(\lambda[\theta(x, \lambda) + m_{\pm}(\lambda)\varphi(x, \lambda)])$$
$$(x \in [0, \infty) \text{ and } \lambda \in L_{\pm}(\theta_0)).$$

Then the following theorem holds:

THEOREM 2 *Let the coefficient q be given satisfying $q \in L_{\mathrm{loc}}[0, \infty)$; let the differential expression $-f'' + qf$ be in the strong limit-point condition at $+\infty$ in $L^2[0, \infty)$; let the domain $D \subset L^2[0, \infty)$ be defined as in (1.8); let the above definitions of, and concerning the m-coefficient hold; then:*

1. *in all cases $\theta_0 \in (0, \frac{1}{2}\pi]$, i.e. $\theta_0 > 0$ and at least one of $\theta_+ > 0, \theta_- > 0$ must hold;*
2. *the inequality on the domain D*

$$\left(\int_0^\infty \{f'(x)^2 + q(x)f(x)^2\}\, dx \right)^2$$

$$\leq K \int_0^\infty f(x)^2\, dx \int_0^\infty \{f''(x) - q(x)f(x)\}^2\, dx \qquad (2.8)$$

   is valid if and only if

$$\theta_0 \in \left(0, \tfrac{1}{2}\pi\right), \ \text{i.e. } \theta_0 \neq \tfrac{1}{2}\pi; \qquad (2.9)$$

3. *if (2.9) is satisfied then the best possible number K in the inequality (2.8) is given by*

$$K = (\cos(\theta_0))^{-2}; \qquad (2.10)$$

4. *if (2.9) is satisfied then all cases of equality in (2.8) are given by one of the three mutually exclusive cases: (i) $f = 0$, (ii) the **weak** case of Section 1 above, i.e. there exists $f \in D$ with $f \neq 0$ and $-f'' + qf = 0$ on $[0, \infty)$ with either $f(0) = 0$ or $f'(0) = 0$ (but not both), in which case both sides of (2.8) are zero, (iii) the **normal** case of Section 1 above, i.e. $E_+ \cup E_-$ is not empty and $f \in D$ is defined by*

$$f(x) := \alpha Y_\pm(x; \rho) \quad (x \in [0, \infty) \text{ and } \rho \in E_+ \cup E_-)$$

   with $\alpha \in \mathbb{R} \backslash \{0\}$, in which case both sides of (2.8) are not zero.

Proof See [9, Sections 6 and 7].

In the general case of the shift parameter τ the $m_\pm$-coefficient above, now for the differential equation (1.5), is replaced by the $m_\pm(\cdot : \tau)$-coefficient where, as a calculation shows,

$$m_\pm(\lambda : \tau) = m_\pm(\lambda + \tau) \quad (\lambda \in \mathbb{C}_\pm \text{ and } \tau \in \mathbb{R}). \qquad (2.11)$$

The results of Theorem 2 above then apply to $m_\pm(\cdot : \tau)$ in order to characterise the general inequality (1.7).

Remark   In the case when the differential equation (2.1) has special functions as solutions it may be possible to obtain an explicit expression for $\{m_+, m_-\}$ in terms of these solutions. Given that the equation is in the strong limit-point case suppose, from the properties of the special solutions, that it is possible to find a solution of (2.1) $\Psi(\cdot, \lambda)$ such that

$$\Psi(\cdot, \lambda) \in L^2[0, \infty) \quad (\lambda \in \mathbb{C}\backslash\mathbb{R}); \tag{2.12}$$

then, using the initial conditions (2.2), a calculation shows that

$$m_\pm(\lambda) = -\frac{\Psi(0, \lambda)}{\Psi'(0, \lambda)} \quad (\lambda \in \mathbb{C}_\pm). \tag{2.13}$$

3   THE CASE $q(x) = x^2$

This example has the differential equation

$$-y''(x) + (x^2 - \tau)y(x) = \lambda y(x) \quad (x \in [0, \infty)) \tag{3.1}$$

and the associated inequality is

$$\left(\int_0^\infty \{f'(x)^2 + (x^2 - \tau)f(x)^2\} \, dx \right)^2$$
$$\leq K(\tau) \int_0^\infty f(x)^2 dx \int_0^\infty \{f''(x) - (x^2 - \tau)f(x)\}^2 \, dx \tag{3.2}$$

with domain

$$D = \{f: [0, \infty) \to \mathbb{R}: f, f' \in AC_{\text{loc}}[0, \infty) \text{ and } f, f'' - x^2 f \in L^2[0, \infty)\}. \tag{3.3}$$

The equation (3.1) has solutions in terms of the Weber parabolic cylinder functions, see [1, Chapter 19], and for a direct treatment [21, Section 4.2]. A calculation shows that

$$m_\pm(\lambda) = \frac{\Gamma(\frac{1}{4} - \frac{1}{4}\lambda)}{2\Gamma(\frac{3}{4} - \frac{1}{4}\lambda)} \quad (\lambda \in \mathbb{C}_\pm) \tag{3.4}$$

so that m_- and m_+ are analytic continuations of each other and then m is a meromorphic function on $\mathbb{C}$.

A full analysis of the consequences of Theorem 2 as applied to this case is given in [12]. The results are here quoted in

THEOREM 3 *Define the set*

$$\Lambda := \{2n + 1 : n \in \mathbb{N}_0 = (0, 1, 2, \ldots)\}.$$

The inequality (3.2) *has the properties*;

1. $K(\tau) = \infty$ *for all* $\tau \in \{\mathbb{R} \backslash \Lambda\}$.
2. *The inequality is valid for all* $\tau \in \Lambda$ *with details as follows*:

  (i) *if* $n = 2p$ $(p \in \mathbb{N}_0)$ *is even then*

$$K(4p + 1) = 4 \quad (p \in \mathbb{N}_0)$$

  and there is equality if and only if

$$f(x) = \alpha \exp\left(-\tfrac{1}{2}x^2\right) H_{2p}(x) \quad (x \in [0, \infty))$$

  where H_{2p} is the Hermite polynomial of order $2p$, and the weak case (ii) *of equality in Theorem 2 holds*; $E_-(4p + 1) = E_+(4p + 1) = \emptyset$;
 (ii) *if* $n = 2p + 1$ $(p \in \mathbb{N}_0)$ *is odd then*

$$K(4p + 7) > K(4p + 3) > 4 \quad (p \in \mathbb{N}_0)$$
$$\lim_{p \to \infty} K(4p + 3) = 4$$

  and there is equality if and only if **either**

$$f(x) = \alpha \exp\left(-\tfrac{1}{2}x^2\right) H_{2p+1}(x) \quad (x \in [0, \infty))$$

  and the weak case (ii) *of equality in Theorem 2 holds*, **or**

$$f(x) = \alpha \operatorname{Im}(r_n \exp(-i\theta_n) j_n \Psi(x, 2n + 1 - r_n \exp(-i\theta_n)))$$

  where Ψ is Weber's parabolic cylinder function, and $r_n \in (0, 4)$, $j_n \in \mathbb{C}$, $\theta_n \in (\tfrac{1}{2}\pi, \tfrac{2}{3}\pi)$ are well-determined numbers, and the normal

case (iii) *of equality of Theorem* 2 *holds*; $E_-(4p+7)=\emptyset$ *and* $E_+(4p+7)=\{r_n\}$; *there are asymptotic formulae*:

$$K(2n+1) = 4 + \frac{3}{\pi^2 n^2} + O\left(\frac{1}{n^3}\right) \quad (n \to \infty)$$

and

$$r_n = \frac{6}{\pi^2 n} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

Proof   See [12] for the analytical details of the proof; see also the accounts in [9, Section 4] and [3, Section 4, Example 6].

   This example was one of the first examples tested in the numerical procedures discussed in [7, Section 4]; the numerical results make accurate comparison with the analytical properties given in this Theorem.

4   THE CASE $q(x) = -x$

This example has the differential equation

$$-y''(x) - (x+\tau)y(x) = \lambda y(x) \quad (x \in [0, \infty)) \tag{4.1}$$

and the associated inequality is

$$\left(\int_0^{\to\infty} \{f'(x)^2 - (x+\tau)f(x)^2\}\,dx\right)^2$$
$$\leq K(\tau)\int_0^\infty f(x)^2 dx \int_0^\infty \{f''(x) + (x+\tau)f(x)\}^2 dx \tag{4.2}$$

with domain

$$D = \{f\colon [0,\infty) \to \mathbb{R}\colon f, f' \in AC_{\text{loc}}[0,\infty) \text{ and } f, f'' + xf \in L^2[0,\infty)\}. \tag{4.3}$$

   The equation (4.1) has solutions in terms of Bessel functions of order $\frac{1}{3}$, see [1, Chapter 10, Section 10.4]. This equation is strong limit-point at ∞ in $L^2[0,\infty)$ but in general the integral on the left-hand side of (4.2)

is only conditionally convergent; for this result see [9, Section 3], and for a more direct treatment [21, Section 4.13]. In this example the two components of the Nevanlinna function $m_{\pm}$ are not analytical continuations of each other and are given by separate formulae

$$m_+(\lambda) = -\frac{H^{(1)}_{\frac{1}{3}}\left(\frac{2}{3}\lambda\sqrt{\lambda}\right)}{\sqrt{\lambda}H^{(1)}_{-\frac{1}{3}}\left(\frac{2}{3}\lambda\sqrt{\lambda}\right)} \quad (\lambda \in \mathbb{C}_+)$$

$$m_-(\lambda) = -\frac{H^{(2)}_{\frac{1}{3}}\left(\frac{2}{3}\lambda\sqrt{\lambda}\right)}{\sqrt{\lambda}H^{(2)}_{-\frac{1}{3}}\left(\frac{2}{3}\lambda\sqrt{\lambda}\right)} \quad (\lambda \in \mathbb{C}_-);$$

here $H^{(r)}_{\nu}$ represents the Hankel–Bessel function of order ν and type r ($r = 1, 2$), and $\sqrt{\cdot}$ is the function defined in Section 2 above.

A full analysis of the consequences of Theorem 2 as applied to this case is given in [10]. The results are here quoted in

THEOREM 4   *For $\tau = 0$ the inequality (4.2) has the properties;*

$$\theta_+ = \tfrac{1}{3}\pi \quad \theta_- = \tfrac{1}{6}\pi \quad E_+ = (0, \infty) \quad E_- = \emptyset$$

and so

$$K(0) = 4.$$

There is a continuum of normal cases of equality in respect of (iii) *of 4, Theorem 2*

$$f(x) = \alpha\,\mathrm{Re}\left(\exp(-\tfrac{1}{3}\pi i)\sqrt{x+\lambda}\,H^{(1)}_{\frac{1}{3}}\left(\tfrac{2}{3}(x+\lambda)\sqrt{x+\lambda}\right)\right)$$
$$(x \in [0, \infty))$$

with $\lambda = \rho\exp(\tfrac{1}{3}\pi i)$ and $\rho \in E_+ = (0, \infty)$.

There are no weak cases of equality in respect of (ii) *of 4, Theorem 2.*

Proof   For details of the proof of this theorem see [10].   □

Remark   For all other values of the shift parameter τ there are no analytical results presently available; however the numerical pro-gramme outlined in [7, Section 4] is particularly successful in this

example. The tested reliability of these numerical techniques leads to making the following analytical conjectures:

1. The inequality (4.2) is valid for all $\tau \in \mathbb{R}$.
2. The best-possible number function $K(\cdot)$ is continuous and monotonic decreasing on $\mathbb{R}$.
3. $K(\tau) > 4$ for all $\tau \in (-\infty, 0)$, and $K(\tau) = 4$ for all $\tau \in [0, \infty)$.
4. $\lim_{\tau \to -\infty} K(\tau) = +\infty$.
5. There are no cases of equality for (4.2) for all $\tau \in \mathbb{R} \backslash \{0\}$, other than the null function.

5 THE CASE $q(x) = x$

This example has the differential equation

$$-y''(x) + (x - \tau)y(x) = \lambda y(x) \quad (x \in [0, \infty)) \tag{5.1}$$

and the associated inequality is

$$\left(\int_0^\infty \{f'(x)^2 + (x - \tau)f(x)^2\} \, dx \right)^2$$
$$\leq K(\tau) \int_0^\infty f(x)^2 \, dx \int_0^\infty \{f''(x) - (x - \tau)f(x)\}^2 \, dx \tag{5.2}$$

with domain

$$D = \{f : [0, \infty) \to \mathbb{R} : f, f' \in AC_{\text{loc}}[0, \infty) \text{ and } f, f'' - xf \in L^2[0, \infty)\}. \tag{5.3}$$

5.1 The Differential Equation

The equation (5.1) has solutions in terms of Bessel functions of order $\frac{1}{3}$, see [21, Section 4.12]; these solutions are linked to the Airy functions $\text{Ai}(\cdot)$ and $\text{Bi}(\cdot)$, see [1, Section 10.4].

For $\tau = 0$ the m-coefficient can be constructed using the result (2.13). From [21, Section 2.12] the solution that is small at $+\infty$ is given, in

terms of the modified Bessel function $K_{\frac{1}{3}}$, by

$$\Psi(x, \lambda) = (x - \lambda)^{\frac{1}{2}} K_{\frac{1}{3}}\left(\tfrac{2}{3}(x - \lambda)^{\frac{3}{2}}\right) \quad (x \in [0, \infty)) \qquad (5.4)$$

where terms such as $(x - \lambda)^{\frac{1}{2}}$ and $(x - \lambda)^{\frac{2}{3}}$ are taken to have arg 0 when $\lambda \in \mathbb{R}$ and $x > \lambda$. If $\lambda \in \mathbb{R}$ and $x < \lambda$ then it is shown in [21, Section 4.12] that (5.4) can be written in the equivalent form

$$\Psi(x, \lambda) = \frac{\pi}{\sqrt{3}}(\lambda - x)^{\frac{1}{2}}\left[J_{\frac{1}{3}}\left(\tfrac{2}{3}(\lambda - x)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\tfrac{2}{3}(\lambda - x)^{\frac{3}{2}}\right)\right]. \qquad (5.5)$$

This last solution of (5.1) can be used to calculate the $m_{\pm}$ coefficient from (2.13); this yields the formulae

$$m_{\pm}(\lambda) = \frac{J_{\frac{1}{3}}\left(\tfrac{2}{3}\lambda^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\tfrac{2}{3}\lambda^{\frac{3}{2}}\right)}{\sqrt{\lambda}\left[J_{\frac{2}{3}}\left(\tfrac{2}{3}\lambda^{\frac{3}{2}}\right) - J_{-\frac{2}{3}}\left(\tfrac{2}{3}\lambda^{\frac{3}{2}}\right)\right]} \quad (\lambda \in \mathbb{C}\backslash\mathbb{R}). \qquad (5.6)$$

This form of the $m_{\pm}$ coefficient turns out to be very inconvenient for the application of Theorem 2 of Section 2 above; for this reason both results in (5.5) and (5.6) are replaced by equivalent forms but using Airy functions.

5.2 The Airy Functions

Firstly we replace (5.4) by the Airy function Ai; for details of this function see [1, Sections 10.4 and 10.5]. Ai($\cdot$) is a holomorphic function on $\mathbb{C}$ and is defined as a solution of the differential equation

$$w''(z) = zw(z) \quad (z \in \mathbb{C}) \qquad (5.7)$$

with initial conditions at the regular point 0 given in [1, Sections 10.4.4 and 10.4.5]. The relationship between Ai($\cdot$) and the Bessel functions $J_{\pm\frac{1}{3}}$ is given by

$$\text{Ai}(-z) = \tfrac{1}{3}\sqrt{z}\left[J_{\frac{1}{3}}(\zeta) + J_{-\frac{1}{3}}(\zeta)\right] \qquad (5.8)$$

where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$ and appropriate branches of the many-valued variables are chosen; see [1, Section 10.4.15].

From (5.6) and (5.8) it may be shown that the small solution Ψ of (5.1), with $\tau = 0$, as given by (5.4) or (5.5) has the form, apart from an unimportant numerical factor,

$$\Psi(x, \lambda) = \text{Ai}(x - \lambda) \quad (x \in [0, \infty) \text{ and } \lambda \in \mathbb{C}). \qquad (5.9)$$

Note that with $\text{Ai}(\cdot)$ holomorphic on $\mathbb{C}$ the solution Ψ given by (5.9) is not troubled with decisions to be made concerning branch points.

An independent check confirms that the formula (5.9) gives an exponentially small solution for fixed λ and large positive values of the variable x. For if $\lambda \in \mathbb{C}$ and x is large enough then $|\arg(x - \lambda)| < \pi$ and the asymptotic formula [1, Section 10.4.59] is valid to give

$$|\text{Ai}(x - \lambda)| = O\left(\exp\left(-\tfrac{2}{3}|x - \lambda|^{\frac{3}{2}}\right)\right)$$

$$= O\left(\exp\left(-\tfrac{2}{3}x^{\frac{3}{2}}\right)\right) \quad (x \to \infty).$$

If we now apply the formula (2.13) to Ψ given by (5.9) then we obtain

$$m_{\pm}(\lambda) = -\frac{\text{Ai}(-\lambda)}{\text{Ai}'(-\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{R}). \qquad (5.10)$$

This result shows that $m_{\mp}$ is the continuation of $m_{\pm}$ and that m is a meromorphic function on $\mathbb{C}$. As in Section 2 we denote this single coefficient by m and note that it is real-valued at all points on $\mathbb{R}$ where it is holomorphic.

To apply the results of Theorem 2 of Section 2 above, it is necessary to have information about the zeros and poles of m as given by (5.10). From the theory of the m-coefficient, see [9], it is known that the poles and zeros of m lie on the real axis $\mathbb{R}$ of $\mathbb{C}$.

From the general theory of the differential equation (5.1), see [1, Section 10.4], or from the spectral theory properties of differential operators generated by (5.1) in the Hilbert function space $L^2[0, \infty)$, see [9], the following results are stated.

THEOREM 5 (i) *the Airy function* Ai$(\cdot)$ *has a countable infinity of real, simple and strictly negative zeros, say* $\{-\mu_n^D : n \in \mathbb{N}_0\}$, *with*

$$0 < \mu_n^D < \mu_{n+1}^D \quad (n \in \mathbb{N}_0) \quad and \quad \lim_{n \to \infty} \mu_n^D = +\infty, \tag{5.11}$$

(ii) *the Airy function* Ai$'(\cdot)$ *has a countable infinity of real, simple and strictly negative zeros, say* $\{-\mu_n^N : n \in \mathbb{N}_0\}$, *with*

$$0 < \mu_n^N < \mu_{n+1}^N \quad (n \in \mathbb{N}_0) \quad and \quad \lim_{n \to \infty} \mu_n^N = +\infty, \tag{5.12}$$

(iii) *the zeros of* Ai$(\cdot)$ *and* Ai$'(\cdot)$ *interlace with*

$$0 < \mu_n^N < \mu_n^D < \mu_{n+1}^N < \mu_{n+1}^D \quad (n \in \mathbb{N}_0). \tag{5.13}$$

Proof See the references cited above. $\qquad\square$

Remark The use of the superscripts N and D relates to the Neumann and Dirichlet differential operators generated by the differential equation (5.1) but this connection need not be further considered in this paper.

COROLLARY 1 *The m-coefficient given by (5.10) has simple poles at the set* $\{\mu_n^N : n \in \mathbb{N}_0\}$ *and simple zeros at the set* $\{\mu_n^D : n \in \mathbb{N}_0\}$; *at all other points of* $\mathbb{C}$ *the function is holomorphic but not zero.*

Proof Clear. $\qquad\square$

COROLLARY 2 *Define, for all* $x \in [0, \infty)$ *and* $n \in \mathbb{N}_0$, *the functions*

$$\varphi_n^N(x) := \mathrm{Ai}(x - \mu_n^N) \quad and \quad \varphi_n^D := \mathrm{Ai}(x - \mu_n^D). \tag{5.14}$$

Then

(i) $\{\varphi_n^N : n \in \mathbb{N}_0\}$ *and* $\{\varphi_n^D : n \in \mathbb{N}_0\}$ *are solutions of the equation (5.1) for* $\tau = \mu_n^N$ *or* μ_n^D, *respectively, for all* $n \in \mathbb{N}_0$.
(ii) *with the inequality domain D defined by (5.3)*

$$\varphi_n^N \in D \quad and \quad \varphi_n^D \in D \quad (n \in \mathbb{N}_0). \tag{5.15}$$

Proof Clear. $\qquad\square$

5.3   The Inequality

We are now in a position to state the inequality theorem for this case; the form of the result is similar to the case discussed in Theorem 3 of Section 3 above; we recall the notation that if the general inequality (1.7) is valid then $K(\tau)$ is the best-possible number in the inequality.

THEOREM 6   *The inequality* (5.2) *has the properties*:

1. $K(\tau) = +\infty$ *for all* $\tau \in \{\mathbb{R}\backslash(\{\mu_n^N : n \in \mathbb{N}\} \cup \{\mu_n^D : n \in \mathbb{N}_0\})\}$.
2. $K(\tau) = 4$ *for all* $\tau \in (\{\mu_n^N : n \in \mathbb{N}\} \cup \{\mu_n^D : n \in \mathbb{N}_0\})$.
3. *If* $\tau = \mu_n^N$ *for some* $n \in \mathbb{N}_0$ *then there is equality in* (5.2) *if and only if*

$$f(x) = \alpha \varphi_n^N(x) \quad (x \in [0, \infty));$$

   here the weak case of equality in Theorem 2 holds, and $E_+(\mu_n^N) = E_-(\mu_n^N) = \emptyset$.
4. *If* $\tau = \mu_n^D$ *for some* $n \in \mathbb{N}_0$ *then there is equality in* (5.2) *if and only if*

$$f(x) = \alpha \varphi_n^D(x) \quad (x \in [0, \infty));$$

   here the weak case of equality in Theorem 2 holds, and $E_+(\mu_n^D) = E_-(\mu_n^D) = \emptyset$.

Proof   This proof is given in several stages now to follow.

5.4   The m-coefficient

Since in this case the m-coefficient is meromorphic on $\mathbb{C}$ we use m alone as given by the formulae

$$m(\lambda) = -\frac{\mathrm{Ai}(-\lambda)}{\mathrm{Ai}'(-\lambda)} \quad (\lambda \in \mathbb{C}\backslash\Lambda) \tag{5.16}$$

where the set $\Lambda \subset \mathbb{R}$ is defined by

$$\Lambda := \{\mu_n^N : n \in \mathbb{N}_0\} \cup \{\mu_n^D : n \in \mathbb{N}_0\}. \tag{5.17}$$

We have to consider the effect of the shift or translation τ and so define as in (2.11)

$$m_\tau(\lambda) := m(\lambda + \tau) \quad (\lambda \in \mathbb{C}), \tag{5.18}$$

and to this m-coefficient we apply the HELP criteria as given in Theorem 2 of Section 2 above.

We first show that the behaviour of the terms $\mp \operatorname{Im}(\lambda^2 m_\tau(\lambda))$ on the lines $L_\pm(\psi)$ for large values of $|\lambda|$ can be made to conform to the criteria provided that ψ is taken to satisfy the condition $\psi \in (\frac{1}{3}\pi, \frac{1}{2}\pi)$.

This done it is sufficient to consider the criteria in a neighbourhood of the origin 0 of the plane $\mathbb{C}$; for the necessary compactness argument see [14, Section 14]. Thus we apply the criteria to m_τ in the neighbourhood of 0, firstly for $\tau \notin \Lambda$ and then, secondly, for $\tau \in \Lambda$.

5.5   Some Formulae

For any $\tau \in \mathbb{R}$ define $f_\tau : (-\infty, 0) \cup (0, \infty) \times (0, \frac{1}{2}\pi] \to \mathbb{R}$ by, see definitions in Section 2 above,

$$f_\tau(\rho, \psi) := -\operatorname{Im}\big(\lambda^2 m(\lambda + \tau)\big) \quad (\lambda \in L_+(\psi))$$
$$:= +\operatorname{Im}\big(\lambda^2 m(\lambda + \tau)\big) \quad (\lambda \in L_-(\psi)). \qquad (5.19)$$

Firstly, for $\rho > 0$ we have, using (5.16),

$$f_\tau(\rho, \psi) = \operatorname{Im}\left(\frac{\rho^2 e^{2i\psi}\operatorname{Ai}(-\rho e^{i\psi} - \tau)}{\operatorname{Ai}'(-\rho e^{i\psi} - \tau)}\right)$$
$$= \frac{\rho^2\big[e^{2i\psi}\operatorname{Ai}(-\rho e^{i\psi} - \tau)\operatorname{Ai}'(-\rho e^{-i\psi} - \tau) - e^{-2i\psi}\operatorname{Ai}(-\rho e^{-i\psi} - \tau)\operatorname{Ai}'(-\rho e^{i\psi} - \tau)\big]}{2i|\operatorname{Ai}'(-\rho e^{i\psi} - \tau)|^2}$$

$$(5.20)$$

and, secondly, for $\rho < 0$ in a similar calculation

$$f_\tau(\rho, \psi)$$
$$= \frac{\rho^2\big[e^{2i\psi}\operatorname{Ai}(\rho e^{i\psi} - \tau)\operatorname{Ai}'(\rho e^{-i\psi} - \tau) - e^{-2i\psi}\operatorname{Ai}(\rho e^{-i\psi} - \tau)\operatorname{Ai}'(\rho e^{i\psi} - \tau)\big]}{2i|\operatorname{Ai}'(\rho e^{i\psi} - \tau)|^2}.$$

$$(5.21)$$

Here we have used the properties $\bar{\operatorname{A}}i(z) = \operatorname{Ai}(\bar{z})$ and $\bar{\operatorname{A}}i'(z) = \operatorname{Ai}'(\bar{z})$ since the function Ai is real-valued on the real axis $\mathbb{R}$ of $\mathbb{C}$.

5.6   Large Values of $|\lambda|$

To settle the position when $|\lambda|$ is large we make use of the asymptotic formula for the m-coefficient m_τ; this result takes the form, for

each $\tau \in \mathbb{R}$,

$$m_\tau(\lambda) = m(\lambda + \tau) = \frac{i}{\sqrt{\lambda}}\left(1 + O(|\lambda|)^{-1}\right) \quad (|\lambda| \to +\infty) \qquad (5.22)$$

where the O-term is uniformly valid for any region of $\mathbb{C}$ determined by

$$\delta \leq \arg(\lambda) \leq \pi - \delta \quad \text{and} \quad \pi + \delta \leq \arg(\lambda) \leq 2\pi - \delta \qquad (5.23)$$

for any $\delta \in (0, \frac{1}{2}\pi)$. This result follows from [9, Theorem 10.1] or from the asymptotic formulae given in [1, 10.4.59 to 10.4.62].

   If the result (5.22) is inserted into the formulae (5.16) then we obtain, for any $\psi \in (\frac{1}{3}\pi, \frac{1}{2}\pi]$ and any $\tau \in \mathbb{R}$,

$$\lim_{|\lambda| \to +\infty} \mp \operatorname{Im}\left(\lambda^2 m(\lambda + \tau)\right) = +\infty \quad \text{for} \ \lambda \in L_\pm(\psi) \qquad (5.24)$$

and uniformly for $\psi \in [\frac{1}{3}\pi + \varepsilon, \frac{1}{2}\pi]$ with $\varepsilon \in (0, \frac{1}{6}\pi)$.

   Similarly we obtain, for any $\psi \in (0, \frac{1}{3}\pi)$ and any $\tau \in \mathbb{R}$,

$$\lim_{|\lambda| \to +\infty} \mp \operatorname{Im}\left(\lambda^2 m(\lambda + \tau)\right) = -\infty \quad \text{for} \ \lambda \in L_\pm(\psi) \qquad (5.25)$$

and uniformly for $\psi \in [\varepsilon, \frac{1}{3}\pi - \varepsilon]$ with $\varepsilon \in (0, \frac{1}{3}\pi)$.

   The results (5.24) and (5.25) show, in the notation of Theorem 2 of Section 2, that $\theta_\pm \in [\frac{1}{3}\pi, \frac{1}{2}\pi]$ and so we obtain

$$\theta_0 \geq \tfrac{1}{3}\pi. \qquad (5.26)$$

   For the validity of the inequality (5.2) the results (5.24) and (5.26) show that it is sufficient to consider the behaviour of f_τ, see (5.19), in the neighbourhood of the origin 0 of $\mathbb{C}$.

5.7   The Case When τ is not a Pole or Zero of m

This is the case when see, (5.17),

$$\tau \notin \Lambda; \qquad (5.27)$$

these are the points for which $m(\cdot)$ is regular but not zero; it follows from a general theorem [14, Section 15, Theorem 1] that the inequality

(5.2) is not valid at such points. However it is interesting to see this result following from the properties of the mapping f_τ since, from (5.20) and (5.21) respectively, we obtain for all $\psi \in [0, \frac{1}{2}\pi]$, but in particular for $\psi \in [\frac{1}{3}\pi, \frac{1}{2}\pi)$,

$$\lim_{\rho \to 0+} \rho^{-2} f_\tau(\rho, \psi) = \frac{\text{Ai}(-\tau)\text{Ai}'(-\tau)}{|\text{Ai}'(-\tau)|^2} \sin(2\psi) \qquad (5.28)$$

$$\lim_{\rho \to 0-} \rho^{-2} f_\tau(\rho, \psi) = -\frac{\text{Ai}(-\tau)\text{Ai}'(-\tau)}{|\text{Ai}'(-\tau)|^2} \sin(2\psi). \qquad (5.29)$$

From (5.27) it follows that

$$0 < |\text{Ai}(-\tau)\text{Ai}'(-\tau)| < +\infty$$

and so one of the two non-zero expressions (5.28) and (5.29) is negative for all $\psi \in [\frac{1}{3}\pi, \frac{1}{2}\pi)$; this implies that, in terms of the definitions given before Theorem 2 of Section 2, $\theta_0 = \frac{1}{2}\pi$ and so the inequality is not valid; this result holds for all $\tau \notin \Lambda$.

5.8 The Case When τ is Either a Pole or Zero of m

This is the case when, see (5.17),

$$\tau \in \Lambda; \qquad (5.30)$$

these are the points for which $m(\cdot)$ has a simple pole or a simple zero; again it follows from a general theorem [14, Section 16, Theorem 1] that the inequality (5.2) is valid at such points. However to complete the proof of Theorem 6 above, we have to obtain the best-possible numbers for the inequality and to decide on the cases of equality, if any, for all $\tau \in \Lambda$. Note that this condition implies that

$$\text{either Ai}(\tau) = 0 \text{ or Ai}'(\tau) = 0, \qquad (5.31)$$

but not both of these two functions are zero.

The numerical results reported in [7, Section 4] suggested that the results given in Theorem 6 above hold and so the criteria given in Theorem 2 are tested analytically on the lines $L_\pm(\frac{1}{3}\pi)$.

Returning to (5.19) we define F on $\mathbb{R}\backslash\{0\}$ by

$$F(\rho) := \rho^{-2} f_\tau(\rho, \tfrac{1}{3}\pi) \quad (\rho \in (-\infty, 0) \cup (0, \infty)) \qquad (5.32)$$

From (5.20) and (5.21)

$$\lim_{\rho \to 0+} F(\rho) = \lim_{\rho \to 0-} F(\rho) = \mathrm{Ai}(-\tau)\mathrm{Ai}\,'(-\tau)\sin\left(\tfrac{2}{3}\pi\right) = 0 \qquad (5.33)$$

using $\tau \in \Lambda$ and the result (5.31). Thus to complete the definition of F we put $F(0) = 0$ and then $F \in C(-\infty, \infty)$.

A calculation, using (5.20) and (5.21), and recalling (5.7) shows that

$$F'(\rho) = |\mathrm{Ai}(-\rho e^{i\pi/3} - \tau)|^2 \tau \sin\left(\tfrac{1}{3}\pi\right) \quad (\rho \in (0, \infty))$$

and

$$F'(\rho) = |\mathrm{Ai}(\rho e^{i\pi/3} - \tau)|^2 \tau \sin\left(\tfrac{1}{3}\pi\right) \quad (\rho \in (-\infty, 0)).$$

Since $\tau > 0$, from $\tau \in \Lambda$, it follows that F is strictly increasing on $(-\infty, 0) \cup (0, \infty)$ and so, with $F(0) = 0$, we obtain

$$F(\rho) < 0 \; (\rho \in (-\infty, 0)) \quad \text{and} \quad F(\rho) > 0 \; ((\rho \in (0, \infty)). \qquad (5.34)$$

From (5.32) and (5.34) it now follows that for all $\tau \in \Lambda$

$$f_\tau\left(\rho, \tfrac{1}{3}\pi\right) > 0 \; (\rho \in (-\infty, 0)) \quad \text{and} \quad f_\tau\left(\rho, \tfrac{1}{3}\pi\right) > 0 \; ((\rho \in (0, \infty)) \quad (5.35)$$

and so the criteria of Theorem 2 above are satisfied on the two rays $L_-(\tfrac{1}{3}\pi)$ and $L_+(\tfrac{1}{3}\pi)$; this implies that, again for $\tau \in \Lambda$,

$$\theta_- \leq \tfrac{1}{3}\pi \quad \text{and} \quad \theta_+ \leq \tfrac{1}{3}\pi;$$

thus

$$\theta_0 \leq \tfrac{1}{3}\pi. \qquad (5.36)$$

Taking together (5.26) and (5.36) we obtain

$$\theta_0 = \tfrac{1}{3}\pi$$

and this result is valid for all $\tau \in \Lambda$. From Theorem 2 above it now follows that for this example

$$K(\tau) = 4 \quad (\tau \in \Lambda).$$

To complete the proof we note that the two results in (5.35) show that

$$E_-(\tau) = E_+(\tau) = \emptyset \quad (\tau \in \Lambda)$$

and so exclude the possibility of any normal case of equality. However the solutions $\{\varphi_n^N : n \in \mathbb{N}_0\}$ and $\{\varphi_n^D : n \in \mathbb{N}_0\}$, as defined in Theorem 5 above, provide all the weak cases of equality as described in the statement of Theorem 6.   □

6   THE CASE $q(x) = -x^2$

This example has the differential equation

$$-y''(x) - (x^2 + \tau)y(x) = \lambda y(x) \quad (x \in [0, \infty)) \qquad (6.1)$$

and the associated inequality is

$$\left(\int_0^{\to \infty} \{f'(x)^2 - (x^2 + \tau)f(x)^2\}\, dx \right)^2$$
$$\leq K(\tau) \int_0^\infty f(x)^2\, dx \int_0^\infty \{f''(x) + (x^2 + \tau)f(x)\}^2\, dx \quad (6.2)$$

with domain

$$D = \{f : [0, \infty) \to \mathbb{R} \colon f, f' \in AC_{\mathrm{loc}}[0, \infty)$$

and

$$f, f'' + x^2 f \in L^2[0, \infty)\}. \qquad (6.3)$$

6.1 The m-coefficient

The equation (6.1) has solutions in terms of the Weber parabolic cylinder functions, see [1, Chapter 19]. As with the example $q(x) = -x$ this equation is strong limit-point at ∞ in $L^2[0, \infty)$ but in general the integral on the left-hand side is only conditionally convergent. Also in this example the two components of the Nevanlinna function $m_{\pm}$ are not analytical continuations of each other and are given by separate formulae

$$m_+(\lambda) = \exp\left(\tfrac{1}{4}\pi i\right) \frac{\Gamma\left(\tfrac{1}{4} - \tfrac{1}{4}i\lambda\right)}{2\Gamma\left(\tfrac{3}{4} - \tfrac{1}{4}i\lambda\right)} \quad (\lambda \in \mathbb{C}_+) \tag{6.4}$$

$$m_-(\lambda) = \exp\left(-\tfrac{1}{4}\pi i\right) \frac{\Gamma\left(\tfrac{1}{4} + \tfrac{1}{4}i\lambda\right)}{2\Gamma\left(\tfrac{3}{4} + \tfrac{1}{4}i\lambda\right)} \quad (\lambda \in \mathbb{C}_-). \tag{6.5}$$

6.2 The Inequality

For this example we have

THEOREM 7   *For $\tau = 0$ the inequality (6.2) has the properties*

$$\theta_+ = \tfrac{3}{8}\pi \qquad \theta_- = \tfrac{1}{8}\pi \qquad E_+ = E_- = \emptyset;$$

thus $\theta_0 = \tfrac{3}{8}\pi$ and so from Theorem 2 of Section 2 it follows that

$$K(0) = 4 + 2\sqrt{2},$$

and the only case of equality is the null function.

Proof   (1) The determination of θ_+.

Let $z = x + iy = re^{i\theta}$ and recall $\lambda = \rho e^{i\psi}$; set $z = -i\lambda/4$ so that for $\psi \in (0, \tfrac{1}{2}\pi]$ we have $\theta \in (-\tfrac{1}{2}\pi, 0]$.

Now define the open, lower-half z-plane $\mathbb{C}_-$ by

$$\mathbb{C}_- := \{z \in \mathbb{C} : y < 0\}$$

and $u : \mathbb{C}_- \to \mathbb{R}$ by, using (6.4)

$$u(z) := \arg\left(\lambda^2 m_+(\lambda)\right) = \arg\left\{-z^2 \frac{\Gamma\left(\tfrac{1}{4} + z\right)}{\Gamma\left(\tfrac{3}{4} + z\right)} e^{i\pi/4}\right\} \quad (z \in \mathbb{C}_-).$$

We choose that value of the argument of u which is $5\pi/4$ on the real positive z-axis; then u is well-defined and continuous in $\mathbb{C}_-$.

We require a number of general results from the theory of harmonic and subharmonic functions; for the general theory see [19], and for application to inequalities of the kind considered in this paper see [12, Sections 2 and 3].

We define an exceptional set (E-set) to be a finite or countably infinite set of isolated points of $\mathbb{R} \subset \mathbb{C}$, together with the limit points $\pm\infty$.

For the two results that follow D is an open set of the complex plane $\mathbb{C}$.

(A) **The extended maximum principle.** *If the function v is bounded and harmonic in D and*

$$\limsup v(z) \leq 0 \tag{6.6}$$

as $z \to \zeta$ from the interior of D for all $\zeta \in \mathbb{R}$, apart from an E-set, then $v(z) < 0$ ($z \in D$) or $v(z) = 0$ ($z \in D$).

(B) **A uniqueness theorem.** *If in (A) the condition (6.6) is replaced by*

$$\lim v(z) = 0 \tag{6.7}$$

then $v(z) = 0$ ($z \in D$).

The next result is taken from [12, Section 3, Lemma 1].

(C) **A property of the Gamma function.** *If*

$$h(z) := \arg\left\{\frac{\Gamma(z)}{\Gamma(z + \frac{1}{2})}\right\} \quad (z \in \mathbb{C}_-)$$

then

$$0 < h(z) < \pi \quad (z \in \mathbb{C}_-).$$

We now return to the function u and consider its boundary values as we approach a point $\zeta \in \mathbb{R}$ from within $\mathbb{C}_-$. These boundary values exist and are locally constant in the open intervals of $\mathbb{R}$ whose end-points are determined by the E-set consisting of the points where the

meromorphic function φ, defined by

$$\varphi(z) := -z^2 \frac{\Gamma(\frac{1}{4}+z)}{\Gamma(\frac{3}{4}+z)} e^{i\pi/4},$$

has a zero or a pole, *i.e*

(a) the origin 0 where φ has a double zero;
(b) the points $\{-n-\frac{1}{4}: n \in \mathbb{N}_0\}$ where φ has simple poles;
(c) the points $\{-n-\frac{3}{4}: n \in \mathbb{N}_0\}$ where φ has simple zeros.

Respectively at these points, and as the variable z passes in $\mathbb{C}_-$ to the left from a point on the real positive axis to points on the negative axis,

(a) $u(z) - 2\arg(z)$ remains continuous at 0
(b) $u(z) + \arg(z+n+\frac{1}{4})$ remains continuous at $-n-\frac{1}{4}$ for all $n \in \mathbb{N}_0$
(c) $u(z) - \arg(z+n+\frac{3}{4})$ remains continuous at $-n-\frac{3}{4}$ for all $n \in \mathbb{N}_0$.

We note also that $\arg(z - \zeta_0)$, considered as a function in $\mathbb{C}_-$, has boundary values $-\pi$ and 0 at real points ζ such that $\zeta < \zeta_0$ and $\zeta > \zeta_0$ respectively.

Thus as the variable z passes from right to left along the real axis across the points of the E-set, the function $u(z)$:

$$\text{decreases by } 2\pi \text{ at the point 0 of (a)}$$
$$\text{increases by } \pi \text{ at the points } \left\{-n-\tfrac{1}{4}\right\} \text{ of (b)}$$
$$\text{decreases by } \pi \text{ at the points } \left\{-n-\tfrac{3}{4}\right\} \text{ of (c)}.$$

Recalling that $u(\zeta) = 5\pi/4$ for all $\zeta > 0$, we deduce that

$$u(z) = -3\pi/4 \text{ for } -\tfrac{1}{4} < \zeta < 0 \text{ and for } -n-\tfrac{1}{4} < \zeta < -n+\tfrac{1}{4} \ (n \in \mathbb{N})$$
$$u(z) = \pi/4 \text{ for } -n-\tfrac{3}{4} < \zeta < -n-\tfrac{1}{4} \ (n \in \mathbb{N}_0). \tag{6.8}$$

We also deduce from (C) above that u is bounded in $\mathbb{C}_-$; more precisely

$$-\tfrac{3}{4}\pi < u(z) < \tfrac{9}{4}\pi \quad (z \in \mathbb{C}_-)$$

since

$$-\tfrac{3}{4}\pi < \arg\left(-z^2 e^{i\pi/4}\right) < \tfrac{5}{4}\pi \quad (z \in \mathbb{C}_-).$$

We now compare u in $\mathbb{C}_-$ with the function $v: \mathbb{C}_- \to \mathbb{R}$ defined by

$$v(z) := 2\arg(z) + \tfrac{5}{4}\pi \quad (z \in \mathbb{C}_-)$$

where the principal value of $\arg(z)$ is chosen. Then v has boundary values $-\tfrac{3}{4}\pi$ and $\tfrac{5}{4}\pi$ on the negative and positive axes respectively, and it now follows from (A) that

$$v(z) < u(z) \quad (z \in \mathbb{C}_-).$$

In particular we have

$$u(z) > \pi \quad \text{for} \quad -\tfrac{1}{8}\pi \le \arg(z) \le 0;$$

also by looking at boundary values and using (A) above it follows that

$$u(z) < \tfrac{5}{4}\pi \quad \text{for} \quad -\tfrac{1}{8}\pi \le \arg(z) \le 0.$$

Thus

$$\operatorname{Im}\left\{ -z^2 \frac{\Gamma(\tfrac{1}{4}+z)}{\Gamma(\tfrac{3}{4}+z)} e^{i\pi/4} \right\} < 0 \quad \text{for} \quad -\tfrac{1}{8}\pi \le \arg(z) \le 0. \tag{6.9}$$

On the other hand we note that the function $u - v$ has boundary values zero on both sides of the origin 0 so that

$$\lim_{\substack{z \to 0 \\ z \in \mathbb{C}_-}} (u(z) - v(z)) = 0.$$

If then we choose $z = re^{i\theta}$ with $\theta = -\tfrac{1}{8}\pi - \delta$, where δ is a small positive number and let r tend to zero we deduce that, since $v(z) = \pi - 2\delta$ on this ray,

$$\lim_{r \to 0+} u(z) = \pi - 2\delta.$$

Thus for this value of θ and for all small r we have

$$\operatorname{Im}\left\{ -z^2 \frac{\Gamma(\tfrac{1}{4}+z)}{\Gamma(\tfrac{3}{4}+z)} e^{i\pi/4} \right\} > 0. \tag{6.10}$$

Translating the results (6.9) and (6.10) back into the λ-plane this yields

$$-\mathrm{Im}\{\lambda^2 m_+(\lambda)\} > 0 \text{ for all } \lambda \text{ with } |\lambda| > 0 \text{ and } \tfrac{3}{8}\pi \leq \arg(\lambda) \leq \tfrac{1}{2}\pi;$$
$$\text{(6.11)}$$

but if $\arg(\lambda) = \tfrac{3}{8}\pi - \delta$, where δ is a small positive number, then when $|\lambda|$ is sufficiently small, depending upon δ, we have

$$-\mathrm{Im}\{\lambda^2 m_+(\lambda)\} < 0. \qquad (6.12)$$

From the results (6.11) and (6.12), and the notations given in Theorem 2 of Section 2 we have

$$\theta_+ = \tfrac{3}{8}\pi \quad \text{and} \quad E_+ = \emptyset. \qquad (6.13)$$

(2) The determination of θ_-.
We now define similarly, using (6.5), but with $z = i\lambda/4$, $z \in \mathbb{C}_-$,

$$u(z) := \arg\{\lambda^2 m_-(\lambda)\} = \arg\left\{-z^2 \frac{\Gamma(\tfrac{1}{4} + z)}{\Gamma(\tfrac{3}{4} + z)} e^{-i\pi/4}\right\}.$$

We are interested in the range $-\pi < \arg(\lambda) < -\tfrac{1}{2}\pi$ and hence $-\tfrac{1}{2}\pi < \arg(z) < 0$.
This time u has boundary values, compare with (6.8),

$$u(\zeta) = \tfrac{3}{4}\pi \text{ for all } \quad \zeta > 0$$
$$u(\zeta) = -\tfrac{5}{4}\pi \text{ for } -\tfrac{1}{4} < \zeta < 0 \text{ and for } -n - \tfrac{1}{4} < \zeta < -n + \tfrac{1}{4} \, (n \in \mathbb{N})$$
$$u(\zeta) = -\tfrac{1}{4}\pi \text{ for } -n - \tfrac{3}{4} < \zeta < -n - \tfrac{1}{4} \, (n \in \mathbb{N}_0).$$

If, as before, we introduce a comparison function $v(z) = 2\arg(z) + \tfrac{3}{4}\pi$, then we see that

$$u(z) > 2\arg(z) + \tfrac{3}{4}\pi \quad (z \in \mathbb{C}_-)$$

but that

$$\lim_{\substack{z \to 0 \\ z \in \mathbb{C}_-}} (u(z) - 2\arg(z)) = \tfrac{3}{4}\pi.$$

These results show that

$$u(z) > 0 \text{ for } \theta = \arg(z) \geq -\tfrac{3}{8}\pi$$
$$\text{or equivalently for } -\tfrac{7}{8}\pi \leq \arg(\lambda) \leq -\tfrac{1}{2}\pi$$

and that the lower bound $-\tfrac{7}{8}\pi$ cannot be increased to a larger negative number.

From these results then we conclude that

$$\theta_- = \tfrac{1}{8}\pi \quad \text{and} \quad E_- = \emptyset.$$

There are no weak cases of equality for the inequality (6.2) since it may be seen from the form of the $m_\pm$-coefficient (6.4) and (6.5) that there are no eigenvalues for either the associated Dirichlet or the Neumann boundary value problems, and hence no non-null solutions to the weak problems of 4 in Theorem 2, Section 2 above.

This concludes the proof of the Theorem. $\qquad \square$

6.3 Remark

For all other values of the shift parameter τ there are no analytical results presently available; however the numerical techniques lead to the following analytical conjectures:

1. The inequality (6.2) is valid for all $\tau \in \mathbb{R}$.
2. The best-possible number function $K(\cdot)$ is continuous and monotonic decreasing on $\mathbb{R}$.
3. There exists a positive number τ_0 such that

$$K(\tau) > 4 \quad (\tau \in (-\infty, \tau_0))$$
$$= 4 \quad (\tau \in [\tau_0, \infty)).$$

4. $\lim_{\tau \to -\infty} K(\tau) = +\infty$.
5. For all $\tau \in \mathbb{R}$ there are no cases of equality other than the null function.

7 THE CASE $q(x) = \tfrac{1}{2}(x+1)^{-2}$

This example is similar in many ways to the original Hardy–Littlewood inequality in that the differential operators generated in the Hilbert function space $L^2[0, \infty)$ have a continuous spectrum on the half-line

$[0, \infty)$ of the complex spectral plane $\mathbb{C}$. However there is one significant difference; the Hardy–Littlewood inequality is valid for $\tau = 0$ whilst the inequality resulting from the above given coefficient q is not valid for $\tau = 0$. The importance of this result lies in the conjecture that had been made that a necessary and sufficient condition for the general inequality (1.7) to be valid is that the shift parameter τ belong to the union of the spectra of the Neumann and Dirichlet differential operators generated by the differential equation (1.5). We do not enter into further discussion on this conjecture; it is sufficient here to state that the example of this Section shows that the conjecture is false.

The inequality resulting from this coefficient q is considered in some detail in [8, Section 2].

7.1   The Background

This example has the differential equation

$$-y''(x) + \left(\tfrac{1}{2}(x + 1)^{-2} - \tau\right)y(x) = \lambda y(x) \quad (x \in (0, \infty)) \qquad (7.1)$$

and the associated inequality is

$$\left(\int_0^\infty \left(f'(x)^2 + \left(\tfrac{1}{2}(x + 1)^{-2} - \tau\right)f(x)^2\right) dx\right)^2$$

$$\leq K(\tau) \int_0^\infty f(x)^2 \, dx \int_0^\infty \left\{f''(x) - \left(\tfrac{1}{2}(x + 1)^{-2} - \tau\right)f(x)\right\}^2 dx \qquad (7.2)$$

with domain

$$D = \{f : [0, \infty) \to \mathbb{R} : f, f' \in AC_{\mathrm{loc}}[0, \infty)$$

and

$$f, f'' - \tfrac{1}{2}(x + 1)^{-2}f \in L^2[0, \infty)\}. \qquad (7.3)$$

7.2   The m-coefficient

The equation (7.1) has solutions in terms of the Bessel functions, see [21, Section 4.10],

$$(x + 1)^{1/2}J_{\sqrt{3}/2}((x + 1)\sqrt{\lambda}) \quad \text{and} \quad (x + 1)^{1/2}J_{-\sqrt{3}/2}((x + 1)\sqrt{\lambda}).$$

$$(7.4)$$

The solution of (7.1) that is in the space $L^2[0, \infty)$ for $\lambda \in \mathbb{C} \backslash \mathbb{R}$ is best expressed in the form

$$(x + 1)^{1/2} H^{(1)}_{\sqrt{3}/2}((x + 1)\sqrt{\lambda}) \qquad (7.5)$$

and this leads to the following determination of the m-coefficient for this example, for all $\tau \in \mathbb{R}$,

$$m_\tau(\lambda) = \frac{-H^{(1)}_{\sqrt{3}/2}(\sqrt{\lambda + \tau})}{\frac{1}{2} H^{(1)}_{\sqrt{3}/2}(\sqrt{\lambda + \tau}) + \sqrt{\lambda + \tau} H^{(1)'}_{\sqrt{3}/2}(\sqrt{\lambda + \tau})} \quad (\lambda \in \mathbb{C} \backslash [0, \infty))$$

$$(7.6)$$

with $\sqrt{(\cdot)}$ determined as before. Note that in this case $m_\mp$ is the continuation of $m_\pm$.

From the properties of the solutions (7.4) and (7.5) it may be seen that the differential equation (7.1) has no solutions in $L^2[0, \infty)$ when $\lambda \geq 0$ and one independent solution in $L^2[0, \infty)$ when $\lambda < 0$. These properties are mirrored in the form of the m-coefficient when $\tau = 0$; $m(\lambda) = m_0(\lambda)$ is holomorphic on $\mathbb{C} \backslash [0, \infty)$, has a branch point at the origin 0 and is discontinuous from above and below on the positive axis $[0, \infty)$ of $\mathbb{C}$.

From [1, 9.1.9 and 9.1.31] we obtain for $\text{Re}(\nu) > 0$, as $|s| \to 0$ with $s \in \mathbb{C} \backslash [0, \infty)$,

$$H^{(1)}_\nu(s) \sim -\frac{i}{\pi} \Gamma(\nu) \left(\frac{2}{s}\right)^\nu$$

and

$$H^{(1)'}_\nu(s) \sim \frac{i\nu}{2\pi} \Gamma(\nu) \left(\frac{2}{s}\right)^{\nu+1};$$

from these results a calculation shows that

$$\lim_{\substack{|\lambda| \to 0+ \\ \lambda \in \mathbb{C} \backslash [0, \infty)}} m(\lambda) = \frac{2}{\sqrt{3} - 1} > 0. \qquad (7.7)$$

On the other hand the m-coefficient for the Hardy–Littlewood inequality, with $m(\lambda) = i/\sqrt{\lambda}$, satisfies

$$\lim_{\substack{|\lambda| \to 0+ \\ \lambda \in \mathbb{C} \setminus [0, \infty)}} |m(\lambda)| = +\infty. \tag{7.8}$$

The difference between the positive limit and the infinite limit of (7.7) and (7.8) respectively is, in part, responsible for the conclusion given in the Theorem that now follows in comparison with the valid inequality (1.1).

7.3   The Inequality

THEOREM 8   *For $\tau \leq 0$ the inequality (7.2) is not valid, i.e. $K(\tau) = +\infty$ for all $\tau \in (-\infty, 0]$.*

Proof   For $\tau = 0$ the analytical proof of this result follows by direct application of the criteria of Theorem 2 of Section 2 above to the limit result (7.7); see also [8, Section 2.1].

For $\tau = 0$ this conclusion also follows from the general result given in [14, Section 15, Theorem 1], since the coefficient q for (7.2) satisfies the condition $q(x) > 0$ $(x \in [0, \infty))$.

For $\tau < 0$ this conclusion follows from the fact that the m-coefficient is holomorphic at all points of the set $(-\infty, 0) \subseteq \mathbb{C}$; see [14, Section 16, Theorem 1].   □

7.4   Remark

For all other values of the shift parameter τ there are no analytical results presently available; however the numerical techniques and results given in [8, Section 2.1] lead to the following analytical conjectures:

1. The inequality (7.2) is valid for all $\tau \in (0, \infty)$.
2. The best possible number function $K(\cdot)$ is continuous and monotonic decreasing on $(0, \infty)$.
3. $0 < \theta_-(\tau) < \frac{1}{3}\pi \leq \theta_+(\tau)$ and $\frac{1}{3}\pi \leq \theta_0(\tau) < \frac{1}{2}\pi$ for all $\tau \in (0, \infty)$.
4. There exists a positive number τ_0, with the numerical value approximately 0.13, such that

(i) $4 < K(\tau) < +\infty$ $(\tau \in (0, \tau_0))$

(ii) $\lim_{\tau \to 0+} K(\tau) = +\infty$

(iii) $K(\tau) = 4$ $(\tau \in [\tau_0, \infty))$.

5. The cases of equality are as follows:

(i) for $\tau \in (0, \tau_0)$ the set $E_+(\tau)$ is a single point $\{\rho(\tau)\}$ so that there is a one-dimensional case of normal equality,

(ii) for $\tau = \tau_0$ there is a continuum of cases of normal equalities,

(iii) for $\tau \in (\tau_0, \infty)$ only the null function,

(iv) for all $\tau \in (0, \infty)$ there are no cases of weak equality.

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